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ON THE MAXIMUM AUTOMORPHISM GROUP OF SELF-COMPLEMENTARY GRAPHS

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ABSTRACT. We find the maximum size of the automorphism group of a self-complementary graph on n vertices.

§1. Introduction

In this note by a graph we always mean a finite graph without loops and multiple edges. For such a graph $G = (V, E)$ an *automorphism* σ of G is a permutation of the set of vertices V such that $\{\sigma(v), \sigma(w)\} \in E$ whenever $\{v, w\} \in E$. The group of all automorphisms of G we denote by $\Sigma(G)$ and set $s(G) = |\Sigma(G)|$. An *anti-automorphism* ψ of $G = (V, E)$ is defined as a permutation of V such that $\{\psi(v), \psi(w)\} \notin E$ for every $\{v, w\} \in E$. Finally, a graph G is *self-complementary* if there exists at least one anti-automorphism of G. Clearly, if $G = (V, E)$ is self-complementary, then $|E| = \frac{1}{2}\binom{|V|}{2}$ must be an integer and thus $|V| = 0$ or $1 \pmod 4$. Properties of self-complementary graphs have been studied by several authors (see, for instance, [2]–[5]). Balińska and Quintas [1] studied the maximum possible value of $s(G_n)$ for a self-complementary graph G_n on n vertices. They noticed that there are self-complementary graphs G_n on $n = 4k$ vertices for which $s(G_n) = 2(k!)^4$ and conjectured that no other graph of this order has larger automorphism group. They also computed the size of the automorphism group for a number of self-complementary graphs, confirming their claim for small values of n. In this note we settle their conjecture in the affirmative proving the following result.

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THEOREM. *Let*

$$s_n = \begin{cases} 1 & \text{if } n = 1\,, \\ 10 & \text{if } n = 5\,, \\ 72 & \text{if } n = 9\,, \\ 2(k\,!)^4 & \text{if } \begin{cases} n = 4k \text{ and } k \geq 1\,, \\ \text{or} \\ n = 4k+1 \text{ and } k \geq 3\,. \end{cases} \end{cases}$$

Then, $s(G_n) \leq s_n$ for all self-complementary graphs G_n on n vertices.

Furthermore, for $n = 1, 4, 5$ and 9 there exists only one (up to isomorphism) extremal graph H_n on n vertices for which $s(H_n) = s_n$; if $n = 4k$, where $k \geq 2$, then the maximum size of the automorphism group is attained for two non-isomorphic self-complementary graphs; while for $n = 4k+1$, $k \geq 3$, there exist four non-isomorphic self-complementary graphs with the automorphism group of size s_n.

§2. The structure of decomposable self-complementary graphs

For our argument we shall need some elementary facts on self-complementary graphs which cannot be decomposed into two self-complementary subgraphs. Thus, let G_n be a self-complementary graph on n vertices. If the set V of vertices of G_n can be partitioned into two parts, V' and V'', such that for each automorphisms $\sigma \in \Sigma(G_n)$ we have $\sigma(V') = V'$ (and $\sigma(V'') = V''$), and V' and V'' also remain invariant under each anti-automorphism of G_n, we say that G_n is *decomposable*. In such a case we call the pair of subgraphs H' and H'', induced in G_n by V' and V'' respectively, a *decomposition* of G_n. Let us start with the following elementary fact.

FACT 1. *If a self-complementary graph G_n can be decomposed into graphs H' and H'', then both H' and H'' are self-complementary and*

$$s(G_n) \leq s(H')s(H'')\,.$$

P r o o f. Let ψ be an anti-automorphism of G_n. Then $\psi|_{V'}$ and $\psi|_{V''}$ are anti-automorphisms of H' and H'' respectively, so both H' and H'' are self-complementary. Furthermore, since V' and V'' remain invariant under every automorphism σ of G_n, $\Sigma(G_n)$ is a subgroup of the direct product of $\Sigma(H')$ and $\Sigma(H'')$ and consequently $|\Sigma(G_n)| \leq |\Sigma(H')||\Sigma(H'')|$. □

It turns out that a self-complementary graph which is not decomposable has a rather special structure. In order to see that let us consider a non-decomposable graph G_n and let $W_1, W_2, \dots, W_m$ denote orbits of the automorphism group $\Sigma(G_n)$, i.e. for every $i = 1, 2, \dots, m$ and $w \in W_i$ we have

$$W_i = \{\sigma(w) : \sigma \in \Sigma(G_n)\}.$$

Moreover, let $\vec{D}[G_n]$ be the digraph (possibly with loops) with vertex set $\{W_1, \dots, W_m\}$ such that $\overrightarrow{W_i W_j}$ is an arc of $\vec{D}[G_n]$ if for some $w_i \in W_i$ and $w_j \in W_j$ there exists an anti-automorphism ψ such that $\psi(w_i) = w_j$. We list properties of the auxiliary digraph $\vec{D}[G_n]$ and their consequences in a series of simple observations.

FACT 2. *If G_n is non-decomposable self-complementary graph then:*

- (i) *each vertex of $\vec{D}[G_n]$ is the tail (and the head) of at least one arc of $\vec{D}[G_n]$ (which, possibly, is a loop);*
- (ii) *the underlying graph of $\vec{D}[G_n]$ is connected;*
- (iii) *if both arcs $\overrightarrow{W_i W_j}$ and $\overrightarrow{W_j W_\ell}$ belong to $\vec{D}[G_n]$ then $W_i = W_\ell$.*

Proof. The fact that G_n is self-complementary and thus has at least one anti-automorphism immediately gives (i).

To see (ii) note that the set of all vertices of G_n which belong to sets from one component of $\vec{D}[G_n]$ is invariant under each automorphism as well as each anti-automorphism of G_n. Finally, let $w_i \in W_i$, $w_j, w'_j \in W_j$, $w'_\ell \in W_\ell$ be vertices of G_n, ψ, ψ' be anti-automorphisms such that $\psi(w_i) = w_j$ and $\psi'(w'_j) = w'_\ell$, and let $\sigma(w_i) = w'_i$ for $\sigma \in \Sigma(G_n)$. Then $\psi'\sigma\psi$ is an automorphism of G_n which maps w_i into w'_ℓ. Thus, w_i and w'_ℓ belong to the same orbit and so $W_i = W_\ell$. □

FACT 3. *If G_n is a non-decomposable self-complementary graph, then $\vec{D}[G_n]$ is either a loop, or a directed cycle of length two. In particular, G_n has at most two orbits.*

Proof. It is enough to notice that the only two connected digraphs with the minimal out-degree at least one and no proper directed paths of length larger than two are a loop and a directed cycle of length two. Thus, Fact 3 is a straightforward consequence of Fact 2. □

FACT 4. *Let G_n be a non-decomposable self-complementary graph on $n = 4k+1$ vertices. Then G_n is a $(2k)$-regular graph whose automorphism group $\Sigma(G_n)$ is transitive, i.e. for every two vertices v, w of G_n there is an automorphism $\sigma \in \Sigma(G_n)$ such that $\sigma(v) = w$.*

Proof. Note that $\vec{D}[G_n]$ cannot be a directed cycle of length two: in such a case G_n would consist of two orbits of the same size (since each anti-automorphism could serve as a bijection between them) while G_n contains an odd number of vertices. Thus, due to Fact 3, $\vec{D}[G_n]$ is a loop and consequently G_n contains only one orbit, i.e. $\Sigma(G_n)$ is transitive. In particular G_n is regular, and since it is self-complementary each of its vertices has degree $2k$. □

Fact 5. *The vertex set of every non-decomposable self-complementary graph G_n on $n = 4k$ vertices can be partitioned into two sets W_1 and W_2 such that:*

(i) *W_1 and W_2 are the only orbits of G_n;*
(ii) *all vertices from W_i, $i = 1, 2$, are of the same degree;*
(iii) *every anti-automorphisms of G_n maps W_1 into W_2 and W_2 into W_1; in particular $|W_1| = |W_2| = 2k$;*
(iv) *all vertices from W_i, $i = 1, 2$, have k neighbours in W_{3-i}.*

P r o o f. Note that in a self-complementary graph G_n on $n = 4k$ vertices each vertex of degree d is mapped by an anti-automorphism into a vertex of degree $4k - d - 1 \neq d$; thus each such graph G_n has at least two orbits. Hence, if G_n is non-decomposable then, due to Fact 3, it contains precisely two orbits, say W_1 and W_2. Clearly, all vertices from one orbit have the same degree. Let ψ be any anti-automorphism of G_n. As we have already noticed a vertex $w \in W_1$ of degree d is mapped by ψ into a vertex of degree $n - 1 - d \neq d$, so $\psi(w) \notin W_1$. Hence, since every anti-automorphism maps an orbit into an orbit, we have $\psi(W_1) = W_2$, $\psi(W_2) = W_1$, and $|W_1| = |W_2| = 2k$. Furthermore, let

$$[W_1, W_2] = \{\{w_1, w_2\} : \ w_1 \in W_1, \ \ w_2 \in W_2\}.$$

Then the mapping defined as

$$\hat{\psi}\colon [W_1, W_2] \to [W_1, W_2] : \quad \{w_1, w_2\} \mapsto \{\psi(w_1), \psi(w_2)\}$$

is a bijection and thus precisely half of the pairs from $[W_1, W_2]$ are edges of G_n. Since all vertices of an orbit have the same number of neighbours in any other orbit, each vertex from W_i, where $i = 1, 2$, must have precisely k neighbours in W_{3-i}. □

§3. Proof of the main result

P r o o f o f T h e o r e m . We shall use induction on n. For $n = 1$ and $n = 4$ there is nothing to prove: K_1 is the only graph with one vertex and the path of length three is the only self-complementary graph on four vertices. There are two self-complementary graphs on five vertices: one with only one non-trivial automorphism, and the cycle of length five whose automorphism group consists of ten elements.

Now let us assume the assertion holds for every self-complementary graph $G_{n'}$ on $n' = 4k'$ vertices, where $k' < k$, and let G_n be a self-complementary graph on $n = 4k$ vertices, for $k \geq 2$. Let us suppose that G_n can be decomposed into two graphs H' and H''. Then, H' and H'' have $4k'$ and $4(k-k')$ vertices respectively, for some k', where $1 \leq k' \leq k-1$. Hence Fact 1 and the induction hypothesis imply that $s(G_n)$ is bounded from above by

$$\max_{1 \leq k' \leq k-1} s_{4k'} s_{4(k-k')} = \max_{1 \leq k' \leq k-1} 2(k'!)^4 2\big[(k-k')!\big]^4 = 4\big[(k-1)!\big]^4 < 2(k!)^4 = s_n ,$$

and the assertion follows.

Thus, it is enough to consider the case when G_n is non-decomposable, with the structure as described in Fact 5. We shall bound from above the number of automorphisms of G_n. Take any vertex v from the set W_1. In order to construct an automorphism of G_n we first choose an image $v' \in W_1$ of v, which can be done in at most $|W_1| = 2k$ ways. Furthermore, all neighbours of v in W_2 should be transformed into neighbours of v' in W_2 (there are $k!$ ways of doing that) and k non-neighbours of v in W_2 into non-neighbours of v' in W_2 (again we have $k!$ possibilities). Now let us take a vertex $w \in W_2$ adjacent to v for which we have already chosen an image $w' \in W_2$. We must decide how to map $k-1$ remaining neighbours of w in W_1 into neighbours of w' in W_2 ($(k-1)!$ possibilities) and vertices of W_1 not adjacent to w into vertices of W_1 not adjacent to w' ($k!$ possibilities). Thus, altogether there are not more than

$$2k \cdot k!\, k!\, (k-1)!\, k! = 2(k!)^4$$

automorphisms of G_n. Furthermore, from the proof it is clear that this maximum is achieved only for non-decomposable graphs, such that for $i = 1, 2$ and each vertex $v \in W_i$:

- all vertices $N(v)$ of W_{3-i} adjacent to v span either a complete subgraph or an independent set;
- the same is true also for the set $W_{3-i} \setminus N(v)$;
- either all pairs of vertices $\{v, w\}$ such that $v \in N(v)$ and $w \in W_{3-i} \setminus N(v)$ are edges of G_n, or none of them is an edge of the graph.

From this description one can immediately identify two extremal graphs $H_{4k}^{(1)}$ and $H_{4k}^{(2)}$ for which the automorphism group has $2(k!)^4$ elements. The vertex set of each of them can be partitioned into four sets V_1, V_2, V_3, V_4, each of k elements. For $j = 1, 2, 3$, every vertex from V_j is adjacent to every vertex from V_{j+1}. Finally, in $H_{4k}^{(1)}$ each of the sets V_1 and V_4 spans complete subgraphs, whereas the sets V_2 and V_3 are independent; in $H_{4k}^{(2)}$ these are sets V_1 and V_4 which are independent, while the sets V_2 and V_3 induce complete subgraphs in $H_{4k}^{(2)}$.

Now consider the case when the number of vertices in a self-complementary graph G_n is equal to $n = 4k+1$, where $k \geq 2$. Assume first that (H', H'') is a decomposition of G_n, where H' has $4k'$ vertices and H'' has $4(k-k')+1$ vertices, for some $1 \le k' \le k$. Then, from Fact 1 and the induction hypothesis, we get

$$s(G_n) \le \max_{1 \le k' \le k-1} s_{4k'} s_{4(k-k')+1} \,.$$

Elementary calculations reveal that the above maximum is not larger that s_{4k+1} and the equality holds if and only if $k' = k \ge 3$ and $s(H') = s_{4k}$, i.e. H' is one of two extremal graphs $H^{(1)}_{4k}$ and $H^{(2)}_{4k}$ described in the first part of the proof. Now it is enough to find all possible ways of adding to each of them a single vertex in such a way that the resulted graph is self-complementary and the size of its automorphism group remains equal to s_{4k}. There are precisely two ways of doing that: either we connect the additional vertex to all vertices from the sets V_1 and V_4, or join it to all vertices from V_2 and V_3. Consequently, one can obtain from $H^{(1)}_{4k}$ two extremal graphs $H^{(1)}_{4k+1}$ and $H^{(2)}_{4k+1}$, and two other graphs $H^{(3)}_{4k+1}$ and $H^{(4)}_{4k+1}$ with $s\big(H^{(3)}_{4k+1}\big) = s\big(H^{(4)}_{4k+1}\big) = 2(k!)^4$ can be constructed out of $H^{(2)}_{4k}$.

Thus, let us suppose that a self-complementary graph $G_n = (V, E)$ on $n = 4k+1$ vertices with $k \ge 2$ is non-decomposable. Then, due to Fact 4, the automorphism group of G_n is transitive. Choose any vertex v_0 of G_n, let ψ be any anti-automorphism of G_n and $\sigma \in \Sigma(G_n)$ be such that $\sigma\big(\psi(v_0)\big) = v_0$. Then, $\sigma\psi$ is an anti-automorphism of G_n which leaves v_0 invariant. Thus, the graph $G_n - v_0$ obtained from G_n by removing v_0 is self-complementary, and clearly

$$s(G_n) \le \sum_{v_0 \in V} s(G_n - v_0) = (4k+1) \max_{v_0 \in V} s(G_n - v_0)\,.$$

As a matter of fact, since $\Sigma(G_n)$ is transitive, for all $v_0 \in V$, graphs $G_n - v_0$ are isomorphic, so it is enough to study properties of one of them.

Note that if $G_n - v_0$ is decomposable then the above inequality and the induction hypothesis give

$$s(G_n) \le (4k+1) \max_{1 \le k' \le k-1} 2(k'!)^4 2\big[(k-k')!\big]^4 = 4(4k+1)\big[(k-1)!\big]^4 ,$$

which, for $k \ge 2$, is less than the value of s_n. Hence, assume that $G_n - v_0$ is non-decomposable. Then, the structure of $G_n - v_0$ is characterized by Fact 5. Note that in the partition (W_1, W_2) described in Fact 5, W_1 must be the set of all neighbours of v_0 in G_n (all these vertices have degree $k-1$ in $G_n - v_0$) and W_2 consists of vertices of $G_n - v_0$ which are not adjacent to v_0 (each of them has degree k in $G_n - v_0$). Consequently, each vertex v of G_n adjacent to v_0 shares with v_0 precisely $k-1$ neighbours in G_n, and for every vertex w non-adjacent to v_0 there is exactly k common neighbours of v_0 and w. Since $\Sigma(G_n)$ is transitive, this fact implies that G_n is a *conference graph*: a

$(2k)$-regular graph on $4k+1$ vertices in which each pair of adjacent vertices has $k-1$ common neighbours, and for each pair of non-adjacent vertices there exist k vertices joined to both of them. We shall show that this property significantly affects the size of $\Sigma(G_n - v_0)$, and thus $s(G_n)$.

Let w', w'' be two neighbours of v_0 and let W_2' and W_2'' denote the sets of vertices of W_2 adjacent to w' and w'' respectively. Since $G_n - v_0$ is non-decomposable, Fact 5(iv) implies that $|W_2'| = |W_2''| = k$. Furthermore, since G_n is a conference graph and both w' and w'' are adjacent to v_0 we must have $W_2' \neq W_2''$. Hence each neighbour of v_0 can be uniquely identified by its neighbourhood in W_2 and so the automorphisms of $G_n - v_0$ are uniquely determined by the automorphisms of the subgraph J_2 of $G_n - v_0$ induced by W_2. But J_2 is a k-regular graph on $2k$ vertices and so

$$s(J_2) \leq 2k(k)!\,(k-1)! = 2(k!)^2\,.$$

Consequently,

$$s(G_n) \leq (4k+1)s(G_n - v_0) = (4k+1)s(J_2) \leq 2(4k+1)(k!)^2\,. \qquad (*)$$

One can easily see that for $k \geq 3$

$$2(4k+1)(k!)^2 < 2(k!)^4 = s_{4k+1}\,,$$

while for $k = 2$ the inequality $(*)$ becomes $s(G_9) \leq 72 = s_9$. Thus, to complete the proof, it is enough to observe that among four self-complementary 4-regular graphs on nine vertices for only one, call it H_9, the automorphism group is transitive and $s(H_9) = 72$. (As a matter of fact, H_9 is also the unique conference graph on nine vertices.) □

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