Book Reviews

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BOOK REVIEWS

Cignoli, R. L. O. — D'Ottaviano, I. M. L. — Mundici, D.:
ALGEBRAIC FOUNDATION OF MANY-VALUED REASONING.
Trends in Logic — Studia Logica Library 7.
Kluwer Academic Publishers, Dordrecht 2000, ix+231 pp.
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MV-algebras entered mathematics in 1958 by C. C. Chang as an algebraic analysis of many-valued logics. They give an algebraic framework for infinite-valued propositional calculus of Łukasiewicz, and today MV-algebras have a well-developed theory with many nice and deep results and with many applications, e.g., in computer science. The accent is put to the algebraic point of view and on this basis, the Łukasiewicz calculus is developed.

The monograph consists of 10 chapters and Bibliography with 250 items.

Chapter 1 is devoted to basic notions on MV-algebras as algebras $(A; \oplus, \neg, 0)$ of type $(2,1,0)$ such that, for all $x, y, z \in A$,

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV2) $x \oplus y = y \oplus x$;
(MV3) $x \oplus 0 = 0$;
(MV4) $\neg\neg x = x$;
(MV5) $x \oplus \neg 0 = \neg 0$;
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Putting $x \leq y$ if and only if $\neg x \oplus y = 1 =: \neg 0$, we have that $(A, \leq)$ is a distributive lattice with the least and the greatest elements 0 and 1, respectively.

For an orthodox example of MV-algebras: Let (G, u) be a unital Abelian ℓ-group with strong unit, then $\Gamma(G,u) := \{g \in G : \ 0 \leq g \leq u\}$ equipped with $g_1 \oplus g_2 := (g_1 + g_2) \wedge u$; $\neg g := u - g$ is a MV-algebra. In particular, if $G = \mathbb{R}$, $u = 1$, we obtain a very important case of MV-algebras.

The fundamental result of Chang, the Subdirect Representation Theorem, is proved here using theory of ideals. In addition, MV-equations are introduced, and it is proved that an MV-equation is satisfied by all MV-algebras if and only if it is satisfied by all MV-chains. This result is considerably strengthened in the second chapter.

Chapter 2 is devoted to Chang's Completeness theorem. The above defined mapping Γ from the category $\mathcal{A}$ whose objects are unital Abelian ℓ-groups into the category of MV-algebras $\mathcal{MV}$ gives a functor which plays a fundamental role, because it defines even a natural equivalence proved in a subsequent chapter.

The main tool is a notion of good sequences which are converted into a partially ordered monoind M_A, and it into a Chang unital ℓ-group (G_A, u_A) with a strong unit u_A; in particular, any MV-algebra A is isomorphic with $\Gamma(G_A, u_A)$. Consequently, Chang's Completeness Theorem saying that an equation holds in $[0,1]$ if and only if it holds in every MV-algebra, is proved.

Chapter 3 is dedicated to free MV-algebras. Special role is played by free MV-algebras $Free_n$ over n generators. $Free_n$ can be easily described by piecewise linear continuous functions over $[0,1]^n$ with values in the interval $[0,1]$. Such functions are so-called McNaughton functions, and they are studied in details. In many cases, very important MV-algebras (simple and semisimple) can be described as MV-subalgebras of continuous fuzzy sets over some compact Hausdorff spaces.

Chapter 4 describes Łukasiewicz ∞-valued calculus, Łukasiewicz, who in early twenties introduced a system of logic in which propositions admit as truth values real numbers between 0 and 1. The main connectives are *implication* $\to$ and *negation* $\neg$, such that $x \to y := \min(1, 1-x+y)$ and $\neg x = 1-x$. These connectives can be rewritten in terms of MV-algebras as $x \to y = \neg x \oplus y$. Using results of Chapter 3, it is proved that all tautologies are obtainable from a certain set of initial tautologies (corresponding to the MV-axioms) by a finite number of applications of *modus ponens*, and an effective procedure to decide whether a proposition is a tautology.

Chapter 5 presents Ulam's game, the variant of Twenty Questions, where $n-2$ lies or errors are allowed. This game goes back to Ulam. In Ulam's game with m lies, our knowledge is presented by the function $\sigma\colon S \to \{0,1,\dots,m-1\}$, where S is a finite set of numbers. Such problems are closely connected with finding an optimal strategy in Ulam game with m lies or with finding an optimal m-error correcting code. MV-algebras can be interpreted as algebras of states of knowledge in generalized Ulam games, where the number of lies may depend on the secret number $x \in S$, and S itself may be infinite. Truth values may be irrational numbers in $[0,1]$, more even nonstandard real numbers.

Chapter 6 is a continuation of lattice-theoretical properties of MV-algebras. Here minimal prime ideals are studied together with Stonean ideals, archimedean elements, and hyperarchimedean algebras. In addition, complete MV-algebras and complete distributivity are studied.

In **Chapter 7**, the crucial fact that the category of MV-algebras and the category of Abelian unital ℓ-groups are categoricaly equivalent via the above defined functor Γ is established (this is a famous Mundici Representation Theorem). As a corollary. a genuine addition can be uniquely recovered from the MV-algebraic structure.

Perfect MV-algebras are MV-algebras A such that each element $x \in A$ belongs either to $\mathrm{Rad}(A)$ or to $\neg\,\mathrm{Rad}(A) := \{\neg a : \ a \in \mathrm{Rad}(A)\}$. For such MV-algebras it si shown that the category of perfect perfect MV-algebras is categoricaly equivalent with the category of all ℓ-groups.

Chapter 8 is dedicated to the description of all varieties of MV-algebras. Komori's classification and varieties generated by finite chains are presented.

Advanced topics are given in **Chapter 9**. Th first part deals with disjunctive minimal forms in the infinite-valued calculus of Łukasiewicz. The relationship between MV-algebras and approximately finite-dimensional C^*-algebras is presented. Finally, an important Di Nola's Representation Theorem is given which says that every MV-algebra A is an algebra of $[0,1]^*$ valued-functions over some set, where $[0,1]^*$ is an ultrapower of $[0,1]$, depending only on he cardinality of A.

The last chapter is Further Readings, where the authors outlined further ways of study of MV-algebras, like states, observables, product, probability, etc.

All chapters contain bibliographical remarks.

The monograph under review is addressed to computer scientists, mathematicians, logicians wishing to get acquainted with a compact body of beautiful theory, results and methodologies on MV-algebras, that have found applications in the handling of uncertainty information connecting many areas of mathematics like lattice-ordered groups, C^*-algebras, lattices, algebra topic, geometry of numbers, model theory, polyhedra, etc.

The book presents a welcome source on MV-algebras and many-valued reasoning by world-known experts on this topic.

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