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ON BICRITICAL SNARKS

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ABSTRACT. Bicritical snarks are the irreducible ones with respect to the reductions considered by Nedela and Škoviera in [NEDELA, R.—ŠKOVIERA, M.: *Decompositions and reductions of snarks*, J. Graph Theory **22** (1996), 253–279]. We show that for some $n \geq 10$ and for each even $n \geq 92$ there is a hypohamiltonian and henceforth bicritical snark of order n. This solves a problem stated in [NEDELA, R.—ŠKOVIERA, M.: *Decompositions and reductions of snarks*, J. Graph Theory **22** (1996), 253–279].

1. Introduction

We are using standard graph theoretical terminology and notation in this paper. We define a *snark* to be a cubic graph with chromatic index $\chi' = 4$. Note that multiple edges and loops are allowed.

There are two main questions which lead to the study of reduction of snarks. The first one is the question about the intrinsic properties of cubic graphs which force them being a snark. The hope is that these properties can be detected in the irreducible snarks, and, since every snark is reducible to an irreducible one, every snark should have this property, too.

The second one is the question about a method to construct all snarks recursively starting from the irreducible snarks. Here the hope is that the reverse operation (of a reduction) can be described without reflecting on the reduction process and hence it would yield such a method.

There are many papers dealing with this topic, cf. [1], [2], [4], [5], [6], [8], [9], and we refer the reader to one of these papers for a more extensive introduction.

In this paper we consider the approach of Nedela and Škoviera [6]. We will give their definitions. A *multipole* $M = \big(V(M), E(M)\big)$ consists of a vertex set $V(M)$, and an edge set $E(M)$. Every edge of $E(M)$ has two ends and every

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end may or may not be incident to a vertex of $V(M)$. An end of an edge not incident to a vertex is called a *semiedge.*

Let M and N be multipoles with semiedges $e_1, \dots, e_k$ and $f_1, \dots, f_k$ ($k \geq 0$), respectively. The *k-junction* $M * N$ is obtained from M and N by identifying f_i and e_i, for $i = 1, \dots, k$. Clearly, $M * N$ is a multipole, and if it has no semiedges, we say it is a graph.

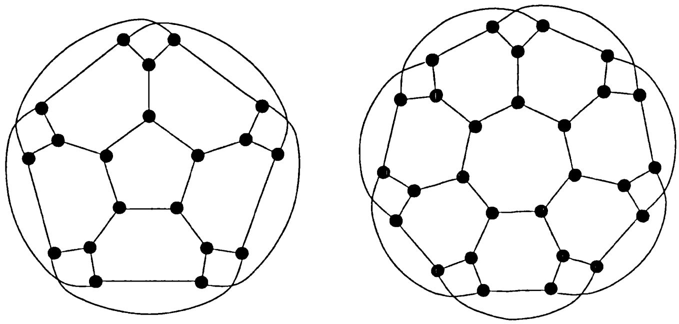
Let $G = \big(V(G), E(G)\big)$ be a snark which is the k-junction of two multipoles M and N. If the chromatic index of one of the multipoles is 4, say $\chi'(M) = 4$, then M can be extended to a snark $M^* = \big(V(M^*), E(M^*)\big)$ with $|V(M^*)| \leq |V(G)|$, and M^* is called a *k-reduction* of G. If $|V(M^*)| < |V(G)|$, then M^* is called a *proper* k-reduction of G.

A snark is called *k-irreducible* if it has no proper m-reduction for each $m < k$, and it is called *irreducible* if it is k-irreducible for each $k \geq 1$.

A snark G is called *bicritical* if $\chi'\big(G - \{v, w\}\big) = 3$ for any two vertices $v, w \in V(G)$.

THEOREM 1.1. ([6]) *A snark is irreducible if and only if it is bicritical.*

It is proved in [6] that the flower snark J_{2k+1} is irreducible, for each $k \geq 2$, where the flower snark J_{2k+1} is the graph with vertex set $V(J_{2k+1}) = \{a_i, b_i, c_i, d_i : i = 0, 1, \dots, 2k\}$ and edge set $E(J_{2k+1}) = \{b_i a_i, b_i c_i, b_i d_i; a_i a_{i+1}; c_i d_{i+1}; d_i c_{i+1} : i = 0, 1, \dots, 2k\}$. In the definition of $E(J_{2k+1})$ the addition in the indices is taken modulo $(2k + 1)$. The graphs J_5 and J_7 are shown in Figure 1.



J_5 $\qquad$ J_7

FIGURE 1.

Thus there are irreducible snarks of order $8k + 4$, for each $k \geq 2$. In [6] it is also stated that there are no irreducible snarks of orders 12, 14, 16 and 24, but there are irreducible snarks of orders 10, 18, and 22. In [1] these results are obtained independently, in fact, it is shown that there are exactly 2 irreducible snarks of order 18, 1 of order 20, 2 of order 22, 111 of order 26 and that there are 33 irreducible snarks of order 28.

Nedela and Škoviera asked the following questions:

PROBLEM 1.2. ([6]) For which even number $n \geq 10$ does there exist an irreducible snark of order n? In particular, does there exist an irreducible snark of each sufficiently large order?

We will give a partial solution of the first and a positive answer to the second question.

2. Families of bicritical snarks

We show that there are irreducible snarks of order n for some $n \geq 10$ and for each even $n \geq 92$. Indeed, we will show the somewhat stronger result, that for some $n \geq 10$ and that for each even $n \geq 92$ there is a hypohamiltonian snark. For the proof we use some results of [3] and [8]. Fiorini proved in [3] a sufficient condition under which the dot product preserves the property of being hypohamiltonian. To state Fiorini's results we give the definitions.

A graph G is *hypohamiltonian* if it is not hamiltonian, but $G - v$ is hamiltonian for every vertex v of G.

Let G_1 and G_2 be snarks. The *dot product* $G_1 \cdot G_2$ is a snark (see [5]) and it is defined as follows:

1. take $G_1' = G_1 - \{ab, cd\}$, where ab, cd are two nonadjacent edges of G_1;
2. take $G_2' = G_2 - \{x, y\}$, where x, y are adjacent vertices in G_2;
3. let w, z and u, v be the other neighbors of x and y in G_2, respectively.

Then $G_1 \cdot G_2$ is the graph $G = (V, E)$ with $V = VG_1' \cup VG_2'$ and $E = EG_1' \cup EG_2' \cup \{aw, bz, cu, dv\}$.

Let $G = (V, E)$ be a graph. A pair of vertices (v, w) is *good in* G if there is a hamiltonian path with terminal vertices v, w.

Two pairs of vertices $\big((v, w), (x, y)\big)$ are *good in* G if there are two disjoint paths in G forming a spanning subgraph of G, with terminal ends v, w and x, y, respectively.

THEOREM 2.1. ([3]) *Let G be a hypohamiltonian snark having two independent edges $e = uv$, $f = xy$ for which*

1. *each of (u, x), (u, y), (v, x), (v, y), $\big((u, v), (x, y)\big)$ is good in G*

and

2. *for each vertex w, one of (u, v), (x, y) is good in $G - w$,*

and let H be a hypohamiltonian snark. Then $G \cdot H$ is a hypohamiltonian snark.

THEOREM 2.2. ([3]) *The flower snark* J_{2k+1} *is hypohamiltonian for each* $k \geq 2$.

Fiorini mentioned without proof that the two Blanuša snarks and the double-star snark D (cf. [9]) are hypohamiltonian, and that J_9 satisfies the conditions of Theorem 2.1. The following lemma verifies and extends the latter result. The proof is given in the appendix.

LEMMA 2.3. *The flower snarks* J_9, J_{11} *and* J_{13} *satisfy the conditions of Theorem* 2.1.

The proof of Theorem 2.5 uses the following result of the author.

THEOREM 2.4. ([8]) *Each hypohamiltonian snark is bicritical.*

The converse is not true. The Goldberg snarks [4] on 22 vertices are bicritical and they are not hypohamiltonian, cf. [3].

THEOREM 2.5. *There is a hypohamiltonian snark of order n*

(1) *for each* $n \in \{m: \ m \geq 64 \ \text{and} \ m \equiv 0 \bmod 8\}$,
(2) *for each* $n \in \{10, 18\} \cup \{m: \ m \geq 98 \ \text{and} \ m \equiv 2 \bmod 8\}$,
(3) *for each* $n \in \{m: \ m \geq 20 \ \text{and} \ m \equiv 4 \bmod 8\}$,
(4) *for each* $n \in \{30\} \cup \{m: \ m \geq 54 \ \text{and} \ m \equiv 6 \bmod 8\}$,
(5) *for each even* $n \geq 92$.

Proof. The flower snark J_{2k+1} is hypohamiltonian for $k \geq 2$. Thus there are hypohamiltonian snarks of orders $8k+4$ for all $k \geq 2$, and hence (3).

Graph J_9 satisfies the conditions of Theorem 2.1. Applying Theorem 2.1 m times we get that the iterated dot product $J_9 \cdot (\cdots (J_9 \cdot (J_9 \cdot J_{2k+1})))$ is a hypohamiltonian snark of order $n = 8k + 4 + 34m$, $k \geq 2$ and $m \geq 0$.

For $m = 1$, we obtain numbers $8n' + 6$ for all $n' \geq 6$, and the double-star snark D of order 30 is hypohamiltonian. Thus (4) holds true.

For $m = 2$, we obtain numbers $8n'$ for all $n' \geq 11$. By Theorem 2.1 and Lemma 2.3 the graphs $J_9 \cdot D$, $J_{11} \cdot D$ and $J_{13} \cdot D$ are hypohamiltonian and they have orders 64, 72, and 80, respectively. Hence (1) holds true.

For $m = 3$, we obtain numbers $8n'+2$ for all $n' \geq 15$. Again by Theorem 2.1 and Lemma 2.3 the graphs $J_9 \cdot (J_9 \cdot D)$, $J_9 \cdot (J_{11} \cdot D)$ and $J_{11} \cdot (J_{11} \cdot D)$, are hypohamiltonian and they have orders 98, 106, and 114, respectively. The Petersen graph and the two Blanuša snarks are hypohamiltonian, see [3], and henceforth (2) holds true.

Combining (1), (2), (3) and (4) we have that there exists a hypohamiltonian snark G_i of order $92 + 2i$ for each $0 \leq i \leq 16$. Thus, for each even $n \geq 126$, there is $0 \leq i \leq 16$ such that the iterated dot product $J_9 \cdot (\cdots (J_9 \cdot (J_9 \cdot G_i)))$ yields a hypohamiltonian snark of order n. Hence there is a bicritical snark of order n, for each even $n \geq 92$. □

COROLLARY 2.6. *There is an irreducible snark of order n*

(1) *for each* $n \in \{m:\ m \geq 64$ *and* $m \equiv 0 \bmod 8\}$,
(2) *for each* $n \in \{10, 18, 26\} \cup \{m:\ m \geq 98$ *and* $m \equiv 2 \bmod 8\}$,
(3) *for each* $n \in \{m:\ m \geq 20$ *and* $m \equiv 4 \bmod 8\}$,
(4) *for each* $n \in \{22, 30\} \cup \{m:\ m \geq 54$ *and* $m \equiv 6 \bmod 8\}$,
(5) *for each even* $n \geq 92$.

Proof. Theorem 1.1 allows to consider bicritical snarks instead of irreducible ones.

We already mentioned in the introduction that there are bicritical graphs of orders 22 and 26. Thus the statement follows from Theorems 2.4 and 2.5. □

Martin Škoviera [7] told me that he can solve Problem 1.2 completely by a different method.

Appendix

Proof of Lemma 2.3.

Let the vertices and edges be denoted as in the definition of J_{2k+1}, $k \geq 2$.

We show that edges b_0c_0 and b_4c_4 satisfy the conditions of Theorem 2.1 in each of J_9, J_{11} and J_{13}.

CLAIM 1. *Vertices* (c_0, c_4) *are good in* J_9, J_{11} *and* J_{13}.

Proof. The Hamilton-paths are for

J_9: $c_0, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, c_5, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_3, b_3, a_3, a_2, b_2, d_2, c_1, d_0, b_0, a_0, a_1, b_1, d_1, c_2, d_3, c_4$.

J_{11}: $c_0, d_{10}, c_9, b_9, a_9, a_{10}, b_{10}, c_{10}, d_9, c_8, d_7, b_7, a_7, a_8, b_8, d_8, c_7, d_6, c_5, b_5, d_5, c_6, b_6, a_6, a_5, a_4, b_4, d_4, c_3, b_3, a_3, a_2, b_2, d_2, c_1, d_0, b_0, a_0, a_1, b_1, d_1, c_2, d_3, c_4$.

J_{13}: $c_0, d_{12}, c_{11}, b_{11}, a_{11}, a_{12}, b_{12}, c_{12}, d_{11}, c_{10}, d_9, b_9, a_9, a_{10}, b_{10}, d_{10}, c_9, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, c_5, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_3, b_3, a_3, a_2, b_2, d_2, c_1, d_0, b_0, a_0, a_1, b_1, d_1, c_2, d_3, c_4$.

□

CLAIM 2. *Vertices* (c_0, b_4) *are good in* J_9, J_{11} *and* J_{13}.

Proof. The Hamilton-paths are for

J_9: $c_0, d_8, b_8, c_8, d_7, b_7, c_7, d_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, c_5, b_5, a_5, a_6, a_7, a_8, a_0, b_0, d_0, c_1, d_2, b_2, c_2, d_1, b_1, a_1, a_2, a_3, a_4, b_4$.

J_{11}: $c_0, d_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_8, b_8, c_8, d_7, b_7, c_7, d_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, c_5, b_5, a_5, a_6, a_7, a_8, a_9, a_{10}, a_0, b_0, d_0, c_1, d_2, b_2, c_2, d_1, b_1, a_1, a_2, a_3, a_4, b_4$.

J_{13}: $c_0, d_{12}, b_{12}, c_{12}, d_{11}, b_{11}, c_{11}, d_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_8, b_8, c_8, d_7, b_7, c_7, d_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, c_5, b_5, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_0, b_0, d_0, c_1, d_2, b_2, c_2, d_1, b_1, a_1, a_2, a_3, a_4, b_4$.

□

CLAIM 3. *Vertices (b_0, c_4) are good in J_9, J_{11} and J_{13}.*

Proof. The Hamilton-paths are for

J_9: $b_0, a_0, a_1, b_1, d_1, c_0, d_8, b_8, a_8, a_7, b_7, c_7, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_5, b_5, d_5, c_6, d_7, a_8, d_0, c_1, d_2, c_3, b_3, a_3, a_2, b_2, c_2, d_3, c_4$.

J_{11}: $b_0, a_0, a_1, b_1, d_1, c_0, d_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, c_7, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_5, b_5, d_5, c_6, d_7, c_8, b_8, d_8, c_9, b_9, d_9, c_{10}, d_0, c_1, d_2, c_3, b_3, a_3, a_2, b_2, c_2, d_3, c_4$.

J_{13}: $b_0, a_0, a_1, b_1, d_1, c_0, d_{12}, b_{12}, a_{12}, a_{11}, a_{10}, a_9, a_8, a_7, b_7, c_7, d_6, b_6, a_6, a_5, a_4, b_4, d_4, c_5, b_5, d_5, c_6, d_7, c_8, b_8, d_8, c_9, b_9, d_9, c_{10}, b_{10}, d_{10}, c_{11}, b_{11}, d_{11}, c_{12}, d_0, c_1, d_2, c_3, b_3, a_3, a_2, b_2, c_2, d_3, c_4$.

□

CLAIM 4. *Vertices (b_0, b_4) are good in J_9, J_{11} and J_{13}.*

Proof. The Hamilton-paths are for

J_9: $b_0, d_0, c_8, b_8, d_8, c_0, d_1, c_2, b_2, a_2, a_3, b_3, d_3, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_0, a_1, b_1, c_1, d_2, c_3, d_4, c_5, b_5, a_5, a_4, b_4$.

J_{11}: $b_0, d_0, c_{10}, a_{10}, d_{10}, c_0, d_1, c_2, b_2, a_2, a_3, b_3, d_3, c_4, d_5, c_6, d_7, b_7, a_7, a_6, b_6, d_6, c_7, d_8, c_9, b_9, d_9, c_8, b_8, a_8, a_9, a_{10}, a_0, a_1, b_1, c_1, d_2, c_3, d_4, c_5, b_5, a_5, a_4, b_4$.

J_{13}: $b_0, d_0, c_{12}, a_{12}, d_{12}, c_0, d_1, c_2, b_2, a_2, a_3, b_3, d_3, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, c_9, d_8, b_8, c_8, d_9, c_{10}, d_{11}, b_{11}, c_{11}, d_{10}, b_{10}, a_{10}, a_{11}, a_{12}, a_0, a_1, b_1, c_1, d_2, c_3, d_4, c_5, b_5, a_5, a_4, b_4$.

□

CLAIM 5. *The two pairs of vertices $((b_0, c_0), (b_4, c_4))$ are good in J_9, J_{11} and J_{13}.*

Proof. The cycles are for

J_9: $c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, b_3, a_3, a_4, a_5, b_5, c_5, d_6, b_6, a_6, a_7, b_7, c_7, d_8, b_8, a_8, a_0, b_0$
and
$c_4, d_5, c_6, d_7, c_8, d_0, c_1, d_2, c_3, d_4, b_4$.

J_{11}: $c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, b_3, a_3, a_4, a_5, b_5, c_5, d_6, b_6, a_6, a_7, a_8, a_9, b_9, c_9, d_{10}, b_{10}, a_{10}, a_0, b_0$
and
$c_4, d_5, c_6, d_7, b_7, c_7, d_8, b_8, c_8, d_9, c_{10}, d_0, c_1, d_2, c_3, d_4, b_4$.

J_{13}: $c_0, d_1, b_1, a_1, a_2, b_2, c_2, d_3, b_3, a_3, a_4, a_5, b_5, c_5, d_6, b_6, a_6, a_7, a_8, a_9, a_{10}, a_{11}, b_{11}, c_{11}, d_{12}, b_{12}, a_{12}, a_0, b_0$
and
$c_4, d_5, c_6, d_7, b_7, c_7, d_8, b_8, c_8, d_9, b_9, c_9, d_{10}, b_{10}, c_{10}, d_{11}, c_{12}, d_0, c_1, d_2, c_3, d_4, b_4$.

□

Claim 6. *For each $v \in VJ_9$ one of (b_0, c_0), (b_4, c_4) is good in $J_9 - v$.*

Proof. A Hamilton-cycle H in $J_9 - a_0$ is:

$c_0, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, a_5, a_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, a_1, a_2, b_2, d_2, c_1, d_0, b_0.$

This cycle contains edges $b_i c_i$ for $i \in \{0,3,4,7,8\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0, c_0) is good in $J_9 - a_j$ for all $j \in \{0,1,2,5,6\}$, and that (b_4, c_4) is good in $J_9 - a_4$.

The following cycle is a Hamilton-cycle in $J_9 - a_8$:

$c_0, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2, a_3, a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7, b_7, a_7, a_6, b_6, d_6, c_7, d_8, b_8, c_8, d_0, c_1, b_1, a_1, a_0, b_0.$

This cycle contains edges $b_0 c_0$, $b_1 c_1$ and $b_5 c_5$. Shifting the indices of the vertices of this cycle from i to $i+j$ yields that (b_0, c_0) is good in $J_9 - a_{j+8}$ for $j \in \{0,4,8\}$.

The following cycle is a Hamilton-cycle in $J_9 - b_1$:

$c_0, d_1, c_2, b_2, d_2, c_1, d_0, c_8, d_7, b_7, a_7, a_8, b_8, d_8, c_7, d_6, c_5, b_5, a_5, a_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, b_4, a_4, a_3, a_2, a_1, a_0, b_0.$

This cycle contains edges $b_i c_i$ for $i \in \{0,2,3,5,6\}$. Thus shifting the indices of the vertices of H from i to $i+j$ yields that (b_0, c_0) is good in $J_9 - b_{j+1}$ for $j \in \{0,3,4,6,7\}$, and that (b_4, c_4) is good in $J_9 - b_{j+1}$ for $j \in \{1,2,8\}$.

The following cycle shows that (b_4, c_4) is good in $J_9 - b_6$:

$c_4, d_3, b_3, a_3, a_4, a_5, a_6, a_7, b_7, c_7, d_6, c_5, b_5, d_5, c_6, d_7, c_8, d_0, b_0, a_0, a_8, b_8, d_8, c_0, d_1, c_2, b_2, a_2, a_1, b_1, c_1, d_2, c_3, d_4, b_4.$

The following cycle is a Hamilton-cycle in $J_9 - c_0$:

$c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0, d_0, c_8, d_7, b_7, a_7, a_8, b_8, d_8, c_7, d_6, c_5, b_5, d_5, c_6, b_6, a_6, a_5, a_4, a_3, b_3, c_3, d_4, b_4.$

This cycle contains edges $b_i c_i$ for $i \in \{1,3,4,5,6,\}$. Thus shifting the indices of the vertices of H from i to $i+j$ yields that (b_4, c_4) is good in $J_9 - c_j$ for $j \in \{0,1,3,7,8\}$, and that (b_0, c_0) is good in $J_9 - c_j$ for $j \in \{4,5,6\}$.

The following cycle shows that (b_4, c_4) is good in $J_9 - c_2$:

$c_4, d_3, b_3, a_3, a_4, a_5, a_6, b_6, d_6, c_5, b_5, d_5, c_6, d_7, c_8, b_8, a_8, a_7, b_7, c_7, d_8, c_0, d_1, b_1, c_1, d_0, b_0, a_0, a_1, a_2, b_2, d_2, c_3, d_4, b_4.$

The following cycle is a Hamilton-cycle in $J_9 - d_0$:

$c_0, d_8, c_7, b_7, d_7, c_8, b_8, a_8, a_7, a_6, a_5, b_5, d_5, c_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0.$

This cycle contains edges $b_i c_i$ for $i \in \{0,1,3,4,6,7,8\}$. Thus shifting the indices of the vertices of H from i to $i+j$ yields that (b_0, c_0) is good in $J_9 - d_j$ for $j \in \{0,1,2,3,5,6,8\}$, and that (b_4, c_4) is good in $J_9 - d_j$ for $j \in \{4,7\}$. □

Claim 7. *For each $v \in VJ_{11}$ one of (b_0, c_0), (b_4, c_4) is good in $J_{11} - v$ for all v.*

Proof. A Hamilton-cycle H in $J_{11} - a_0$ is:

$c_0, d_{10}, c_9, b_9, d_9, c_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, c_7, d_8, b_8, c_8, d_7, c_6, d_5, b_5, a_5, a_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, a_1, a_2, b_2, d_2, c_1, d_0, b_0.$

This cycle contains edges b_ic_i for $i \in \{0,3,4,7,8,9,10\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{11}-a_j$ for all $j \in \{0,1,2,3,4,7,8\}$, and that (b_4,c_4) is good in $J_{11}-a_j$ for all $j \in \{5,6\}$.

The following cycle is a Hamilton-cycle H in $J_{11}-a_{10}$:

c_0, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2, a_3, a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, d_9, c_8, b_8, d_8, c_9, d_{10}, b_{10}, c_{10}, d_0, c_1, b_1, a_1, a_0, b_0.

This cycle contains edge b_1c_1. Shifting the indices of the vertices from i to $i+10$ yields that (b_0,c_0) is good in $J_{11}-a_9$.

The following cycle is a Hamilton-cycle in $J_{11}-b_1$:

c_0, d_1, c_2, b_2, d_2, c_1, d_0, c_{10}, d_9, b_9, c_9, d_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, d_7, c_8, b_8, d_8, c_7, d_6, c_5, b_5, a_5, a_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, b_4, a_4, a_3, a_2, a_1, a_0, b_0.

This cycle contains edges b_ic_i for $i \in \{0,2,3,5,6,8,9\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{11}-b_{j+1}$ for all $j \in \{0,2,3,5,6,8,9\}$, and that (b_4,c_4) is good in $J_{11}-b_{j+1}$ for all $j \in \{1,4,6,7,10\}$.

The following cycle is a Hamilton-cycle in $J_{11}-c_0$:

c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0, d_0, c_{10}, d_9, b_9, a_9, a_{10}, b_{10}, d_{10}, c_9, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, c_5, d_6, b_6, a_6, a_5, a_5, a_4, a_3, b_3, c_3, d_4, b_4.

This cycle contains edges b_ic_i for $i \in \{0,1,3,4,5,7,8\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{11}-c_j$ for all $j \in \{0,3,4,6,7,8,10\}$, and that (b_4,c_4) is good in $J_{11}-c_1$.

The following cycle is a Hamilton-cycle in $J_{11}-c_2$:

c_4, d_3, b_3, a_3, a_4, a_5, a_6, b_6, d_6, c_5, b_5, d_5, c_6, d_7, c_8, b_8, d_8, c_7, b_7, a_7, a_8, a_9, a_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_{10}, c_0, d_1, b_1, c_1, d_0, b_0, a_0, a_1, a_2, b_2, d_2, c_3, d_4, b_4.

This cycle contains edges b_1c_1, and b_4c_4, shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_4,c_4) is good in $J_{11}-c_{j+2}$ for all $j \in \{0,3\}$, and that (b_0,c_0) is good $J_{11}-c_9$.

The following cycle is a Hamilton-cycle in $J_{11}-d_0$:

c_0, d_{10}, c_9, b_9, a_9, a_{10}, b_{10}, c_{10}, d_9, c_8, d_7, b_7, c_7, d_8, b_8, a_8, a_7, a_6, a_5, b_5, d_5, c_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0.

This cycle contains edges b_ic_i for $i \in \{0,1,3,4,6,7,9,10\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{11}-d_j$ for all $j \in \{0,1,2,4,5,7,8,10\}$, and that (b_4,c_4) is good in $J_{11}-d_j$ for all $j \in \{3,6,9\}$. □

CLAIM 8. *For each $v \in VJ_{13}$ one of (b_0,c_0), (b_4,c_4) is good in $J_{13}-v$ for all v.*

P r o o f. A Hamilton-cycle H in $J_{13}-a_0$ is:

c_0, d_{12}, c_{11}, b_{11}, d_{11}, c_{12}, b_{12}, a_{12}, a_{11}, a_{10}, a_9, b_9, d_9, c_{10}, b_{10}, d_{10}, c_9, d_8, c_7, b_7, a_7, a_8, b_8, c_8, d_7, c_6, d_5, b_5, a_5, a_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, a_1, a_2, b_2, d_2, c_1, d_0, b_0.

This cycle contains edges b_ic_i for $i \in \{0,3,4,7,8,10,11,12\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-a_j$ for all $j \in \{0,1,2,3,5,6,9,10\}$, and that (b_4,c_4) is good in $J_{13}-a_j$ for all $j \in \{4,7\}$.

The following cycle is a Hamilton-cycle in $J_{13}-a_0$:

c_0, d_1, c_2, b_2, a_2, a_1, b_1, c_1, d_2, c_3, d_4, b_4, c_4, d_3, b_3, a_3, a_4, a_5, a_6, b_6, c_6, d_5, b_5, c_5, d_6, c_7, d_8, b_8, c_8, d_7, b_7, a_7, a_8, a_9, a_{10}, b_{10}, c_{10}, d_9, b_9, c_9, d_{10}, c_{11}, d_{12}, b_{12}, a_{12}, a_{11}, b_{11}, d_{11}, c_{12}, d_0, b_0.

This cycle contains edges b_1c_1, b_2c_2 and b_5c_5. Thus shifting the indices of the vertices of the cycle from i to $i+8$, $i+11$ and $i+12$ yields that (b_0,c_0) is good in $J_{13}-a_8$, $J_{13}-a_{11}$ and $J_{13}-a_{12}$, respectively.

The following cycle is a Hamilton-cycle in $J_{13}-b_1$:

c_0, d_1, c_2, b_2, d_2, c_1, d_0, c_{12}, d_{11}, b_{11}, a_{11}, a_{12}, b_{12}, d_{12}, c_{11}, d_{10}, c_9, b_9, d_9, c_{10}, b_{10}, a_{10}, a_9, a_8, a_7, b_7, d_7, c_8, b_8, d_8, c_7, d_6, c_5, b_5, a_5, a_6, b_6, c_6, d_5, c_4, d_3, b_3, c_3, d_4, b_4, a_4, a_3, a_2, a_1, a_0, b_0.

This cycle contains edges b_ic_i for $i \in \{0,2,3,5,6,8,9,10\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-b_{j+1}$ for all $j \in \{0,3,4,5,7,8,10,11\}$, and that (b_4,c_4) is good in $J_{13}-b_{j+1}$ for all $j \in \{1,2,9,12\}$.

The following Hamilton-cycle shows that (b_4,c_4) is good in $J_{13}-b_7$:

b_4, a_4, a_5, b_5, c_5, d_4, c_3, d_2, b_2, c_2, d_3, b_3, a_3, a_2, a_1, a_0, b_0, d_0, c_1, b_1, d_1, c_0, d_{12}, c_{11}, b_{11}, a_{11}, a_{12}, b_{12}, c_{12}, d_{11}, c_{10}, d_9, b_9, a_9, a_{10}, b_{10}, d_{10}, c_9, d_8, c_7, d_6, b_6, a_6, a_7, a_8, b_8, c_8, d_7, c_6, d_5, c_4.

The following cycle is a Hamilton-cycle in $J_{13}-c_1$:

c_0, d_{12}, b_{12}, a_{12}, a_0, a_1, b_1, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2, a_3, a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, d_9, c_8, b_8, d_8, c_9, d_{10}, c_{11}, b_{11}, a_{11}, a_{10}, b_{10}, c_{10}, d_{11}, c_{12}, d_0, b_0.

This cycle contains edges b_ic_i for $i \in \{0,3,4,5,7,8,10,11\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-c_{j+1}$ for all $j \in \{0,2,3,5,6,8,9,10\}$, and that (b_4,c_4) is good in $J_{13}-c_{j+1}$ for all $j \in \{1,4,7,12\}$.

The following cycle shows that (b_0,c_0) is good in $J_{13}-c_{12}$:

b_0, d_1, c_2, d_3, b_3, c_3, d_2, b_2, a_2, a_3, a_4, a_5, b_5, c_5, d_4, b_4, c_4, d_5, c_6, d_7, b_7, c_7, d_6, b_6, a_6, a_7, a_8, a_9, b_9, c_9, d_8, b_8, c_8, d_9, c_{10}, d_{11}, b_{11}, a_{11}, a_{10}, b_{10}, d_{10}, c_{11}, d_{12}, b_{12}, a_{12}, a_0, a_1, b_1, c_1, d_0, b_0.

The following cycle is a Hamilton-cycle in $J_{13}-d_0$:

c_0, d_{12}, c_{11}, b_{11}, d_{11}, c_{12}, b_{12}, a_{12}, a_{11}, a_{10}, a_9, b_9, c_9, d_{10}, b_{10}, c_{10}, d_9, c_8, d_7, b_7, c_7, d_8, b_8, a_8, a_7, a_6, a_5, b_5, d_5, c_6, b_6, d_6, c_5, d_4, c_3, b_3, a_3, a_4, b_4, c_4, d_3, c_2, d_1, b_1, c_1, d_2, b_2, a_2, a_1, a_0, b_0.

The cycle contains edges b_ic_i for $i \in \{0,1,3,4,6,7,9,10,11,12\}$. Thus shifting the indices of the vertices of the cycle from i to $i+j$ yields that (b_0,c_0) is good in $J_{13}-d_j$ for all $j \in \{0,1,2,3,4,6,7,9,10,12\}$, and that (b_4,c_4) is good in $J_{13}-d_j$ for all $j \in \{5,8,11\}$. □

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