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GALOIS TRIANGLE THEORY FOR CERTAIN FREE MODULES

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ABSTRACT. The aim of this paper is to generalize the Galois triangle theory for free modules over local rings of a special type.

I. Introduction

Let a local ring $\mathbf{A}$ be given and let $\mathbf{M}$ be a free $\mathbf{A}$-module. The purpose of this paper is to find $1-1$ correspondences between the ordered set of submodules of $\mathbf{M}$ and the set of left (right) anihilators of the ring of endomorphisms of $\mathbf{M}$.

The solution of this problem is well known for example when $\mathbf{M}$ is a vector spaces ([1]) or a totally reducible module ([6]).

In this paper we will consider a linear algebra $\mathbf{A}$ which as a vector space over a field $\mathbf{T}$ has a basis

$$\{1, \eta, \eta^2, \dots, \eta^{m-1}\} \qquad \text{with} \quad \eta^m = 0. \tag{1}$$

($\mathbf{A}$ is isomorphic to the factor ring of polynomials $\mathbf{T}[x]/(x^m)$).

Evidently, $\mathbf{A}$ is a local ring with the maximal ideal $\eta\mathbf{A}$. The all ideals of $\mathbf{A}$ are just $\eta^j\mathbf{A}$, $1 \le j \le m$.

II. A-spaces and their endomorphisms

Let $\mathbf{M}$ be a free finite dimensional module over $\mathbf{A}$. It is well known that all bases of $\mathbf{M}$ have the same number of elements (called the $\mathbf{A}$-*dimension of* $\mathbf{M}$) and from every system of generators of $\mathbf{M}$ we may select a basis of $\mathbf{M}$ (see [2]).

Moreover in our case the module $\mathbf{M}$ has the following qualities (proved in [4]):

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1. Any linearly independent system can be completed to a basis of $\mathbf{M}$.
2. A submodule of $\mathbf{M}$ is a free module if and only if it is a direct summand of $\mathbf{M}$.

Remark. Free finitely dimensional modules over local ring $\mathbf{R}$ are called $\mathbf{R}$-*spaces* (see e.g. [2]) and their direct summands $\mathbf{R}$-*subspaces.*

We get that in our case $\mathbf{A}$-subspaces of the $\mathbf{A}$-space $\mathbf{M}$ are just the free submodules of $\mathbf{M}$.

In what follows, let $\mathbf{M}$ denote an arbitrary but fixed n-dimensional $\mathbf{A}$-space.

Let us define an endomorphism η on an $\mathbf{A}$-space $\mathbf{M}$ by the relation:

$$(\forall \mathbf{x} \in \mathbf{M})\big(\eta(\mathbf{x}) = \eta \cdot \mathbf{x}\big). \tag{1}$$

3. Proposition. *If S is a nontrivial submodule of $\mathbf{M}$, then there exists a system $\mathcal{B}_0, \dots, \mathcal{B}_r$ of subsets of $\mathbf{M}$ such that*

(a) $\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{r-1} \cup \mathcal{B}_r$ *is a basis of the $\mathbf{A}$-space $\mathbf{M}$,*
(b) $\eta^{m-r}\mathcal{B}_0 \cup \eta^{m-r+1}\mathcal{B}_1 \cup \dots \cup \eta^{m-1}\mathcal{B}_{r-1}$ *is a set of generators of the $\mathbf{A}$-module S,*
(c) $S \subseteq \operatorname{Ker}\eta^r \wedge S \not\subseteq \operatorname{Ker}\eta^{r-1}$.

Proof. Let us define $\vartheta \in \operatorname{End}\mathbf{M}$ by $\vartheta = \eta|_S$. As S is η-invariant, $\vartheta \in \operatorname{End} S$.

Using [5] ($\mathbf{M}$ is a free $\mathbf{A}$-module) we get $\eta^j\mathbf{M} = \operatorname{Ker}\eta^{m-j}$, which implies that

$$S \cap \eta^j\mathbf{M} = \operatorname{Ker}\vartheta^{m-j}, \qquad 1 \le j \le m-1. \tag{2}$$

As $\mathbf{T} \subseteq \mathbf{A}$ it is clear that $\mathbf{M}$ (as well as every submodule of $\mathbf{M}$) is a vector space over $\mathbf{T}$.

The operator η is a nilpotent endomorphism on the vector space $\mathbf{M}$. It is well known (see [3]) that the following kernels form a chain of inclusions

$$\{\mathbf{o}\} = \operatorname{Ker}\eta^0 \subset \operatorname{Ker}\eta \subset \dots \subset \operatorname{Ker}\eta^{r-1} \subset \operatorname{Ker}\eta^r \subset \dots \subset \operatorname{Ker}\eta^{m-1} \subset \operatorname{Ker}\eta^m = \mathbf{M}.$$

For any nontrivial submodule S of $\mathbf{M}$ there is a uniquely determined integer r, $1 \le r \le m$, such that $S \subseteq \operatorname{Ker}\eta^r \wedge S \not\subseteq \operatorname{Ker}\eta^{r-1}$.

Thus we have the following chain for the endomorphism $\vartheta = \eta|_S$ on S:

$$\{\mathbf{o}\} = \operatorname{Ker}\vartheta^0 \subset \operatorname{Ker}\vartheta \subset \dots \subset \operatorname{Ker}\vartheta^{r-1} \subset \operatorname{Ker}\vartheta^r = S.$$

Viewing these submodules as well as factor modules $S/\operatorname{Ker}\vartheta^{r-1}$, $\operatorname{Ker}\vartheta^{r-1}/\operatorname{Ker}\vartheta^{r-2}, \dots, \operatorname{Ker}\vartheta/\operatorname{Ker}\vartheta^0$ as vector spaces over $\mathbf{T}$ we have guaranteed the existence of elements
$\mathbf{w}_1, \dots, \mathbf{w}_{s_0}, \mathbf{w}_{s_0+1}, \dots, \mathbf{w}_{s_1}, \mathbf{w}_{s_1+1}, \dots, \mathbf{w}_{s_2}, \dots, \mathbf{w}_{s_{r-2}+1}, \dots, \mathbf{w}_{s_{r-1}}$ of S such

that $\mathbf{w}_1, \dots, \mathbf{w}_{s_0}$ is a $\mathbf{T}$-basis (i.e. a basis of this module considered as a vector space over $\mathbf{T}$) of S relatively (= modulo) to $\operatorname{Ker} \vartheta^{r-1}$,
$\eta \mathbf{w}_1, \dots, \eta \mathbf{w}_{s_0}, \mathbf{w}_{s_0+1}, \dots, \mathbf{w}_{s_1}$ is a $\mathbf{T}$-basis of $\operatorname{Ker} \vartheta^{r-1}$ relatively to $\operatorname{Ker} \vartheta^{r-2}$,
....
$\eta^{r-k} \mathbf{w}_1, \dots, \eta^{r-k} \mathbf{w}_{s_0}, \eta^{r-k-1} \mathbf{w}_{s_0+1}, \dots, \eta^{r-k-1} \mathbf{w}_{s_1}, \dots, \mathbf{w}_{s_{r-k-1}+1}, \dots, \mathbf{w}_{s_{r-k}}$ is a $\mathbf{T}$-basis of $\operatorname{Ker} \vartheta^k$ relatively to $\operatorname{Ker} \vartheta^{k-1}$, $1 \le k < r-1$,
....
$\eta^{r-1} \mathbf{w}_1, \dots, \eta^{r-1} \mathbf{w}_{s_0}, \eta^{r-2} \mathbf{w}_{s_0+1}, \dots, \eta^{r-2} \mathbf{w}_{s_1}, \dots, \mathbf{w}_{s_{r-2}+1}, \dots, \mathbf{w}_{s_{r-1}}$, is a $\mathbf{T}$-basis of $\operatorname{Ker} \vartheta$.

Further, the union of the above set forms a $\mathbf{T}$-basis of S.

Since $\{\mathbf{w}_1, \dots, \mathbf{w}_{s_0}\} \subseteq \operatorname{Ker} \eta^r$ and $\operatorname{Ker} \eta^r = \eta^{m-r} \mathbf{M}$ (by (2)) we obtain the existence of elements $\mathbf{u}_1, \dots, \mathbf{u}_{s_0}$ of $\mathbf{M}$ such that

$$\mathbf{w}_i = \eta^{m-r} \mathbf{u}_i, \qquad 1 \le i \le s_0. \tag{3}$$

Similarly, having in mind that $\{\mathbf{w}_{s_{r-k-1}+1}, \dots, \mathbf{w}_{s_{r-k}}\} \subseteq \operatorname{Ker} \eta^k$ and $\operatorname{Ker} \eta^k = \eta^{m-k} \mathbf{M}$ we obtain the set of elements $\mathbf{u}_{s_{r-k-1}+1}, \dots, \mathbf{u}_{s_{r-k}}$ of $\mathbf{M}$ satisfying

$$\mathbf{w}_i = \eta^{m-k} \mathbf{u}_i, \quad s_{r-k-1} + 1 \le i \le s_{r-k}, \qquad \text{for} \quad k = 1, \dots, r-1. \tag{4}$$

Now, put $\mathcal{C}_0 = \{\mathbf{u}_1, \dots, \mathbf{u}_{s_0}\}$ and $\mathcal{C}_{r-k} = \{\mathbf{u}_{s_{r-k-1}+1}, \dots, \mathbf{u}_{s_{r-k}}\}$, $k = r-1, \dots, 1$.

We will prove that this system of sets has properties (a), (b) of the theorem.

Firstly, let us prove the linear independence of the union $\mathcal{C}_0 \cup \dots \cup \mathcal{C}_{r-1}$.

Supposing

$$\sum_{1 \le i \le s_{r-1}} \xi_i \mathbf{u}_i = \mathbf{o}, \tag{5}$$

where (by (1) in I) $\xi_i = \sum_{j=0}^{m-1} x_{ij} \eta^j$, $1 \le i \le s_{r-1}$ and all $x_{ij} \in \mathbf{T}$, we get that

$$\sum_{0 \le j \le m-1} \eta^j \sum_{1 \le i \le s_{r-1}} x_{ij} \mathbf{u}_i = \mathbf{o}. \tag{6}$$

Multiplying this equality by η^{m-1} we get

$$\mathbf{o} = \sum_{0 \le j \le m-1} \eta^{m-1+j} \sum_{1 \le i \le s_{r-1}} x_{ij} \mathbf{u}_i = \eta^{m-1} \sum_{1 \le i \le s_{r-1}} x_{i0} \mathbf{u}_i.$$

This may be expressed as

$$\sum_{1 < i \le s_0} x_{i0}(\eta^{m-1} \mathbf{u}_i) + \sum_{s_0 < i \le s_1} x_{i0}(\eta^{m-1} \mathbf{u}_i) + \dots + \sum_{s_{r-2} < i \le s_{r-1}} x_{i0}(\eta^{m-1} \mathbf{u}_i) = \mathbf{o},$$

which (according to (3), (4)) gives

$$\sum_{1\le i\le s_0} x_{i0}(\eta^{r-1}\mathbf{w}_i) + \sum_{s_0<i\le s_1} x_{i0}(\eta^{r-2}\mathbf{w}_i) + \cdots + \sum_{s_{r-2}<i\le s_{r-1}} x_{i0}\mathbf{w}_i = \mathbf{o}\,.$$

Since it is a linear combination of elements of the $\mathbf{T}$-basis of S (the coefficients of which belong to $\mathbf{T}$) we obtain that $x_{i0} = 0$ for $i = 1,\dots,s_{r-1}$.

This implies that (6) may be written as

$$\sum_{1\le j\le m-1} \eta^j \sum_{1\le i\le s_{r-1}} x_{ij}\mathbf{u}_i = \mathbf{o}\,.$$

Now, multiplying this equality by η^{m-2} and using (3), (4) we have

$$\begin{aligned}\mathbf{o} &= \eta^{m-1} \sum_{1\le i\le s_{r-1}} x_{i1}\mathbf{u}_i \\ &= \sum_{1\le i\le s_0} x_{i1}(\eta^{r-1}\mathbf{w}_i) + \sum_{s_0<i\le s_1} x_{i1}(\eta^{r-2}\mathbf{w}_i) + \cdots + \sum_{s_{r-2}<i\le s_{r-1}} x_{i1}\mathbf{w}_i\,,\end{aligned}$$

which (as in the previous step) yields $x_{i1} = 0$ for $i = 1,\dots,s_{r-1}$. Thus (6) becomes

$$\sum_{2\le j\le m-1} \eta^j \sum_{1\le i\le s_{r-1}} x_{ij}\mathbf{u}_i = \mathbf{o}\,.$$

If we multiply (6) by $\eta^{m-3},\dots,\eta$, successively, then in the same way we may deduce that all coefficients x_{ij} are zero and $\xi_1 = \xi_2 = \cdots = \xi_{s_{r-1}} = 0$, consequently. The linear independence of the union $\mathcal{C}_0 \cup \cdots \cup \mathcal{C}_{r-1}$ is proved.

By Proposition 1 we may complete this set to an $\mathbf{A}$-basis of an $\mathbf{A}$-space $\mathbf{M}$ by a subset $\mathcal{C}_r$.

Secondly, we will prove that $\eta^{m-r}\mathcal{C}_0 \cup \eta^{m-r+1}\mathcal{C}_1 \cup \cdots \cup \eta^{m-1}\mathcal{C}_{r-1}$ is a set of generators (over $\mathbf{A}$) of the $\mathbf{A}$-module S.

Using the notation of the elements of basis of factormodules of the first part of this proof and having in mind (3), (4) we may write

$$\begin{aligned}
\mathbf{x} &= \sum_{\substack{1\le i\le s_0\\ 0\le j\le r-1}} x_{ij}(\eta^j\mathbf{w}_i) + \sum_{\substack{s_0<i\le s_1\\ 0\le j\le r-2}} x_{ij}(\eta^j\mathbf{w}_i) + \dots \\
&\qquad \dots + \sum_{\substack{s_{r-3}<i\le s_{r-2}\\ 0\le j\le 1}} x_{ij}(\eta^j\mathbf{w}_i) + \sum_{s_{r-2}<i\le s_{r-1}} x_{ij}\mathbf{w}_i \\
&= \sum_{\substack{1\le i\le s_0\\ 0\le j\le r-1}} x_{ij}(\eta^{j+m-r}\mathbf{u}_i) + \sum_{\substack{s_0<i\le s_1\\ 0\le j\le r-2}} x_{ij}(\eta^{j+m-r+1}\mathbf{u}_i) + \dots \\
&\qquad \dots + \sum_{\substack{s_{r-3}<i\le s_{r-2}\\ 0\le j\le 1}} x_{ij}(\eta^{j+m-2}\mathbf{u}_i) + \sum_{s_{r-2}<i\le s_{r-1}} x_{ij}\eta^{m-1}\mathbf{u}_i \\
&= \sum_{\substack{1\le i\le s_0\\ 0\le k\le m-1}} (x_{ik}\eta^k)(\eta^{m-r}\mathbf{u}_i) + \sum_{\substack{s_0<i\le s_1\\ 0\le k\le m-1}} (x_{ik}\eta^k)(\eta^{m-r+1}\mathbf{u}_i) + \dots \\
&\qquad \dots + \sum_{\substack{s_{r-3}<i\le s_{r-2}\\ 0\le k\le m-1}} (x_{ik}\eta^k)(\eta^{m-2}\mathbf{u}_i) + \sum_{\substack{s_{r-2}<i\le s_{r-1}\\ 0\le k\le m-1}} (x_{ik}\eta^k)(\eta^{m-1}\mathbf{u}_i) \\
&= \sum_{1\le i\le s_0} \xi_i(\eta^{m-r}\mathbf{u}_i) + \sum_{s_0<i\le s_1} \xi_i(\eta^{m-r+1}\mathbf{u}_i) + \dots \\
&\qquad \dots + \sum_{s_{r-3}<i\le s_{r-2}} \xi_i(\eta^{m-2}\mathbf{u}_i) + \sum_{s_{r-2}<i\le s_{r-1}} \xi_i(\eta^{m-1}\mathbf{u}_i)\,.
\end{aligned}$$

Obviously, this implies that an arbitrary element of $\mathbf{M}$ which may be expressed as a linear combination over $\mathbf{T}$ of elements of $\mathbf{T}$-basis of S may also be written as a linear combination of elements of $\eta^{m-r}\mathcal{C}_0\cup\eta^{m-r+1}\mathcal{C}_1\cup\cdots\cup\eta^{m-1}\mathcal{C}_{r-1}$ with coefficients from $\mathbf{A}$ and vice versa.

Now, we may prove that the system of sets $\mathcal{C}_0,\dots,\mathcal{C}_{r-1},\mathcal{C}_r$ has all the demanded properties. □

4. THEOREM. *Let S be a submodule of an $\mathbf{A}$-space $\mathbf{M}$. Then there exist endomorpisms f, g of $\mathbf{M}$ such that*

$$\operatorname{Ker} f = S\,, \qquad \operatorname{Im} g = S\,.^{1)}$$

P r o o f . Evidently, if S is trivial, then the theorem holds.

Let S be nontrivial. Let us construct a system of subsets $\mathcal{B}_0,\dots,\mathcal{B}_r$ as in Proposition 3.

[1] Let us remark that in general this theorem does not hold for modules over an arbitrary ring (for example the set of integers $\mathbb{Z}$ may be considered as a (free) module over $\mathbb{Z}$. The submodule of even numbers is not kernel of any endomorphism of the module Z).

Denoting by $\mathbf{M}_j$ the $\mathbf{A}$-subspace with basis $\mathcal{B}_j$ for all j, $0 \le j \le r$, we get a system of $\mathbf{A}$-subspaces of $\mathbf{M}$ with

$$\mathbf{M}_0 \oplus \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_r = \mathbf{M}, \tag{7}$$

$$\eta^{m-r}\mathbf{M}_0 + \eta^{m-r+1}\mathbf{M}_1 + \cdots + \eta^{m-r+j}\mathbf{M}_j + \cdots + \eta^{m-1}\mathbf{M}_{r-1} = S. \tag{8}$$

Let us define the endomorphism f on $\mathbf{M}$ by

$$f|_{\mathbf{M}_j} = \eta^{r-j}, \qquad 0 \le j \le r.^{2)} \tag{9}$$

Clearly, (8) implies that $S \subseteq \operatorname{Ker} f$.

Let $\mathbf{x} \in \mathbf{M}$, $\mathbf{x} = \sum\limits_{j=0}^{r} \mathbf{x}_j$, $\mathbf{x}_j \in \mathbf{M}_j$. Supposing $\mathbf{x} \in \operatorname{Ker} f$ we get (by (9))

$$\mathbf{o} = f(\mathbf{x}) = \sum_{j=0}^{r} f(\mathbf{x}_j) = \sum_{j=0}^{r} (\eta^{r-j}\mathbf{x}_j).$$

Since $\eta^{r-j}\mathbf{x}_j \in \mathbf{M}_j$ we obtain (by (7)) $\eta^{r-j}\mathbf{x}_j = \mathbf{o}$, $0 \le j \le r$. Having in mind that all $\mathbf{M}_j$ are $\mathbf{A}$-spaces we have (as in the proof of 3) $\operatorname{Ker}(\eta|_{\mathbf{M}_j})^k = \eta^{m-k}\mathbf{M}_j$, $0 \le k \le m$, $0 \le j \le r$. This yields that $\mathbf{x}_j \in \eta^{m-r+j}\mathbf{M}_j$ and thus (by (8)) $\mathbf{x}_j \in S$, $0 \le j \le r$. Consequently, $\mathbf{x} \in S$.

The kernel of this endomorphism f is equal to S.

Now define an endomorphism g on $\mathbf{M}$ by

$$g|_{\mathbf{M}_j} = \eta^{m-r+j}, \qquad 0 \le j \le r. \tag{10}$$

Evidently, $\operatorname{Im} g \subseteq S$ (by (7) and (8)).

If $\mathbf{x} \in S$, then (by (8) and (10)) we may write

$$\mathbf{x} = \sum_{j=0}^{r} \eta^{m-r+j}\mathbf{y}_j{}^{3)} = \sum_{j=0}^{r} g(\mathbf{y}_j) = g(\mathbf{y}),$$

where

$$\mathbf{y} = \sum_{j=0}^{r} \mathbf{y}_j,$$

which gives $\mathbf{x} \in \operatorname{Im} g$. □

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2) By η we denote the endomorphism defined by (1).

3) where, of course, $\mathbf{y}_j \in \mathbf{M}_j$

III. Galois triangle theory for A-spaces

Let $\mathbf{M}$ be an $\mathbf{A}$-space as in the previous section.

1. Notation.

1.1. We will denote by $\mathbf{P}$ the ring of endomorphisms of $\mathbf{M}$, $\mathbf{P} = \operatorname{End}\mathbf{M}$, and we will define the composition of $f, g \in \mathbf{P}$ by $(fg)(\mathbf{x}) = g(f(\mathbf{x}))$.

1.2. Let $J \subseteq \mathbf{P}$. Then we will denote by $\mathbf{L}(J)$ the left anihilator of J, i.e. $\mathbf{L}(J) = \{f \in \mathbf{P} : (\forall g \in J)(fg = o)\}$ and by $\mathbf{R}(J)$ the right anihilator of J, i.e. $\mathbf{R}(J) = \{f \in \mathbf{P} : (\forall g \in J)(gf = o)\}$.

1.3. We will denote by $\mathcal{L}(\mathbf{P})$ the set of the all left anihilators of the ring $\mathbf{P}$, by $\mathcal{R}(\mathbf{P})$ the set of the all right anihilators of $\mathbf{P}$ and by $\mathcal{U}(\mathbf{M})$ the set of the all submodules of the $\mathbf{A}$-space $\mathbf{M}$.

1.4. For every submodule $S \in \mathcal{U}(\mathbf{M})$ let us denote

$$\mathbf{N}(S) = \{f \in \mathbf{P} : (\forall \mathbf{x} \in S)(f(\mathbf{x}) = \mathbf{o})\},$$
$$\mathbf{Q}(S) = \{f \in \mathbf{P} : (\forall \mathbf{x} \in \mathbf{M})(f(\mathbf{x}) \in S)\}.$$

(Equivalently,

$$\mathbf{N}(S) = \{f \in \mathbf{P} : S \subseteq \operatorname{Ker} f\}, \qquad \mathbf{Q}(S) = \{f \in \mathbf{P} : \operatorname{Im} f \subseteq S\}.)$$

1.5. For every subset J of $\mathbf{P}$ let us denote

$$\mathbf{K}(J) = \{\mathbf{x} \in \mathbf{M} : (\forall f \in J)(f(\mathbf{x}) = \mathbf{o})\},$$
$$\mathbf{M}(J) = \{\mathbf{x} \in \mathbf{M} : (\exists f \in J)(\exists \mathbf{y} \in \mathbf{M})(\mathbf{x} = f(\mathbf{y}))\}.$$

(In the same way as in 1.4,

$$\mathbf{K}(J) = \bigcap_{f \in J} \operatorname{Ker} f, \qquad \mathbf{M}(J) = \bigcup_{f \in J} \operatorname{Im} f.)$$

2. Remark. It is easy to see that, for every $J \subseteq \mathbf{P}$ and every $S \in \mathcal{U}(\mathbf{M})$, $\mathbf{L}(J)$ and $\mathbf{Q}(S)$ are left ideals of $\mathbf{P}$ and $\mathbf{R}(J)$ and $\mathbf{N}(S)$ are right ideals of $\mathbf{P}$.

It is also easy to derive that for every $U, S \in \mathcal{U}(\mathbf{P})$, $J, H \subseteq \mathbf{P}$,

$$J \subseteq H \implies \mathbf{K}(J) \supseteq \mathbf{K}(H), \ \mathbf{M}(J) \subseteq \mathbf{M}(H), \ \mathbf{R}(J) \supseteq \mathbf{R}(H), \ \mathbf{L}(J) \supseteq \mathbf{L}(H),$$
$$U \subseteq S \implies \mathbf{N}(U) \supseteq \mathbf{N}(S), \ \mathbf{Q}(U) \subseteq \mathbf{Q}(S).$$

3. THEOREM. *For every submodule* $\forall S \in \mathcal{U}(\mathbf{M})$ *we have:*

$$\mathbf{K}(\mathbf{N}(S)) = S, \qquad \mathbf{M}(\mathbf{Q}(S)) = S.$$

Proof. It follows from the definition of $\mathbf{K}$ and $\mathbf{N}$, respectively $\mathbf{M}$ and $\mathbf{Q}$, that $S \subseteq \mathbf{K}(\mathbf{N}(S))$, respectively $S \supseteq \mathbf{M}(\mathbf{Q}(S))$. Let us prove the reverse inclusions. According to Theorem I.4 there exist endomorphisms f, g s.t. $S = \operatorname{Ker} f = \operatorname{Im} g$.

a) Using the fact $S = \operatorname{Ker} f$ we have that $f \in \mathbf{N}(S)$ (by 1.4).

Let $\mathbf{s}$ be an arbitrary element of $\mathbf{K}(\mathbf{N}(S))$. Then (by 1.5)

$$(\forall h \in \mathbf{N}(S))(h(\mathbf{s}) = \mathbf{o}),$$

which gives $f(\mathbf{s}) = \mathbf{o}$, of course. As $S = \operatorname{Ker} f$, then $\mathbf{s}$ belongs to S.

b) Since $S = \operatorname{Im} g$, we have (by 1.4) $g \in \mathbf{Q}(S)$.

If $\mathbf{s}$ be an arbitrary element of S, then it may be written as $\mathbf{s} = g(\mathbf{x})$, $\mathbf{x} \in \mathbf{M}$. This implies that $\mathbf{s} \in \mathbf{M}(\mathbf{Q}(S))$ (by 1.5). □

Using Definitions 1.4, 1.5 we may prove the following proposition as in the case when $\mathbf{M}$ is a vector space (see [1]).

4. PROPOSITION. *For every subset* $J \subseteq \mathbf{P}$ *we have:*

$$\mathbf{N}(\mathbf{M}(J)) = \mathbf{R}(J), \qquad \mathbf{Q}(\mathbf{K}(J)) = \mathbf{L}(J).$$

5. PROPOSITION. *For every submodule*

$$(\forall S \in \mathcal{U}(\mathbf{M}))\Big(\mathbf{N}(S) = \mathbf{R}(\mathbf{Q}(S)) \ \& \ \mathbf{Q}(S) = \mathbf{L}(\mathbf{N}(S))\Big).$$

This proposition is a consequence of Propositions 3 and 4.

6. Remark. It follows from this proposition that $\mathbf{N}(S)$ is an element of $\mathcal{R}(\mathbf{P})$ and $\mathbf{Q}(S)$ is an element of $\mathcal{L}(\mathbf{P})$ for every $S \in \mathcal{U}(\mathbf{M})$.

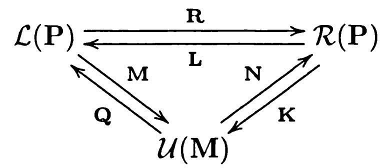
Using the Propositions 3 and 4 we may prove the following proposition as in the case $\mathbf{M}$ is a vector space (see [1]).

7. PROPOSITION. *For every right anihilator* $H \in \mathcal{R}(\mathbf{P})$, $\mathbf{N}(\mathbf{K}(H)) = H$, *for every left anihilator* $J \in \mathcal{L}(\mathbf{P})$, $\mathbf{Q}(\mathbf{M}(J)) = J$.

Now, considering operators $\mathbf{N}$, $\mathbf{K}$, $\mathbf{Q}$, $\mathbf{M}$, $\mathbf{L}$, $\mathbf{R}$ as mappings of corresponding ordered sets we may formulate the fundamental theorem of the Galois triangle theory.

THEOREM.

1. *The operators* **N** *and* **K** *are mutually inverse antiisomorphisms of the ordered sets* $(\mathcal{U}(\mathbf{M}), \subseteq)$ *and* $(\mathcal{R}(\mathbf{P}), \subseteq)$.
2. *The operators* **Q** *and* **M** *are mutually inverse isomorphisms of the ordered sets* $(\mathcal{U}(\mathbf{M}), \subseteq)$ *and* $(\mathcal{L}(\mathbf{P}), \subseteq)$.
3. *The operators* **L** *and* **R** *are mutually inverse antiisomorphisms of the ordered sets* $(\mathcal{R}(\mathbf{P}), \subseteq)$ *and* $(\mathcal{L}(\mathbf{P}), \subseteq)$.
4. *The following diagram is commutative.*



P r o o f. This theorem follows from Propositions 3, 4, 5 and 7, and Remarks 2, 6 as in the case when **M** is a vector space. □

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