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TORSION CLASSES AND
TORSION PRIME SELECTORS OF $hl$-GROUPS

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(Communicated by Stanislav Jakubec)

ABSTRACT. In this paper we introduce two notions: A torsion class of $hl$-groups is a class closed under taking convex $hl$-subgroups, joins of convex $hl$-subgroups and $hl$-homomorphic images; a torsion prime selector of $hl$-groups is a function assigning to each $hl$-group $G$ some subset $M(G)$ of $P(G)$. We show that there exists a complete lattice isomorphism from the family of torsion classes into the family of torsion prime selectors.

1. Introduction

M. Giraudet and F. Lucas introduced a new concept of half $l$-groups in [4]. The concept of half $l$-groups is a natural generalization of $l$-groups. For the definitions and standard results concerning $l$-groups, the reader is referred to [1], [2], [3], [5].

Let $G$ be a group with unit $e$ and a non-trivial ordered underlying set. Set

$$G^+ = \{ g \in G \mid x \leq y \implies gx \leq gy \text{ for all } x, y \in G \},$$
$$G^- = \{ g \in G \mid x \leq y \implies gx \geq gy \text{ for all } x, y \in G \}. $$

$G^+$ is called the increasing part of $G$ and $G^-$ the decreasing part of $G$. $G$ is called a half $l$-group (abbreviated: $hl$-group), if

1. $x \leq y$ implies $xg \leq yg$ for all $x, y$ and $g \in G$;
2. $G = G^+ \cup G^-$;
3. $G^+$ is an $l$-group.

For example, the set $M(\omega)$ of all monotonic permutations of a chain $\omega$ is an $hl$-group. Let $G_1$ be the set of all $hl$-groups (and similarly for $G_2$). Let $G$ be an

1991 Mathematics Subject Classification: Primary 06F15.
Key words: $hl$-group, torsion class, torsion prime selector, lattice isomorphism.

The author wishes to express his appreciation to the University of Main in France for its hospitality during his visit in Le Mans when this paper was prepared.
hl-group and $G \downarrow \neq \emptyset$, then the index $(G,G\uparrow) = 2$, so $G\uparrow$ is normal in $G$. An element in $G\uparrow$ and an element in $G\downarrow$ are never comparable. $G\uparrow$ is isomorphic to $G \downarrow$ as a lattice. $G = G\uparrow \cup aG\uparrow$, where $a \in G \downarrow$ can be selected to be an element of order 2 ([4], [9]). Put $E(G) = \{x \in G \mid x^2 = e, \; x \neq e\}$.

A subgroup $H$ of an hl-group $G$ is said to be a half $l$-subgroup (abbreviated: hl-subgroup) if $H\uparrow = H \cap G\uparrow$ is an $l$-subgroup of $G\uparrow$. An hl-subgroup $H$ of $G$ is called convex, if $H\uparrow$ is convex in $G\uparrow$. A normal convex hl-subgroup of $G$ is called an hl-ideal of $G$. $G\uparrow$ is an hl-ideal of $G$. We denote by $\mathcal{C}(G)$ the set of all convex hl-subgroups of $G$. Let $X \subseteq G$ and $a \in G$. We denote by $G(X)$ the convex hl-subgroup of $G$ generated by $X$, which is the smallest convex hl-subgroup of $G$ containing $X$, and $G(X,a)$ the convex hl-subgroup of $G$ generated by $\{X,a\}$. Let $H$ be an $l$-group and $G$ an hl-group with $G\uparrow - H$; then $G$ is called an $h$-extension of $H$.

A mapping $\phi$ from an hl-group $G$ onto an hl-group $G'$ is called an hl-homomorphism, if

1. $\phi$ is a group homomorphism,
2. $\phi|G\uparrow$ is a lattice homomorphism of $G\uparrow$ onto $G'\uparrow$.

A 1–1 hl-homomorphism is called an hl-isomorphism. It is denoted by $G \approx G'$.

The join in a lattice $L$ is denoted by $\vee_L$.

**Proposition 1.1.** Let $G$ be an hl-group and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$. Then $\bigcap_{\lambda \in \Lambda} G_\lambda$ is also a convex hl-subgroup of $G$; moreover, $\left( \bigcap_{\lambda \in \Lambda} G_\lambda \right)\uparrow = \bigcap_{\lambda \in \Lambda} G_\lambda\uparrow$.

The assertion of this proposition is obvious and we omit the proof.

Let $G$ be an hl-group and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$. By Proposition 1.1, we can define meets and joins in $\mathcal{C}(G)$ as follows:

$$\bigwedge_{\lambda \in \Lambda} G_\lambda = \bigcap_{\lambda \in \Lambda} G_\lambda,$$

$$\bigvee_{\lambda \in \Lambda} G_\lambda = \bigcap \left\{ K \in \mathcal{C}(G) \mid K \supseteq \bigcup_{\lambda \in \Lambda} G_\lambda \right\}.$$

Thus, $\mathcal{C}(G)$ becomes a complete lattice. Let $H$ be an $l$-group and $X \subseteq H$. We denote by $\langle X \rangle_H$ the convex $l$-subgroup of $H$ generated by $X$.

**Proposition 1.2.** Let $G$ be an hl-group and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$, $G_\lambda \uparrow \cup a_\lambda G_\lambda \uparrow$ with $a_\lambda \in E(G_\lambda)$ for each $\lambda \in \Lambda$. Then

$$\left( \bigvee_{\lambda \in \Lambda} G_\lambda \right)\uparrow = \left( \bigcup_{\lambda \in \Lambda} G_\lambda\uparrow \cup \{a_\lambda a_\mu \mid \lambda, \mu \in \Lambda\} \right)_{G\uparrow}$$

and

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\[
\bigvee_{\lambda \in \Lambda} G_\lambda = \left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow \cup a_\lambda \left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow \quad \text{for any } a_\lambda \in E(G_\lambda)
\]

\[
= \left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow \cup b \left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow \quad \text{for any } b \in \bigcup_{\lambda \in \Lambda} a_\lambda G_\lambda \uparrow.
\]

Proof. Put $H = \bigvee_{\lambda \in \Lambda} G_\lambda$. Let $C \in \mathcal{C}(G)$. Then $C \uparrow \supseteq H \uparrow$ if and only if

\[
C \uparrow \supseteq \bigcup_{\lambda \in \Lambda} G_\lambda \uparrow \cup \left( \bigcup_{\lambda, \mu \in \Lambda} a_\lambda G_\lambda \uparrow a_\mu G_\mu \uparrow \right) = \bigcup_{\lambda \in \Lambda} G_\lambda \uparrow \cup \left( \bigcup_{\lambda, \mu \in \Lambda} a_\lambda a_\mu G_\mu \uparrow \right),
\]

if and only if

\[
cC \uparrow \supseteq \left( \bigcup_{\lambda \in \Lambda} G_\lambda \uparrow \cup \left\{ a_\lambda a_\mu \mid \lambda, \mu \in \Lambda \right\} \right) \uparrow.
\]

So we get (1.1). For any $\lambda, \mu \in \Lambda$,

\[
a_\mu G_\mu \uparrow = a_\lambda a_\lambda a_\mu G_\mu \uparrow \subseteq a_\lambda H \uparrow.
\]

Hence for any $\lambda, \mu \in \Lambda$, $a_\mu H \uparrow = a_\lambda H \uparrow$. So we have (1.2) and (1.3). \qed

COROLLARY 1.3. Let $G$ be an $hl$-group and $\left\{ G_\lambda \mid \lambda \in \Lambda \right\} \subseteq \mathcal{C}(G)$, $G_\lambda = G_\lambda \uparrow \cup a_\lambda G_\lambda \uparrow$ with $a_\lambda \in E(G_\lambda)$ such that $G_\lambda \uparrow = H$ for any $\lambda \in \Lambda$. Then

\[
\left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow = \bigvee_{\lambda \in \Lambda} G_\lambda \uparrow \quad \text{if and only if } \bigcap_{\lambda \in \Lambda} a_\lambda G_\lambda \uparrow \neq \emptyset \quad \text{if and only if } G_\lambda = G_\mu \quad \text{for any } \lambda, \mu \in \Lambda.
\]

Proof. If $\left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow = \bigvee_{\lambda \in \Lambda} G_\lambda \uparrow$, then $a_\lambda, a_\mu \in \bigvee_{\lambda \in \Lambda} G_\lambda = H$ for any $\lambda, \mu \in \Lambda$ by (1.1). Since $a_\lambda \in G_\lambda \downarrow$, so $a_\mu \in G_\lambda \downarrow$. Hence $G_\mu = G_\mu \uparrow \cup a_\mu G_\mu \uparrow = H \cup a_\mu H = G_\lambda$ for any $\lambda, \mu \in \Lambda$. Hence $\bigcap_{\lambda \in \Lambda} a_\lambda G_\lambda \neq \emptyset$. Conversely, if there exists $a \in \bigcap_{\lambda \in \Lambda} a_\lambda G_\lambda \uparrow$, let $a' = a \vee a^{-1}$. Then $a' \in E(G_\lambda)$ and $G_\lambda = G_\lambda \uparrow \cup a' G_\lambda \uparrow = H \cup a' H$ for each $\lambda \in \Lambda$. It follows from (1.1) that $\left( \bigvee_{\lambda \in \Lambda} G_\lambda \right) \uparrow = \bigvee_{\lambda \in \Lambda} G_\lambda \uparrow = H$. \qed

2. Torsion classes of $hl$-groups

A family $\mathcal{R}$ of $hl$-groups is called a torsion class if it is closed under

(1) taking convex $hl$-subgroups,
(2) forming joins of convex $hl$-subgroups,
(3) taking $hl$-homomorphic images.

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Let $\mathcal{R}$ be a torsion class of $hl$-groups, and $G$ be an $hl$-group. Then there exists a largest convex $hl$-subgroup $\mathcal{R}(G)$ of $G$ belonging to $\mathcal{R}$. $\mathcal{R}(G)$ is called a torsion radical of $G$. It is invariant under all $hl$-automorphisms of $G$, and in particular, it is an $hl$-ideal of $G$. The mapping $G \rightarrow \mathcal{R}(G)$ is called a torsion radical mapping. Let $T$ denote the family of all torsion classes of $hl$-groups and $T^l$ the complete lattice of all torsion classes of $l$-groups. The notion of torsion classes of $hl$-groups is a generalization of torsion classes of $l$-groups. Torsion classes of $l$-groups were studied by M. Giraudet and J. Rachůnek [Varieties of half lattice-ordered groups of monotonic permutations in chains, Prepublication No 57, Paris 7CNRS LOGIQUE, 1996]. Let $\mathcal{R}$ be a family of $hl$-groups. Put

$$\mathcal{R}^l = \{H \in G_2 \mid H = G^\uparrow \text{ for some } G \in \mathcal{R}\}.$$ 

**Theorem 2.1.** Let $\mathcal{R}$ be a torsion class of $hl$-groups, and let $G$ be an $hl$-group. Then

1. $\mathcal{R}^l$ is a torsion class of $l$-groups,
2. $\mathcal{R}^l(G^\uparrow)$ has at most one $h$-extension in $G$ belonging to $\mathcal{R}$,
3. $\mathcal{R}(G)^\uparrow = \mathcal{R}^l(G^\uparrow)$.

**Proof.**

(1) is clear, because $G_2 \subseteq G_1$ and $G^\uparrow \in C(G)$ for any $hl$-group $G$.

(2) Let $G_1$ and $G_2$ be two $hl$-subgroups of $G$ belonging to $\mathcal{R}$ such that $G_1^\uparrow = G_2^\uparrow = \mathcal{R}^l(G^\uparrow)$, $G_1^\downarrow \neq 0 \neq G_2^\downarrow$ and $G_1^\downarrow \neq G_2^\downarrow$. Then $G_1 \vee G_2 \in \mathcal{R}$. If there exists $a \in G_1^\downarrow \cap G_2^\downarrow$, then $G_1^\downarrow = aG_1^\uparrow = aG_2^\uparrow = G_2^\downarrow$, which is a contradiction. So $G_1^\downarrow \cap G_2^\downarrow = \emptyset$. Hence $(G_1 \vee G_2)^\uparrow \supset \mathcal{R}^l(G^\uparrow)$ by Corollary 1.3. But $(G_1 \vee G_2)^\uparrow \in \mathcal{R}$, which is a contradiction.

(3) Since $\mathcal{R}(G)$ is the largest convex $hl$-subgroup of $G$ belonging to $\mathcal{R}$, $\mathcal{R}(G) \supseteq \mathcal{R}^l(G^\uparrow)$ and so $\mathcal{R}(G)^\uparrow = \mathcal{R}(G) \cap G^\uparrow \supseteq \mathcal{R}^l(G^\uparrow)$. On the other hand, $\mathcal{R}(G) \in \mathcal{R}$ and $\mathcal{R}(G)^\uparrow \in C(\mathcal{R}(G))$ imply $\mathcal{R}(G)^\uparrow \subseteq \mathcal{R}^l(G^\uparrow)$. □

Theorem 2.1 tells us that, for a torsion class $\mathcal{R}$ of $hl$-groups, the torsion radical $\mathcal{R}(G)$ of an $hl$-group $G$ is uniquely determined by the torsion radical $\mathcal{R}^l(G^\uparrow)$ of the increasing part $G^\uparrow$ of $G$. This fact is very useful in what follows.

**Theorem 2.2.** Suppose that $\mathcal{R}$ is a torsion class of $hl$-groups and $G$ is an $hl$-group. Then

I. if $A \in C(G)$, then $\mathcal{R}(A) = A \cap \mathcal{R}(G)$;

II. if $\varphi: G \rightarrow H$ is a surjective $hl$-homomorphism, then $\varphi[\mathcal{R}(G)] \subseteq \mathcal{R}(H)$.

Conversely, any mapping $\phi$ associating to each $hl$-group $G$ an $hl$-ideal and satisfying properties (I) and (II) always defines a unique torsion class $\mathcal{R}$ of $hl$-groups such that $\mathcal{R}(G) = \phi(G)$. 34
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Proof. By the above Theorem 2.1(3) and [7; Proposition 1.1] for any \( A \in \mathcal{C}(G) \) we have

\[
\mathcal{R}(A)^\uparrow = \mathcal{R}^l(A^\uparrow) = A^\uparrow \cap \mathcal{R}^l(G^\uparrow) = A^\uparrow \cap \mathcal{R}(G)^\uparrow = (A \cap \mathcal{R}(G))^\uparrow.
\]

So \( \mathcal{R}(A) \) and \( A \cap \mathcal{R}(G) \) are all h-extensions of \( \mathcal{R}^l(A^\uparrow) \), and Theorem 2.1(2) implies that \( \mathcal{R}(A) = A \cap \mathcal{R}(G) \).

If \( \varphi: G \rightarrow H \) is an onto hl-homomorphism, then \( \mathcal{R}(G) \in \mathcal{R} \) and so \( \varphi[\mathcal{R}(G)] \in \mathcal{R} \). Hence \( \varphi[\mathcal{R}(G)] \subseteq \mathcal{R}(H) \), because \( \mathcal{R}(H) \) is the largest convex hl-subgroup belonging to \( \mathcal{R} \).

Conversely, suppose that the mapping \( \phi \) satisfies (I) and (II). Let \( \mathcal{R} = \{ G \in \mathcal{G}_2 \mid \phi(G) = G \} \). It is easy to show that \( \mathcal{R} \) is a torsion class of hl-groups. For each hl-group \( G \), \( \phi(\phi(G)) = \phi(G) \) implies \( \phi(G) \in \mathcal{R} \) and \( \phi(G) \subseteq \mathcal{R}(G) \).

On the other hand, \( \mathcal{R}(G) = \phi(\mathcal{R}(G)) = \mathcal{R}(G) \cap \phi(G) \). Hence \( \mathcal{R}(G) = \phi(G) \).

Suppose that \( \{ \mathcal{R}_\lambda \mid \lambda \in \Lambda \} \subseteq T \). Since the intersection of a family of torsion classes of hl-groups is also a torsion class, we can define

\[
\bigwedge_{\lambda \in \Lambda} \mathcal{R}_\lambda = \bigcap_{\lambda \in \Lambda} \mathcal{R}_\lambda,
\]

\[
\bigvee_{\lambda \in \Lambda} \mathcal{R}_\lambda = \bigvee \{ \mathcal{R} \in T \mid \mathcal{R} \supseteq \mathcal{R}_\lambda \text{ for all } \lambda \in \Lambda \}.
\]

Thus, \( T \) becomes a complete lattice and we have

\[
\left( \bigvee_{\lambda \in \Lambda} \mathcal{R}_\lambda \right)^l = \bigcap \{ \mathcal{R}^l \in T^l \mid \mathcal{R}^l \supseteq \mathcal{R} \} = \bigvee_{\lambda \in \Lambda} \mathcal{R}_\lambda^l.
\]

\[
\mathbf{THEOREM \ 2.3. \ If } \{ U_\lambda \mid \lambda \in \Lambda \} \subseteq T. \text{ Then for any hl-group } G
\]

\[
\left( \bigvee_{\lambda \in \Lambda} U_\lambda \right)(G) = \bigvee_{\lambda \in \Lambda} U_\lambda(G).
\]

The proof is similar to that used in [7].

3. Torsion prime selectors of hl-groups

The prime subgroups are the most important subgroups of an l-group in the theory of l-groups. All representation theorems and most structure results come from properties of prime subgroups. So we want to define a similar concept in an hl-group. Let \( L \) be a lattice. An element \( a \in L \) is called meet irreducible, if \( a = a_\lambda \) implies \( a = a_\lambda \) for some \( \lambda \in \Lambda \); \( a \) is called finitely meet irreducible, if \( a = a_{i_1} \cap \cdots \cap a_{i_m} \) implies \( a = a_k \) for some \( k \) \((1 \leq k \leq n)\).
A convex $hl$-subgroup $P$ of an $hl$-group $G$ is prime, if whenever $e \leq a$, $e \leq b$ and $a \lor b \in P$, then either $a \in P$ or $b \in P$. Let $P(G)$ be the set of all prime subgroups of $G$.

**Theorem 3.1.** Let $P$ be a convex $hl$-subgroup of an $hl$-group $G$. Then the following conditions are equivalent:

1. $P$ is prime,
2. $P^\uparrow$ is prime in $G^\uparrow$ as an $l$-group,
3. if $g \land h = e$, then $g \in P$ or $h \in P$,
4. if $g, h \in G^+, P$, then $g \land h \not\in P$,
5. $\{A \in \mathcal{C}(G) \mid A \supseteq P\}$ is a chain,
6. $P$ is finitely meet irreducible in $\mathcal{C}(G)$,
7. $g, h \in G^+ \setminus P$ implies $g \land h \in G^+ \setminus P$.

**Proof.**

$(1) \iff (2)$ is evident.

It is clear that $(1) \implies (3) \implies (4)$.

Now suppose that $(4)$ is valid and $A, B \in \mathcal{C}(G)$, $A \supseteq P$ and $B \supseteq P$. If $A^\uparrow$ and $B^\uparrow$ are incomparable, then there exist $e < a \in A^\uparrow \setminus B^\uparrow$ and $e < b \in B^\uparrow \setminus A^\uparrow$. Then $a = a'(a \land b)$ and $b = b'(a \land b)$, where $e < a' \in G^+ \setminus P$ and $e < b' \in G^+ \setminus P$ and $a' \land b' = e$, which is absurd. If $A^\uparrow \subseteq B^\uparrow$ and $A^\downarrow \subseteq B^\downarrow$, then $A \subseteq B$. If $A^\uparrow \subseteq B^\uparrow$ and $A^\downarrow \supseteq B^\downarrow$, let $A^\downarrow = fA^\uparrow$ with $f \in A^\downarrow$ and $B^\downarrow = gB^\uparrow$ with $g \in E(B) \subseteq A^\downarrow$. Then $A^\uparrow = B^\downarrow = gB^\uparrow$. This implies that $A^\uparrow \supseteq B^\uparrow$. Hence $A^\uparrow = B^\uparrow$ and $A \supseteq B$.

$(5) \implies (6)$ is also clear.

$(6) \implies (7)$ is shown by the fact that $P \subseteq G(P, g) \cap G(P, h) = [P \lor G(g)] \cap [P \lor G(h)] = P \lor G(g \land h) = G(P, g \land h)$.

For $(7) \implies (1)$, if $e < a \land b \in P$, then clearly $a \in P$ or $b \in P$. \qed

Now we shall give a special kind of prime subgroups for an $hl$-group. Let $G$ be an $hl$-group and $e \neq g \in G$. By Zorn’s Lemma there exists a maximal convex $hl$-subgroup $G_g$ of $G$ not containing $g$. $G_g$ is called a value of $g$ and is also called a regular subgroup of $G$. The convex $hl$-subgroup $G(G_g, g)$ generated by $\{G_g, g\}$ is a cover of $G_g$. As in $[1; \text{Theorem 1.2.8}]$ we can prove that a convex $hl$-subgroup $P$ of an $hl$-group $G$ is meet irreducible in $\mathcal{C}(G)$ if and only if $P$ is regular. The proof of the following lemma is similar to that for $[1; \text{Theorem 1.2.13}]$.

**Lemma 3.2.** Let $G$ be an $hl$-group and $H \in \mathcal{C}(G)$. Then $\rho: P \rightarrow P' = P \cap H$ is a $1-1$ correspondence from $\{P \in P(G) \mid H \not\subseteq P\}$ onto $P(H)$. 

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A function $M$ assigning to each $hl$-group $G$ a subset $M(G)$ of $P(G)$ is called a torsion prime selector of $hl$-groups if the following is true:

1. if $A \in C(G)$ and $P \in P(G)$, then
   $$M(A) = \{P \cap A \mid P \in M(G) \text{ and } A \not\subseteq P\},$$
2. if $\varphi: G \to H$ is an onto $hl$-homomorphism, then
   $$M(H) \supseteq \{\varphi(P) \mid P \in M(G) \text{ and } P \supseteq \text{Ker}(\varphi)\}.$$

Now let $M$ be a torsion prime selector of $hl$-groups. Set
$$R(M) = \{G \in \mathcal{G}_1 \mid M(G) = P(G)\}.$$

**Theorem 3.3.** For each torsion prime selector $M$ of $hl$-groups, $R(M)$ is a torsion class of $hl$-groups.

The proof is similar to that for $l$-groups.

Let $\mathcal{R}$ be a torsion class of $hl$-groups. We define a function
$$M(\mathcal{R}): G \to \{H \in P(G) \mid \mathcal{R}(G) \not\subseteq H\}.$$

**Theorem 3.4.** For each torsion class $\mathcal{R}$ of $hl$-groups, $M(\mathcal{R})$ is a torsion prime selector of $hl$-groups; moreover, for any $hl$-group $G$ we have $G \in \mathcal{R}$ if and only if $M(\mathcal{R})(G) = P(G)$.

The proof is analogous to that for $l$-groups.

### 4. Connection between torsion classes and torsion prime selectors

Let $M$ and $M^*$ be two torsion prime selectors of $hl$-groups. We define $M < M^*$ if $M(G) \subseteq M^*(G)$ for any $hl$-group $G$. Let $\{M_i \mid i \in I\}$ be a family of torsion prime selectors of $hl$-groups. We define $M_1(G) = \bigcap_{i \in I} M_i(G)$ and $M_2(G) = \bigcup_{i \in I} M_i(G)$ for any $hl$-group $G$.

**Theorem 4.1.** $M_1$ and $M_2$ are all torsion prime selectors of $hl$-groups.

**Proof.** We prove that $M_1$ and $M_2$ satisfy conditions (1) and (2).

1. Let $G$ be an $hl$-group and let $A \in C(G)$. If $Q \in M_1(A) = \bigcap_{i \in I} M_i(A)$, then for each $i \in I$ we have $Q \in M_i(A)$, and there exists $Q_i \in M_i(G)$ such that $A \not\subseteq Q_i$ and $Q = Q_i \cap A$. So $Q_i \uparrow \in P(G) \uparrow$ for each $i \in I$. But
   $$Q_i \uparrow \cap A = (Q_i \cap A) \uparrow = Q \uparrow = (Q_i \cap A) \uparrow = Q_i \uparrow \cap A \uparrow.$$
So $Q'_{i} = Q_{i} \uparrow$ for any $i \neq j \in I$ by [1; Theorem 1.2.13]. Hence $Q'_{i} = Q_{i} \uparrow \cup aQ'_{j} \uparrow = Q'_{j} \uparrow \cup aQ'_{j} \uparrow$ for any $i \neq j$, where $a \in Q \downarrow$. Let $Q' = Q'_{i}$ for any $i \in I$. Then $Q' \in \bigcap_{i \in I} M_{i}(G) = M_{1}(G)$, and so $Q \in \{ P \cap A \mid P \in M_{1}(G) \text{ and } A \not\subseteq P \}$.

Conversely, it is clear that $\{ P \cap A \mid P \in M_{1}(G) \text{ and } A \not\subseteq P \} \subseteq M_{1}(A)$. Therefore

$$M_{1}(A) = \{ P \cap A \mid P \in M_{1}(G) \text{ and } A \not\subseteq P \}.$$ We have proved that $M_{1}$ satisfies the condition (1).

(2) Suppose that $\varphi$ is an $hl$-homomorphism of an $hl$-group $G$ onto an $hl$-group $H$. Since each $M_{i}$ is a torsion prime selector of $hl$-groups,

$$M_{i}(H) = \bigcap_{P \in M_{i}(G) \text{ and } P \supseteq \text{Ker}(\varphi)} \{ \varphi(P) \mid P \in M_{i}(G) \text{ and } P \subseteq \text{Ker}(\varphi) \}$$

for each $i \in I$, and so

$$M_{i}(H) = \bigcap_{P \in \bigcap_{i \in I} M_{i}(G) \text{ and } P \supseteq \text{Ker}(\varphi)} \{ \varphi(P) \mid P \in M_{i}(G) \text{ and } P \subseteq \text{Ker}(\varphi) \}$$

for each $i \in I$. Hence

$$M_{1}(H) = \bigcap_{i \in I} M_{i}(H) = \bigcap_{i \in I} \{ \varphi(P) \mid P \in M_{1}(G) \text{ and } P \subseteq \text{Ker}(\varphi) \}.$$ We have proved that $M_{1}$ satisfies the condition (2).

We can prove that $M_{2}$ satisfies the conditions (1) and (2) similarly. □

Now we define

$$M_{1} = \bigwedge_{i \in I} M_{i} \quad \text{and} \quad M_{2} = \bigvee_{i \in I} M_{i}.$$ Thus, the set $S$ of all torsion prime selectors of $hl$-groups is a complete lattice. And we have the mappings

$$R: S \to T \quad \text{and} \quad M: T \to S.$$ A mapping $\varphi$ from a lattice $L_{1}$ into a lattice $L_{2}$ is called a complete lattice homomorphism if, whenever $\bigvee_{\alpha \in A} a_{\alpha}$ and $\bigwedge_{\beta \in B} b_{\beta}$ exist in $L_{1}$, $\varphi\left( \bigvee_{\alpha \in A} a_{\alpha} \right) = \bigvee_{\alpha \in A} \varphi(a_{\alpha})$ and $\varphi\left( \bigwedge_{\beta \in B} b_{\beta} \right) = \bigwedge_{\beta \in B} \varphi(b_{\beta})$. A 1–1 complete lattice homomorphism is called a complete lattice isomorphism.

**Theorem 4.2.** Let $U$ be a torsion class of $hl$-groups. Then $R(M(U)) = U$.

**Proof.** By Theorem 3.4, $G \in U$ if and only if $M(U(G)) = P(G)$, that is, $G \in U$ if and only if $G \in R(M(U))$. □

By Theorem 4.2, we see that $RM = 1_{T}$, where $1_{T}$ is the identity mapping on $T$. So $R$ is onto and $M$ is 1–1.
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THEOREM 4.3. $M$ is a complete lattice isomorphism of $T$ into $S$.

Proof. Suppose that $\{R_\lambda \mid \lambda \in \Lambda\} \subseteq T$. It is clear that for any $hl$-group $G$ and $H \in P(G)$, $\bigvee_{\lambda \in \Lambda} R_\lambda \not\subseteq H$ if and only if $R_\lambda(G) \not\subseteq H$ for some $\lambda \in \Lambda$. By Theorem 2.3 we have

$$\left( \bigvee_{\lambda \in \Lambda} R_\lambda \right)(G) = \bigvee_{\lambda \in \Lambda} R_\lambda(G).$$

Hence $\left\{ H \in P(G) \mid \bigvee_{\lambda \in \Lambda} R_\lambda(G) \not\subseteq H \right\} = \bigcup_{\lambda \in \Lambda} \left\{ H \in P(G) \mid R_\lambda(G) \not\subseteq H \right\}$. That is,

$$M\left( \bigvee_{\lambda \in \Lambda} R_\lambda \right)(G) = \bigcup_{\lambda \in \Lambda} M(R_\lambda)(G)$$

for any $hl$-group $G$. So

$$M\left( \bigvee_{\lambda \in \Lambda} R_\lambda \right) = \bigvee_{\lambda \in \Lambda} M(R_\lambda)$$

and $M$ preserves arbitrary joins.

Now consider meets. Let $\{R_\lambda \mid \lambda \in \Lambda\} \subseteq T$. Assume that $H \in P(G)$. If

$$\bigcap_{\lambda \in \Lambda} M(R_\lambda)(G) \not\subseteq H,$$

then $M(R_\lambda)(G) \not\subseteq H$ for $\lambda \in \Lambda$. Conversely, if $M(R_\lambda)(G) \not\subseteq H$ for all $\lambda \in \Lambda$, then

$$\bigcap_{\lambda \in \Lambda} M(R_\lambda)(G) \not\subseteq H$$

by the meet irreducibility of regular subgroups in $C(G)$. Hence

$$M\left( \bigwedge_{\lambda \in \Lambda} R_\lambda \right)(G) = \bigcap_{\lambda \in \Lambda} M(R_\lambda(G))$$

for any $hl$-group $G$. That means

$$M\left( \bigwedge_{\lambda \in \Lambda} R_\lambda \right) = \bigwedge_{\lambda \in \Lambda} M(R_\lambda),$$

and $M$ preserves any meets. $\square$

Note that Theorem 4.3 generalizes some results in [6], [8].

REFERENCES

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Received January 27, 1997

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