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THE SIZE OF THE SET OF ZEROS OF A WRONSKIAN

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ABSTRACT. If a set $S \subseteq \mathbb{R}$ is such that the identical vanishing on S of any real-valued derivative function f' implies the identical vanishing of f' on $\mathbb{R}$, then the identical vanishing on S of any at least two-dimensional Wronskian W implies the identical vanishing of W on $\mathbb{R}$. A consequence is a simple proof of Curtiss' theorem that the identical vanishing of a Wronskian is not affected by including new functions in it.

1. Introduction

The Wronskian in the title is the k-dimensional functional determinant $W(g_1,\dots,g_k;x)$ whose ith row is the vector $g_1^{(i-1)}(x),\dots,g_k^{(i-1)}(x)$ of the $(i-1)$st derivatives of k real-valued functions $g_1,\dots,g_k$ of a real variable x (see [2]). Banas and El-Sayed [1] proved in this journal that in contrast to derivative functions, Wronskians of dimension at least two do not have the Darboux property in general. (A one-dimensional Wronskian can be any function and thus it is uninteresting.) Consequently, some Wronskians of dimension at least two are not derivative functions. Nevertheless, in this paper we prove that Wronskians of dimension at least two do have the following important property: If a set $S \subseteq \mathbb{R}$ has the property that the identical vanishing of a derivative function on S implies its identical vanishing on $\mathbb{R}$, then S has the same property for Wronskians of dimension at least two in place of derivative functions. Equivalently, if a set admits an at least two-dimensional Wronskian not identically zero that vanishes at all of its points, then it admits such a derivative function, as well. The converse is obvious, since $f' = W(1,f)$.

An immediate consequence of these results is that an at least two-dimensional Wronskian that vanishes almost everywhere, actually vanishes everywhere. This

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is a generalization of a result of Meisters [9]. Another less immediate consequence is that an identically vanishing Wronskian remains such upon including further functions in it. This theorem has a long history (see [3] – [6], [9], [10]). Our proof provided here (including the proof of the property of two-dimensional Wronskians mentioned above) is simpler than Chaundy's [4] simplification of Curtiss' [5], [6] first complete proof.

Our proofs depend on a classical identity of Christoffel [7]. Although we consider real-valued functions of a real variable, our results, including their proofs, extend automatically to complex-valued functions (of a real variable).

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2. How large is the set of zeros of a Wronskian?

Unlike in the Introduction, we state our theorem about the size of the set of zeros of an at least two-dimensional Wronskian on an arbitrary interval rather than just on the entire real line:

THEOREM 1. *Let I be a nondegenerate interval of the real line $\mathbb{R}$ and let S be a subset of I. Properties* (i) – (ii) *below are equivalent.*

(i) *If f is a real-valued function with a finite first derivative defined on I and $f' \equiv 0$ on S, then $f' \equiv 0$ on I.*

(ii) *Let $f_1, \dots, f_n$, $n \geq 2$, be real-valued functions with finite $(n-1)$st derivatives on I. If $W(f_1, \dots, f_n) \equiv 0$ on S, then $W(f_1, \dots, f_n) \equiv 0$ on I.*

The simple lemma below shows that (i) implies an apparently stronger property. This observation will be needed in the proof of Theorem 1.

LEMMA 2. *If a subset S of an interval I has Property* (i), *then the subset $S \cap J$ of any subinterval J of I has Property* (i) *with I replaced by J.*

Proof. Let a subset S of the interval I have Property (i) and let J be a nondegenerate subinterval of I. We consider a differentiable function f on J such that $f' \equiv 0$ on $J \cap S$. Let $a, b \in J \cap S$, $a < b$. Then $f'(a) = f'(b) = 0$. The function $f_{a,b}$, equal to f on the compact interval $[a, b]$, to $f(a)$ on $(-\infty, a)$, and to $f(b)$ on (b, ∞), is defined and differentiable on $\mathbb{R}$ and $f'_{a,b} = 0$ on S. By Property (i) of $S \subseteq I$ we obtain that $f'_{a,b} = 0$ on I, that is, $f_{a,b}$ is constant on I. The desired constancy of f on J can be obtained from this by letting a (b) go to the left (right) endpoint (which may be infinite) of J. This is possible because otherwise the subset S of I would not have Property (i), as is easy to see. □

An example of a set S with Property (i) is any subset of I obtained from I by deleting finitely many points of it. Such a set S does indeed have Property (i) because the continuity of a function f like in (i) implies that f must be equal to one and the same constant on the components of $S \cap I$. A more general set S with Property (i) is one whose complement with respect to I is of Lebesgue measure zero. That an almost everywhere vanishing derivative function vanishes everywhere, is Denjoy's result [8]. Therefore, this classical result of Denjoy, combined with the (i) $\Longrightarrow$ (ii) part of Theorem 1, immediately implies the following generalization of Meisters' result [9; p. 852, Theorem 4].

Corollary 3. *If $f_1, \dots, f_n$ are $n-1$ times differentiable functions on a nondegenerate interval I and $W(f_1, \dots, f_n) = 0$ almost everywhere on I, then $W(f_1, \dots, f_n) = 0$ everywhere on I.*

Meisters' assumptions are equivalent to the following:

1. $W(f_1, \dots, f_n)$ vanishes on a dense open subset of I;
2. $W(f_1, \dots, f_n)$ is quasicontinuous almost everywhere on I.

Since Meisters easily proves that any $W(f_1, \dots, f_n)$ satisfying 1. vanishes at every point where it is quasicontinuous, his result is an easy consequence of Corollary 3. (A function f is quasicontinuous at a point x if the interior of $f^{-1}(V) \cap U$ is nonempty whenever U (V) is a neighborhood of x $(f(x))$.)

We prove Theorem 1 by induction, relying on Christoffel's [7; p. 297–299] identity

$$W(g_1, \dots g_k) = g_1^k W\big((g_2/g_1)', \dots, (g_k/g_1)'\big) \qquad \text{if} \quad g_1 \neq 0, \tag{1}$$

which we prove at the end of this section for the sake of completeness.

Proof of Theorem 1. Given a differentiable function f, we have $f' = W(1, f)$. Therefore, Property (ii) implies (i). Conversely, let us assume that a set $S \subseteq I$ has Property (i). We prove that S has Property (ii). To this end, let $f_1, \dots, f_n$ be such that $W(f_1, \dots, f_n) \equiv 0$ on S. Let $\xi \in I$. We must prove that $W(f_1, \dots, f_n; \xi) = 0$. If $f_1(\xi) = \cdots = f_n(\xi) = 0$, then $W(f_1, \dots, f_n; \xi) = 0$ obviously. If $f_j(\xi) \neq 0$ for some j, then $f_j(x) \neq 0$ for all x in a nondegenerate interval J containing ξ because f_j is continuous since $n \geq 2$. Since $\xi \in J$ and we only want to prove that $W(f_1, \dots, f_n; \xi) = 0$, by Lemma 2 we can and do assume that $I = J$. We also assume that $j = 1$, since this can be achieved by reindexing. Consequently, we assume that $f_1 \neq 0$ everywhere on I.

We prove (ii) (assuming (i)) by induction on n. First let $n = 2$. Then $W(f_1, f_2)/f_1^2 = (f_2/f_1)'$ is 0 everywhere on S by the hypotheses of (ii). Consequently, it is 0 everywhere on I by (i). Now let $n \geq 3$ and assume that (ii) holds for $n-1$ instead of n. By the hypotheses of (ii) and by (1), $W\big((f_2/f_1)', \dots, (f_n/f_1)'\big) = 0$ on S. Then $W\big((f_2/f_1)', \dots, (f_n/f_1)'\big) = 0$ on the entire I by the induction hypothesis, and thus $W(f_1, \ldots, f_n) = 0$ on the entire I by (1). □

QUESTION. Property (ii) in Theorem 1 stipulates that if an at least two-dimensional Wronskian vanishes on S, then it vanishes on I. Because of the additivity of differentiation, (i) states that if given differentiable functions f, g satisfy $f' \equiv g'$ on S, then $f' \equiv g'$ on I. This suggests the following question: If a set S has Property (ii) of Theorem 1 and two Wronskians W, $\tilde{W}$ of the same dimension (≥ 2) satisfy $W \equiv \tilde{W}$ on S, then is it true that $W \equiv \tilde{W}$ on I?

Proof of (1). First we note that

$$W(\varphi g_1, \ldots, \varphi g_k) = \varphi^k W(g_1, \ldots, g_k). \tag{2}$$

We prove (2) by induction. It is obvious for $k = 1$. Let $k \geq 2$ and assume that (2) holds when k is replaced with $k-1$. We expand the $(k-1)$st derivatives in the last row of $W(\varphi g_1, \ldots, \varphi g_k)$ by Leibniz' rule:

$$(\varphi g_j)^{(k-1)} = \sum_{s=0}^{k-1} \binom{k-1}{s} \varphi^{(s)} g_j^{(k-1-s)}, \qquad j = 1, \ldots, k. \tag{3}$$

Using this expansion, we write $W(\varphi g_1, \ldots, \varphi g_k)$ as the sum of k determinants from which we pull out $\binom{k-1}{s}\varphi^{(s)}$:

$$W(\varphi g_1, \ldots, \varphi g_k) = \sum_{s=0}^{k-1} \binom{k-1}{s} \varphi^{(s)} D_s, \tag{4}$$

where D_s is the k-dimensional determinant obtained from $W(\varphi g_1, \ldots, \varphi g_k)$ by replacing its last row with $g_1^{(k-1-s)}, \ldots, g_k^{(k-1-s)}$. If we expand D_s by its last row and apply the induction hypothesis to the cofactors, we obtain φ^{k-1} times the expansion, by its last row, of the determinant $\bar{D}_s$ obtained from $W(g_1, \ldots, g_k)$ by replacing its last row with $g_1^{(k-1-s)}, \ldots, g_k^{(k-1-s)}$. The determinant $\bar{D}_s$ has two equal rows unless $s = 0$, and $\bar{D}_0 = W(g_1, \ldots, g_k)$. Therefore, (2) follows from (4).

The identity (1) follows from (2) with $\varphi = g_1$ and g_j replaced by g_j/g_1. Then the left-hand side of (2) is $W(g_1, \ldots, g_k)$. The right-hand side is g_1^k times a determinant whose entries in the first column are $1, 0, \ldots, 0$ and the cofactor of 1 here is $W((g_2/g_1)', \ldots, (g_k/g_1)')$. The proof is completed by expanding this determinant by its first column. □

3. $W(f_1, \dots, f_{n-1}) \equiv 0 \implies W(f_1, \dots, f_n) \equiv 0$

The section title is a concise statement of the following theorem.

THEOREM 4. (Curtiss [5], [6]) *Let $f_1, \dots, f_n$ be real-valued functions of a real variable with finite $(n-1)$st derivatives throughout a nondegenerate interval I. If the Wronskian $W(f_1, \dots, f_{n-1})$ of the first $n-1$ of the functions vanishes everywhere on I, then so does the Wronskian $W(f_1, \dots, f_n)$ of all n functions.*

This theorem is not obvious because the identical vanishing of $W(f_1, \dots$ $\dots, f_{n-1})$ implies the linear dependence of $f_1, \dots, f_{n-1}$ only on each component of a dense open subset of I (see Meisters [9]). Hence $W(f_1, \dots, f_n)$ is identically 0 if it is continuous. This is Meisters' [9; p. 853] proof of Theorem 4 under the assumption of continuity of $W(f_1, \dots, f_n)$. The first proof of this was given by Bôcher [3; p. 148, Theorem VIII]. He raised the question of whether the continuity of $W(f_1, \dots, f_n)$ is really needed. Curtiss [5; p. 484, the paragraph after Theorem IV], [6; p. 293, easy consequence of Theorem VII] was the first to prove Theorem 4 in its full generality. His quite complicated proof was simplified by Chaundy [4]. Chaundy's proof is complete only up to $n = 4$, passed that it is vague.

In this section we give a new proof of the full-fledged Theorem 4, simpler than Chaundy's. The new feature of our proof is its reliance on Theorem 1. In some way, Theorem 1 substitutes for the fact that if $W(f_1, \dots, f_n)$ is continuous and vanishes on a dense subset, then it vanishes everywhere. In the proof of Theorem 4 we use the implication (i) $\implies$ (ii) only for sets S obtained from I by deleting one point. We use Christoffel's identity (1) again. Chaundy, too, but not Bôcher or Curtiss, used (1). We intend to give another simple proof of Theorem 4 in another paper. However, that proof relies on a theorem whose proof is more complicated than that of Theorem 1.

Proof of Theorem 4. We prove Theorem 4 by induction on n (≥ 2). The case $n = 2$ is obvious. Let $n \geq 3$, let $W(f_1, \dots, f_{n-1}) = 0$ identically on I, and assume that Theorem 4 holds for $n-1$ in place of n. Let $\xi \in I$. If ξ is such that $f_1(\xi) \neq 0$, then by continuity, $f_1(x) \neq 0$ for all x in a nondegenerate subinterval J of I, containing ξ. Therefore,

$$W(f_1, \dots, f_i) = f_1^i W((f_2/f_1)', \dots, (f_i/f_1)'), \qquad f_1 \neq 0, \tag{5}$$

on J for $i = n-1, n$ by (1). Then $W((f_2/f_1)', \dots, (f_{n-1}/f_1)') = 0$ everywhere on J by the hypotheses of the theorem. By the induction hypothesis, $W((f_2/f_1)', \dots, (f_n/f_1)') = 0$ everywhere on J. Therefore, $W(f_1, \dots, f_n) = 0$ everywhere on J by (5). In particular, $W(f_1, \dots, f_n; \xi) = 0$.

If $f_1(\xi) = \cdots = f_1^{(n-1)}(\xi) = 0$, then $W(f_1, \dots, f_n; \xi) = 0$ obviously. If $f_1(\xi) = 0$ but not all of $f_1'(\xi), \dots, f_1^{(n-1)}(\xi)$ are zero, then ξ is an isolated zero of f_1. Consequently, by the beginning of the proof there is a nondegenerate subinterval J of I, containing ξ, on which $W(f_1, \dots f_n)$ vanishes identically, except possibly at ξ. The proof of Theorem 4 can be completed by applying Theorem 1 on J with $S = J \setminus \{\xi\}$. $\square$

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