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*Mathematica Slovaca*, Vol. 49 (1999), No. 1, 63--69

Persistent URL: [http://dml.cz/dmlcz/136741](http://dml.cz/dmlcz/136741)

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AN ABSTRACT UNIFORM BOUNDEDNESS RESULT

CHARLES SWARTZ

(Communicated by Miloslav Duchoň)

ABSTRACT. We prove a uniform boundedness result for spaces which have a family of projection operators satisfying certain properties. The result is used to show that the space of Pettis integrable functions is barrelled.

In [DFP1] Drewnowski, Florencio and Paul established an abstract uniform boundedness result and used the result to establish the fact that the space of Pettis integrable functions with respect to a finite measure is a barrelled space. They later extended their result to an arbitrary measure in [DFP2]. Their proof is based on the fact that the space of Pettis integrable functions has a family of “good projections”. The projections in this case, as well as other typical applications to function spaces, are multiplications by characteristic functions. In this section we establish a result similar to the ones in [DFP1] and [DFP2] and also use the result to establish the barrelledness of the space of Pettis integrable functions.

Throughout this section let $E$ be a Hausdorff locally convex TVS and let $A$ be an algebra of subsets of $S$. We assume that there exists a map $P: A \rightarrow L(E)$, the space of continuous linear operators on $E$. We denote the value of $P$ at $A \in A$ by $P_A$; if $y' \in E'$, $x \in E$, let $y'Px$ be the set function $A \mapsto \langle y', P_Ax \rangle$ from $A$ into $\mathbb{R}$ and let $v(y'Px)$ denote the variation of $y'Px$. We assume that $P$ satisfies the following additivity properties:

(i) $P_{\emptyset} = 0$, $P_S = I$,
(ii) $P$ is finitely additive.

We consider the following additional properties for $P$:

(D) For every $y' \in E'$, $x \in E$, the finitely additive set function $y'Px$ satisfies the decomposition property:

for every $\varepsilon > 0$ there exists a partition $\{B_1, \ldots, B_k\}$ of $S$ with $B_i \in A$ such that $v(y'P_{B_i}x) < \varepsilon$ for $i = 1, \ldots, k$ (Rao and Rao refer to the decomposition property (D) as “strongly continuous” ([RR])).

Key words: projection, Banach-Mackey space, Pettis integral.
Remark 1.

(a) If \( y'Px \) is bounded and non-atomic for every \( y' \in E' \), \( x \in E \) and \( A \) is a \( \sigma \)-algebra, then (D) is satisfied ([RR; 5.1.6]).

(b) If \( A \) is a \( \sigma \)-algebra and \( \mu \) is a \( \sigma \)-finite, non-atomic measure on \( A \) such that \( y'Px \) is \( \mu \)-continuous (i.e., \( \lim_{\mu(A) \to 0} y'P_{A}x = 0 \)), then (a) (and, therefore (D)) holds; in particular, if the \( E \)-valued set function \( P_{x} \) is \( \mu \)-continuous for every \( x \in E \), then (a) holds.

We further consider a gliding hump property for \( P \):

\[ (GHP) \quad \text{If} \quad \{A_n\} \text{ is a pairwise disjoint sequence from} \quad A, \quad \{x_j\} \text{ is a null sequence from} \quad E \quad \text{and} \quad H \quad \text{is a countable} \quad \sigma(E', E) \text{ bounded subset from} \quad E', \quad \text{then there is an increasing sequence} \quad \{n_j\} \text{ such that the series} \quad \sum_{j=1}^{\infty} P_{A_{n_j}}x_{n_j} \quad \text{is} \quad \sigma(E, H) \quad \text{convergent to some} \quad x \in E. \]

Further, we say that \( P \) satisfies the strong \( (GHP) \) property if the series above converges in the original topology of \( E \).

Remark 2. For example, let \( E \) be a complete metrizable space whose topology is generated by a quasi-norm \( || \cdot || \), and suppose that \( \{P_{A_j}\} \) is equicontinuous for every pairwise disjoint sequence \( \{A_j\} \subset A \). If \( \{x_j\} \) is a null sequence in \( E \), then \( P_{A_j}x_j \to 0 \) in \( E \) so there is a subsequence such that \( \sum P_{A_{n_j}}x_{n_j} \) converges in \( E \). Hence, strong \( (GHP) \) is satisfied.

For an example where \( (GHP) \) is satisfied but strong \( (GHP) \) is not, let \( \ell^\infty \) be equipped with \( \sigma(\ell^\infty, ba) \). Let \( A \) be the power set of \( N \) and for \( A \in A \) define \( P_{A}: \ell^\infty \to \ell^\infty \) by \( P_{A}x = C_{A}x \) for \( x \in \ell^\infty \) where \( C_{A} \) is the characteristic function of \( A \) and \( C_{A}x \) is the coordinatewise product of \( C_{A} \) and \( x \). Then (i) and (ii) are satisfied. Then \( o^j \to 0 \) in \( \sigma(\ell^\infty, ba) \) but no sub-series \( \sum o^{n_j} \) is \( \sigma(\ell^\infty, ba) \) convergent so strong \( (GHP) \) is not satisfied. However, let \( x^k \to 0 \) in \( \sigma(\ell^\infty, ba) \), let \( \{A_k\} \subset A \) be pairwise disjoint and let \( \{\nu_j\} \subset ba \). By Drewnowski's Lemma ([D], [DU; I.6]) there is a subsequence \( \{A_{n_j}\} \) such that each \( \nu_i \) is countably additive on the \( \sigma \)-algebra \( \Sigma \) generated by \( \{A_{n_j}\} \). Since \( x^k \to 0 \) in \( \sigma(\ell^\infty, ba) \), there exists \( M > 0 \) such that \( \|x^k\|_\infty \leq M \). Let \( x \) be the coordinatewise sum of \( \sum_{k=1}^{\infty} C_{A_{n_k}}x^{n_k} \). We claim that \( x = \sum_{k=1}^{\infty} C_{A_{n_k}}x^{n_k} \) in the topology \( \sigma(\ell^\infty, \{\nu_j\}) \). This follows since for every \( j \),

\[
\left| \langle \nu_j, x - \sum_{k=1}^{i} C_{A_{n_k}}x^{n_k} \rangle \right| = \left| \int_{N} \sum_{k=i+1}^{\infty} C_{A_{n_k}}x^{n_k} \, d\nu_j \right| \leq M|\nu_j|(\bigcup_{k=i+1}^{\infty} A_{n_k}) \to 0 \quad \text{as} \quad i \to \infty
\]

by the countable additivity of \( \nu_j \) on \( \Sigma \). Hence, \( (GHP) \) is satisfied.

We give further examples of spaces, including the space of Pettis integrable functions, satisfying \( (GHP) \) later.
THEOREM 3. Assume that $P$ satisfies $(D)$ and $(GHP)$. If $B \subseteq E'$ is $\sigma(E',E)$ bounded, then $B$ is $\beta(E',E)$ (strongly) bounded, i.e., $E$ is a Banach-Mackey space ([Wi; 10.4]).

Proof. Suppose the conclusion fails. Then there exists a null sequence $\{x_j\}$ in $E$ such that
\[ \sup \{|(y', x_j)| : y' \in B, \ j \in \mathbb{N} \} = \infty. \]

Pick $y'_1 \in B$, $n_1$ such that $|(y'_1, x_{n_1})| = |y'_1 P_{S} x_{n_1}| > 2$. From (D), there is a partition $\{B_1, \ldots, B_k\}$ of $S$ such that $v(y'_1 P_{B_i} x_{n_1}) < 1$ for $i = 1, \ldots, k$. From (ii) and the $\sigma(E',E)$ boundedness of $B$, we may assume that
\[ \sup \{|y'_i P_{B_1} x_{j}\} : y' \in B, \ j > n_1 \} = \infty. \]

Set $A_1 = S \setminus B_1$ and note from (ii) that $|y'_i P_{A_1} x_{n_1}| > 1$.

Now if we treat $B_1$ as $S$ was treated above there exist a partition $(A_2, B_2)$ of $B_1$, $y'_2 \in B$ and $n_2 > n_1$ such that $\sup \{|y'_i P_{B_2} x_{j}\} : y' \in B, \ j > n_2 \} = \infty$ and $|y'_2 P_{A_2} x_{n_2}| > 2$. Continuing this construction produces a pairwise disjoint sequence $\{A_j\}$ from $A$, $\{y'_j\} \subset B$ and a subsequence $\{x_{n_j}\}$ such that $|y'_i P_{A_j} x_{n_j}| > j$ for every $j$.

Now consider the matrix $M = [m_{ij}] = \{1 \ y'_i P_{A_j} x_{n_j}\}$. The columns of $M$ converge to 0 by the $\sigma(E',E)$ boundedness of $B$. Given any increasing sequence $\{r_j\}$ by $(GHP)$ there is a subsequence $\{s_j\}$ such that the series $\sum_{j=1}^{\infty} P_{A_{s_j}} x_{n_{s_j}}$ is $\sigma(E,\{y'_j\})$ convergent to some $x \in E$. Hence, $\sum m_{i,s_j} = (\frac{1}{i} y'_i, x) \to 0$ and $M$ is a $K$-matrix ([AS; §2]). By the Antosik-Mikusinski Theorem ([AS; 2.2]) the diagonal of $M$ converges to 0 contradicting the construction above. \hfill $\square$

We now give two examples which point out the importance of conditions $(D)$ and $(GHP)$.

EXAMPLE 4. Let $E$ be an arbitrary Hausdorff locally convex TVS. Let $\mathcal{P}$ be the power set of $\mathbb{N}$ and define $P: \mathcal{P} \to L(E)$ by $P_A = I$ if $1 \in A$ and $P_A = 0$ if $1 \notin A$. ($P$ is an operator version of the Dirac measure at 1). Then $y' P x$ is a “Dirac measure” with mass $(y', x)$ at 1. So property $(D)$ clearly fails to hold. Note, however, that conditions (i), (ii) and $(GHP)$ do hold. Thus, if we take $E$ to be any space which is not a Banach-Mackey space, Theorem 3 will fail.

EXAMPLE 5. Let $B$ be the Borel sets in $[0,1]$, and let $E$ be the space of all $B$-simple functions equipped with the $L^2$-norm with respect to Lebesgue measure. For $A \in B$ let $P_A$ be the projection defined by $P_A f = C_A f$. Since $f P g$ is non-atomic for every $f, g \in L^2[0,1]$, condition $(D)$ is satisfied (as well as (i) and (ii)). However, condition $(GHP)$ fails (take any pairwise disjoint sequence
of Borel sets \( \{A_j\} \) with positive measure and set \( f_j = \frac{1}{j} C_{A_j} \). Clearly \( E \) is not a Banach-Mackey space.

Drewnowski, Florencio and Paul proved results analogous to Theorem 3 in [DFP1] and [DFP2]. They assume that \( \mathcal{A} \) is a \( \sigma \)-algebra and the map \( P: \mathcal{A} \rightarrow L(E) \) is projection-valued. The conditions imposed on the map \( P \) are quite different from those in (D) and (GHP). In particular, in [DFP2] they assume that \( P(\mathcal{A}) \) is equicontinuous. On the other hand, they do not require any condition analogous to condition (D). Condition (D) effectively limits the applications of Theorem 3 to non-atomic measures. Condition (GHP) can be viewed as a continuous version of a gliding hump property for sequence spaces, called the zero gliding hump property (see [LS]).

We now give an application of Theorem 3 to the space of Pettis integrable functions. Let \( X \) be a Banach space, let \( \Sigma \) be a \( \sigma \)-algebra of subsets of \( S \) with \( \mu \) a measure on \( \Sigma \). A function \( f: S \rightarrow X \) is said to be weakly measurable if \( x' f \) is measurable for every \( x' \in X' \) and is said to be weakly \( \mu \)-integrable if \( x' f \) is \( \mu \)-integrable for every \( x' \in X' \). If \( f \) is weakly \( \mu \)-integrable, then for every \( A \in \Sigma \), \( x' \mapsto \int x' f \, d\mu \) defines a continuous linear functional \( x''_A \in X'' \) (apply the Closed Graph Theorem to show the linear map \( F: X' \rightarrow L^1(\mu) \), \( F(x') = x' f \), is continuous and then observe that \( x''_A = F'(C_A) \)). The element \( x''_A \) is sometimes called the Gelfand or Dunford integral of \( f \) over \( A \) and is denoted by \( \int_A f \, d\mu \).

Let \( \mathcal{G}^1(\mu, X) \) be the space of all Gelfand integrable functions; equip \( \mathcal{G}^1(\mu, X) \) with the semi-norm \( \| f \|_1 = \sup \left\{ \int_S |x' f| \, d\mu : \|x'\| \leq 1 \right\} \) (this quantity is finite by the continuity of the map \( F \) defined above). A function \( f \) is said to be Pettis integrable if \( f \) is Gelfand integrable and \( \int_A f \, d\mu \in X \) for every \( A \in \Sigma \). Let \( \mathcal{P}^1(\mu, X) \) be the space of all Pettis integrable functions equipped with the semi-norm \( \| \|_1 \). (See [DU] or [HP] for discussions of these integrals.) It is known that, in general, \( \mathcal{P}^1(\mu, X) \) is not complete ([Pe; 9.4]). However, using Theorem 3 we show that \( \mathcal{P}^1(\mu, X) \) is barrelled if \( \mu \) is \( \sigma \)-finite and non-atomic.

For \( A \in \Sigma \) define a projection \( P_A \) on \( \mathcal{P}^1(\mu, X) \) by \( P_A f = C_A f \). The map \( P \) obviously satisfies conditions (i) and (ii) above. We first show that \( P \) satisfies the strong (GHP) condition.

**Theorem 6.** \( \mathcal{P}^1(\mu, X) \) satisfies the strong (GHP) property.

**Proof.** Let \( \{A_j\} \) be a pairwise disjoint sequence from \( \Sigma \) and let \( \{f_j\} \) converge to 0 in \( \mathcal{P}^1(\mu, X) \). Pick a subsequence \( \{n_j\} \) satisfying \( \|f_{n_j}\|_1 < 1/2^j \). Let \( f \) be the pointwise sum of the series \( \sum_{j=1}^{\infty} C_{A_{n_j}} f_{n_j} \); \( f \) is obviously weakly \( \mu \)-measurable.
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If \( x' \in X' \), then \( x'f = \sum_{j=1}^{\infty} C_{A_{n_j}} x' f_{n_j} \) and \( |x'f| = \sum_{j=1}^{\infty} C_{A_{n_j}} |x'f_{n_j}| \) pointwise, and 
\[
\int_S |x'f| \, d\mu = \sum_{j=1}^{\infty} \int_{A_{n_j}} |x'f_{n_j}| \, d\mu \leq \|x'\| \sum_{j=1}^{\infty} \|f_{n_j}\|_1 < \infty
\]
implies that \( f \) is weakly \( \mu \)-integrable.

We next claim that \( \int f \, d\mu \in X \) for \( A \in \Sigma \). Since 
\[
\sum_{j=1}^{\infty} \left\| \sum_{j=1}^{\infty} f_{n_j} \right\|_1 < \infty,
\]
the series \( \sum_{j=1}^{\infty} \int_{A_{n_j}} f_{n_j} \, d\mu \) converges to some \( x_A \in X \) by the completeness of \( X \). Therefore, \( \langle x', x_A \rangle = \sum_{j=1}^{\infty} \int_{A_{n_j}} x'f_{n_j} \, d\mu \). Since \( |x'f| \geq \sum_{j=1}^{n} C_{A_{n_j}} x'f_{n_j} \) for every \( n \), the Dominated Convergence Theorem implies that 
\[
\int_A x'f \, d\mu = \sum_{j=1}^{\infty} \int_{A_{n_j}} x'f_{n_j} \, d\mu.
\]
Hence, \( \int f \, d\mu = x_A \in X \) as desired, and \( f \) is Pettis \( \mu \)-integrable.

Last, we claim that the series \( \sum_{j=1}^{\infty} C_{A_{n_j}} f_{n_j} \) converges to \( f \) is the norm of \( \mathcal{P}^1(\mu, X) \). This follows from
\[
\left\| f - \sum_{j=1}^{n} C_{A_{n_j}} f_{n_j} \right\|_1 = \sup \left\{ \int_S x' \left( \sum_{j=n+1}^{\infty} C_{A_{n_j}} f_{n_j} \right) \, d\mu : \|x'\| \leq 1 \right\}
\leq \sup \left\{ \sum_{j=n+1}^{\infty} \int_{A_{n_j}} |x'f_{n_j}| \, d\mu : \|x'\| \leq 1 \right\}
\leq \sum_{j=n+1}^{\infty} \|f_{n_j}\|_1 \to 0.
\]

The proof of Theorem 6 also establishes:

**Corollary 7.** The subspace of \( \mathcal{P}^1(\mu, X) \) consisting of the strongly \( \mu \)-measurable (countably valued) functions satisfies the strong (GHP) property.

We next consider property (D).

**Proposition 8.** If \( \mu \) is \( \sigma \)-finite and non-atomic, then \( \mathcal{P}^1(\mu, X) \) satisfies (D).

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Proof. Fix \( f \in \mathcal{P}^1(\mu, X) \). The indefinite Pettis integral of \( f \) is \( \mu \)-continuous, i.e.,

\[
\lim_{{\mu(A) \to 0}} \left\| \int_A f \, d\mu \right\| = \lim_{{\mu(A) \to 0}} \sup \left\{ \left\| \int_A x' f \, d\mu : \|x'\| \leq 1 \right\} \right. = 0
\]

([Pe], [HP]) so by [DS; III.1.5 and III.2.15],

\[
\lim_{{\mu(A) \to 0}} \sup \left\{ \int_A |x' f| \, d\mu : \|x'\| \leq 1 \right\} = \lim_{{\mu(A) \to 0}} \|C_A f\|_1 = 0.
\]

The result now follows from Remark 1(b).

From Proposition 8, Corollary 7 and Theorem 3, we obtain:

**Corollary 9.** If \( \mu \) is a \( \sigma \)-finite, non-atomic measure, then \( \mathcal{P}^1(\mu, X) \) and the subspace of strongly measurable (countably valued) functions is barrelled.

Drewnowski, Florencio and Pául generalize Corollary 9 to arbitrary (\( \sigma \)-finite) measures by decomposing the measure into its non-atomic and purely atomic parts (see [DFP1] or [DFP2]).

We can also use Theorem 3 to establish an interesting barrelledness result for the space of Bochner integrable functions. We refer the reader to [DU], [HP] for the basic properties of the Bochner integrable which we employ. Let \( Y \) be a subspace of \( X \) and let \( L^1(\mu, X) \) be the space of \( X \)-valued Bochner \( \mu \)-integrable functions equipped with the norm \( \|f\| = \int \|f(\cdot)\| \, d\mu \). Let \( L^1(\mu, Y) \) be the subspace of \( L^1(\mu, X) \) consisting of the \( Y \)-valued functions. For \( A \in \Sigma \) let \( P_A \) be the projection on \( L^1(\mu, X) \) defined by \( P_A f = C_A f \). The function \( P \) obviously satisfies conditions (i) and (ii). We consider conditions (D) and (GHP).

**Proposition 10.** \( L^1(\mu, Y) \) satisfies the strong (GHP) property.

Proof. Let \( \{f_k\} \) be a null sequence in \( L^1(\mu, Y) \) and let \( \{A_k\} \) be a pairwise disjoint sequence from \( \Sigma \). Pick a subsequence \( \{n_k\} \) such that \( \|f_{n_k}\| < 1/2^k \). Let \( f \) be the pointwise limit of the series \( \sum_{k=1}^{\infty} C_{A_{n_k}} f_{n_k} \). Then \( f \) is clearly strongly measurable, \( Y \)-valued, and since \( \int_S \|f(\cdot)\| \, d\mu = \sum_{k=1}^{\infty} \int_{A_{n_k}} \|f_{n_k}(\cdot)\| \, d\mu \leq \sum_{k=1}^{\infty} 1/2^k \), the series converges to \( f \) in \( L^1(\mu, Y) \).

**Proposition 11.** If \( \mu \) is \( \sigma \)-finite and non-atomic, then \( L^1(\mu, Y) \) satisfies (D).

Proof. For \( f \in L^1(\mu, Y) \), the map \( A \mapsto C_A f \) is \( \mu \)-continuous so the result follows from Remark 1(b).

We thus have
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**COROLLARY 12.** If $\mu$ is $\sigma$-finite and non-atomic, then $L^1(\mu, Y)$ is barrelled.

This is an interesting result in the sense that even though $Y$ may not be barrelled the space $L^1(\mu, Y)$ is barrelled. More general results are given in [DFP1] and [DFP2].

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Received October 5, 1995

Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003
U. S. A

E-mail: cswartz@nmsu.edu

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