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A NOTE ON SUMMABILITY METHODS

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ABSTRACT. The purpose of this paper is to establish some relations between the $|C,\alpha;\delta|_k$ and $|R,p_n;\delta|_k$ summability methods, where $\alpha > 0$ and $k \geq 1$.

1. Introduction

Let $\sum a_n$ be a given infinite series with (s_n) as the sequence of its nth partial sums. We denote by t_n^α the nth Cesaro means of order α, with $\alpha > -1$, of the sequence (na_n), i.e.,

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \,, \tag{1}$$

where

$$A_n^\alpha = O(n^\alpha)\,, \qquad A_0^\alpha = 1 \qquad \text{and} \qquad A_{-n}^\alpha = 0 \quad \text{for } n > 0\,, \qquad \alpha > -1\,. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha;\delta|_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if (see [3])

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_n^\alpha|^k < \infty\,. \tag{3}$$

If we take $\delta = 0$ (resp. $\delta = 0$ and $\alpha = 1$), then $|C,\alpha;\delta|_k$ summability is the same as $|C,\alpha|_k$ (resp. $|C,1|_k$) summability.

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty\,, \qquad (P_{-i} = p_{-i} = 0\,, \quad i \geq 1)\,. \tag{4}$$

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The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (T_n) of the (R, p_n) means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k < \infty, \tag{6}$$

and it is said to be summable $|R, p_n; \delta|_k$, $k \geq 1$, and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |\Delta T_{n-1}|^k < \infty, \tag{7}$$

where

$$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \qquad n \geq 1. \tag{8}$$

If we take $\delta = 0$, then $|R, p_n; \delta|_k$ summability reduces to $|R, p_n|_k$ summability.

The following theorems are known.

THEOREM A. ([5]) *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(n^\alpha p_n) \qquad \text{as} \quad n \to \infty. \tag{9}$$

If the series $\sum a_n$ is summable $|R, p_n|_k$, then it is also summable $|C, \alpha|_k$, $k \geq 1$ and $0 < \alpha < 1$.

THEOREM B. ([5]) *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(n p_n) \qquad \text{as} \quad n \to \infty. \tag{10}$$

If the series $\sum a_n$ is summable $|R, p_n|_k$, then it is also summable $|C, \alpha|_k$, $k \geq 1$ and $\alpha \geq 1$.

2.

The aim of this paper is to generalize above theorems for $|R, p_n; \delta|_k$ and $|C, \alpha; \delta|_k$ summability methods. Now, we shall prove the following theorem.

THEOREM 1. *Let (p_n) be a sequence of positive numbers which satisfy condition (9) of Theorem A. If the series $\sum a_n$ is summable $|R, p_n; \delta|_k$, then it is also summable $|C, \alpha; \delta|_k$, $k \geq 1$, $0 < \alpha < 1$ and $0 \leq \delta k < 1$.*

THEOREM 2. *Let* (p_n) *be a sequence of positive numbers which satisfy condition* (10) *of Theorem* B. *If the series* $\sum a_n$ *is summable* $|R, p_n; \delta|_k$, *then it is also summable* $|C, \alpha; \delta|_k$, $k \geq 1$, $\alpha \geq 1$ *and* $0 \leq \delta k < 1$.

It should be noted that if we take $\delta = 0$ in Theorem 1 and Theorem 2, then we get Theorem A and Theorem B, respectively.

We need the following lemma for the proof of our theorems.

LEMMA. ([6]) *If* $\sigma > \beta > 0$, *then*

$$\sum_{n=v+1}^{\infty} \frac{(n-v)^{\beta-1}}{n^{\sigma}} = O(v^{\beta-\sigma}). \tag{11}$$

3. Proof of Theorem 1

Let t_n^α be the nth (C, α) means of the sequences (na_n), with $0 < \alpha < 1$. By (8), we have that

$$a_n = -\frac{P_n}{p_n}\Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta T_{n-2}. \tag{12}$$

If we put (12) in (1), then we have that

$$\begin{aligned}
t_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v \left\{ -\frac{P_v}{p_v}\Delta T_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2} \right\} \\
&= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} - \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} v A_{n-v}^{\alpha-1}\frac{P_v}{p_v}\Delta T_{v-1} \\
&\qquad + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}(v+1)A_{n-v-1}^{\alpha-1}\frac{P_{v-1}}{p_v}\Delta T_{v-1} \\
&= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\frac{1}{p_v}\Delta T_{v-1}\left\{-vP_v A_{n-v}^{\alpha-1} + (v+1)A_{n-v-1}^{\alpha-1}P_{v-1}\right\}.
\end{aligned}$$

Since

$$-vP_v A_{n-v}^{\alpha-1} + (v+1)A_{n-v-1}^{\alpha-1}P_{v-1} = -vP_v\Delta_v A_{n-v}^{\alpha-1} - vp_v A_{n-v-1}^{\alpha-1} + P_{v-1}A_{n-v-1}^{\alpha-1},$$

we have

$$\begin{aligned}
t_n^\alpha &= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} - \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} v\frac{P_v}{p_v}\Delta_v A_{n-v}^{\alpha-1}\Delta T_{v-1} \\
&\qquad - \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} vA_{n-v-1}^{\alpha-1}\Delta T_{v-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}\Delta T_{v-1} \\
&= t_{n,1}^\alpha + t_{n,2}^\alpha + t_{n,3}^\alpha + t_{n,4}^\alpha.
\end{aligned}$$

Since

$$|t_{n,1}^{\alpha}+t_{n,2}^{\alpha}+t_{n,3}^{\alpha}+t_{n,4}^{\alpha}|^k \le 4^k\left(|t_{n,1}^{\alpha}|^k+|t_{n,2}^{\alpha}|^k+|t_{n,3}^{\alpha}|^k+|t_{n,4}^{\alpha}|^k\right),$$

to complete the proof of Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{m} n^{\delta k-1}|t_{n,r}^{\alpha}|^k = O(1) \qquad \text{as} \quad m\to\infty,\quad r=1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}\sum_{n=1}^{m} n^{\delta k-1}|t_{n,1}^{\alpha}|^k &= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}(P_n/n^{\alpha}p_n)^k|\Delta T_{n-1}|^k\\ &= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|\Delta T_{n-1}|^k\\ &= O(1) \qquad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 1.

Since $P_n = O(n^{\alpha}p_n)$ for $0<\alpha<1$ implies $P_n = O(np_n)$, when $k>1$, by Hölder's inequality, we have that

$$\begin{aligned}&\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,2}^{\alpha}|^k\\ &\le \sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} v\frac{P_v}{p_v}|\Delta_v A_{n-v}^{\alpha-1}||\Delta T_{v-1}|\right\}^k\\ &=O(1)\sum_{n=2}^{m+1}\frac{1}{n^{\alpha k-\delta k+1}}\left\{\sum_{v=1}^{n-1} v^k\left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2}|\Delta T_{v-1}|^k\right\}\times\\ &\qquad\qquad\times\left\{\sum_{v=1}^{n-1}(n-v)^{\alpha-2}\right\}^{k-1}\\ &=O(1)\sum_{n=2}^{m+1}\frac{1}{n^{\alpha k-\delta k+1}}\sum_{v=1}^{n-1} v^k\left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2}|\Delta T_{v-1}|^k\\ &=O(1)\sum_{v=1}^{m} v^k\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-2}}{n^{\alpha k-\delta k+1}}\\ &=O(1)\sum_{v=1}^{m} v^k\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k v^{\delta k-\alpha k-1}\sum_{n=v+1}^{m+1}(n-v)^{\alpha-2}\\ &=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{v^{\alpha}p_v}\right)^k v^{\delta k+k-1}|\Delta T_{v-1}|^k\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}|\Delta T_{v-1}|^k \\
&= O(1) \qquad \text{as} \quad m\to\infty ,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 1.

Also we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,3}^{\alpha}|^k \\
&\leq \sum_{n=2}^{m+1} \frac{n^{\delta k-1}}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} vA_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{n^{\delta k-1}}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} vA_{n-v}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k-1-\alpha}\left\{\sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right\}\left\{\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1} A_{n-v}^{\alpha-1}\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-1}}{n^{\alpha+1-\delta k}} \\
&= O(1)\sum_{v=1}^{m} v^{\delta k-1}v^k|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}|\Delta T_{v-1}|^k \\
&= O(1) \qquad \text{as} \quad m\to\infty ,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 1 and Lemma.

Finally, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,4}^{\alpha}|^k \\
&\leq \sum_{n=2}^{m+1} \frac{n^{\delta k-1}}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{n^{\delta k-1}}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} \frac{P_v}{p_v}A_{n-v}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{n^{\delta k-1}}{A_n^{\alpha}}\left\{\sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right\}\left\{\frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1} A_{n-v}^{\alpha-1}\right\}^{k-1}
\end{aligned}
$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k-1-\alpha}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-1}|\Delta T_{v-1}|^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-1}}{n^{\alpha+1-\delta k}}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k v^{\delta k-1}|\Delta T_{v-1}|^k$$
$$= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}|\Delta T_{v-1}|^k$$
$$= O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 1 and Lemma. Therefore, we get that

$$\sum_{n=1}^{m} n^{\delta k-1}|t_{n,r}^{\alpha}|^k = O(1) \qquad \text{as} \quad m\to\infty, \quad r=1,2,3,4.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

The case $\alpha = 1$ is easy, so consider $\alpha > 1$. We show only that

$$\sum_{n=1}^{m} n^{\delta k-1}|t_{n,r}^{\alpha}|^k = O(1) \qquad \text{as} \quad m\to\infty, \quad r=1,2,$$

since the other case is the same as in Theorem 1. We have that

$$\sum_{n=1}^{m} n^{\delta k-1}|t_{n,1}^{\alpha}|^k \le \sum_{n=1}^{m} n^{\delta k+k-1}(P_n/n^{\alpha}p_n)^k|\Delta T_{n-1}|^k.$$

By the fact that $P_n = O(np_n)$ implies $P_n = O(n^{\alpha}p_n)$ for $\alpha \geq 1$, it follows that

$$\sum_{n=1}^{m} n^{\delta k-1}|t_{n,1}^{\alpha}|^k = O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|\Delta T_{n-1}|^k$$
$$= O(1) \qquad \text{as} \quad m\to\infty.$$

If $\alpha = 1$, then $\Delta_v A_{n-v}^{\alpha-1} = 0$, hence $t_{n,2}^{\alpha} = 0$. Now, we shall consider the case $\alpha > 1$. Since

$$\sum_{v=1}^{n-1}(n-v)^{\alpha-2} = O(1)\int_1^{n-1}(n-x)^{\alpha-2}\,dx = O(n^{\alpha-1}),$$

by Hölder's inequality, we have for $k > 1$

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1}|t_{n,2}^{\alpha}|^k &\le \sum_{n=2}^{m+1} \frac{n^{\delta k-1}}{(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} v\frac{P_v}{p_v}|\Delta_v A_{n-v}^{\alpha-1}||\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{\alpha k-\delta k+1}}\left\{\sum_{v=1}^{n-1} v^k\left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2}|\Delta T_{v-1}|^k\right\}\times \\
&\qquad\times\left\{\sum_{v=1}^{n-1}(n-v)^{\alpha-2}\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{\alpha+k-\delta k}}\sum_{v=1}^{n-1} v^k\left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m} v^k\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-2}}{n^{\alpha+k-\delta k}} \\
&= O(1)\sum_{v=1}^{m} v^{\delta k-1}\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}|\Delta T_{v-1}|^k \\
&= O(1) \qquad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma.

REFERENCES

[1] BOR, H.: *On the relative strength of two absolute summability methods*, Proc. Amer. Math. Soc. **113** (1991), 1009–1012.

[2] BOR, H.: *A note on two absolute summability methods*, Proc. Indian Acad. Sci. **102** (1992), 125–128.

[3] FLETT, T. M.: *Some more theorems concerning the absolute summability of Fourier series*, Proc. London Math. Soc. **8** (1958), 357–387.

[4] HARDY, G. H.: *Divergent Series*, Clarendon Press, Oxford, 1949.

[5] SARIGŐL, M. A.—BOR, H.: *On two summability methods*, Math. Slovaca **43** (1993), 317–325.

[6] SULAIMAN, W. T.: *A study on a relation between two summability methods*, Proc. Amer. Math. Soc. **115** (1992), 303–312.

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