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ON CHAINS IN MV-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be an MV-algebra. Further, let $L = \ell(\mathcal{A})$ be the lattice corresponding to $\mathcal{A}$. In the present paper we deal with maximal convex chains in L containing the zero element of L. Next, we investigate maximal chains in intervals of the lattice L.

Introduction

The motivation for introducing the notion of MV-algebra was to construct an algebraic basis for the Łukasziewicz theory of multivalued logics; cf. [1], [2], [4]. MV-algebras are called also Wajsberg algebras (cf. [5], [15]).

For MV-algebras we use the notation as in [5] and [10]. Thus an MV-algebra is a system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$, where A is a nonempty set, $\oplus$, $*$ are binary operations, $\neg$ is a unary operation and 0, 1 are nulary operations on A such that the identities (m_1)–(m_9) from [5] are satisfied.

If no misunderstanding can occur, then we write often A instead of $\mathcal{A}$. Direct product decompositions of MV-algebras have been investigated in [3], [10], [11], [12]. By means of the basic operations mentioned above, there were defined binary operations $\vee$ and $\wedge$ on A under which A turns out to be a distributive lattice with the least element 0 and with the greatest element 1; we denote this lattice by $\ell(A)$.

Let $\mathcal{C}$ be the system of all convex chains X in A such that $0 \in X$ and $\operatorname{card} X > 1$. The system $\mathcal{C}$ is partially ordered by inclusion. Next let $\mathcal{C}_m$ be the system of all maximal elements of $\mathcal{C}$.

In Section 2 of the present paper we deal with the relations between elements of $\mathcal{C}_m$ and direct product decompositions of $\mathcal{A}$.

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Analogous questions for lattice ordered groups have been investigated in [9].

The generalized Jordan-Dedekind condition (briefly: condition (JD)) for a lattice L requires that whenever $u, v \in L$, $u < v$ and C_1, C_2 are maximal chains in the interval $[u,v]$ of L, then C_1 and C_2 have the same cardinality. This condition has been investigated by S z á s z [14] and the author [7], [7a], [8].

In Section 3 we generalize some results of [14] and [6] concerning maximal chains in a distributive lattice L. These results can be applied to the case when $L = \ell(A)$. If A is a finite MV-algebra, then the lattice $\ell(A)$ satisfies condition (JD). We show that there exist infinite MV-algebras for which this condition is rather strongly violated. One of the results of Section 3 is as follows:

(A) For each cardinal $\alpha > \aleph_0$ there exists an MV-algebra A_α having elements u, v with $u < v$ such that
- (i) for each cardinal β with $\aleph_0 \leqq \beta \leqq \alpha$ there exists a maximal chain C_β in $[u,v]$ whose cardinality is β;
- (ii) the lattice $\ell(\mathcal{A})$ is completely distributive;
- (iii) no element $x \in A$ with $x \neq 0$ is boolean.

1. Preliminaries

Let $\mathcal{A}$ be an MV-algebra. For each $x, y \in A$ we put (see [1])

$$x \vee y = (x * \neg y) \oplus y\,, \qquad x \wedge y = \neg(\neg x \vee \neg y)\,.$$

Then $\ell(A) = (A; \vee, \wedge)$ is a lattice with the least element 0 and the greatest element 1. (Cf. [1].) Further, the lattice $\ell(A)$ is distributive (see [6]). We consider the partial order $\leqq$ on A which is defined in [1] by means of operations $\vee$ and $\wedge$ on A.

For $a, b \in A$ with $a \leqq b$ let $[a,b]$ be the interval in A with the endpoints a and b. A subset S of A is called *convex* if, whenever $a, b \in S$ and $a \leqq b$, then $[a,b] \subseteq S$. In what follows, $\mathcal{C}$ and $\mathcal{C}_m$ are as above. We suppose that $A \neq \{0\}$.

The notion of direct product of MV-algebras is defined in the usual way. For the definition of internal direct product decomposition of $\mathcal{A}$ and internal direct factor of $\mathcal{A}$ cf. [10]. To each direct product decomposition of $\mathcal{A}$ there corresponds in a natural way an internal direct product decomposition of $\mathcal{A}$. In the present paper we consider only internal direct product decompositions and internal direct factors of $\mathcal{A}$, therefore the word "internal" will be omitted.

Each MV-algebra $\mathcal{A}$ can be represented by means of an appropriate abelian lattice ordered group G with a strong unit u (cf. [13], or [10; 1.3, 1.4]); in this connection we shall use the notation from [10]; a different notation has been used in [4].

An element $x \in A$ is called *boolean* if the interval $[0,x]$ of $\ell(\mathcal{A})$ is a Boolean algebra.

2. Direct product decompositions

Let $\mathcal{A}$ be an MV-algebra and let G be as in Section 1 (i.e., $\mathcal{A} = \mathcal{A}_0(G,u)$).

2.0. LEMMA. ([9; Lemma 8]) *Let R be a maximal convex chain in a lattice ordered group H, $0 \in H$. Then R is a subgroup of the group H.*

2.1. PROPOSITION. *Let $Y \in \mathcal{C}_m$. Then Y is closed with respect to the operation $\oplus$.*

Proof. Since A is a convex subset of G we infer that Y is a convex chain in G. From Axiom of Choice we obtain that there exists a maximal convex chain Z in G such that $Y \subseteq Z$. According to 2.0, Z is closed with respect to the group operation $+$ of G. Let $y_1, y_2 \in Y$, $y_1 + y_2 = z$. Thus $z \geqq 0$. Since Z is convex and $0 \in Z$, we get $z \wedge u \in Z$. Moreover, $z \wedge u \in A$, and clearly $Z \cap A = Y$, whence $z \wedge u \in Y$. We have $y_1 \oplus y_2 = (y_1 + y_2) \wedge u$, therefore $y_1 \oplus y_2 \in Y$. □

2.2. LEMMA. *Let $Y \in \mathcal{C}_m$ and suppose that Y has a greatest element. Let Z be as in the proof of* 2.1. *Then Z is not bounded in G.*

Proof. Let y^0 be the greatest element of Y. Then $0 < y^0$ and hence $2y^0 > y^0$. Put $z = 2y^0$. Since Z is an ℓ-subgroup of G, we have $z \in Z$. By way of contradiction, suppose that Z is bounded in G. Hence there exists a positive integer n such that $z_1 \leqq nu$ for each $z_1 \in Z$. Put $z_1 = nz$; then $z_1 \in Z$. From $nz \leqq nu$ we obtain $0 \leqq n(u - z)$, whence $0 \leqq u - z$. This yields that $z \in A$, whence $z \in Y$, which is a contradiction. □

2.3. LEMMA. *Let Y and Z be as in* 2.2. *Then Z is a direct factor of G.*

Proof. This is a consequence of 2.2 and [9; Theorem 1]. □

Under the assumptions as in 2.2, let us denote by Z' the convex ℓ-subgroup of Z generated by the greatest element y^0 of Y. Hence y^0 is a strong unit of Z'. Thus we can construct the MV-algebra $\mathcal{A}_0(Z', y^0)$; the underlying set of this MV-algebra is Y.

2.4. THEOREM. *Let $Y \in \mathcal{C}_m$. Then the following conditions are equivalent:*

(i) *Y is a direct factor of $\mathcal{A}$.*
(ii) *Y has a greatest element.*

Proof.

a) Let (i) be valid. Since $\mathcal{A}$ has a greatest element, the same is valid for Y.

b) Let (ii) hold and let Z be as in 2.3. According to 2.3, there is a direct product decomposition

$$G = Z \times G_1 . \tag{1}$$

Thus in view of [10; Lemma 3.2], we have a direct product decomposition

$$\mathcal{A} = (Z \cap A) \times (G_1 \cap A) . \tag{1'}$$

Since $Z \cap A = Y$, we obtain that Y is a direct factor of $\mathcal{A}$. □

Let us remark that if Y is as in 2.2, moreover, we have, in fact, the operation $\oplus$ in Y which is inherited from $\mathcal{A}$ (cf. 2.1), and moreover, we have the corresponding operation (let us denote it by $\oplus_1$), which is due to the fact that $Y = \mathcal{A}_0(Z', y^0)$.

2.5. PROPOSITION. *Let Y be as in 2.2. Then the operations $\oplus$ and $\oplus_1$ on Y coincide.*

Proof. Consider the direct product decomposition (1'). For each $a \in A$ let a_1 and a_2 be the component of a in Y and in $G_1 \cap A$, respectively. Then a_1 or a_2 is, at the same time, the component of a in Z or in G_1, respectively (with regard to (1)). Since the operations $\vee$ and $\wedge$ are expressed by the basic operation of the MV-algebra $\mathcal{A}$, the relation (1') can be taken also with respect to $\ell(\mathcal{A})$. Let u be as above (i.e., u is the strong unit of G, and hence it is the greatest element of $\ell(\mathcal{A})$). Hence u_1 must be equal to y^0 and u_2 is the greatest element of $G_1 \cap A$; moreover, $y^0 \wedge u_2 = 0$. If $y \in Y$, then $y_1 = y$ and $y_2 = 0$. Thus for y and y' in Y we have

$$\begin{aligned} y \oplus y' &= (y + y') \wedge u = (y + y') \wedge (y^0 + u_2) = (y + y') \wedge (y^0 \vee u_2) \\ &= \big((y + y') \wedge y_0\big) \vee \big((y + y') \wedge u_2\big) = (y + y') \wedge y_0 = y \oplus_1 y' . \end{aligned}$$

□

For the definition of the archimedean property in MV-algebras, see [12].

2.6. THEOREM. *If $\mathcal{A}$ is archimedean, then each element of $\mathcal{C}_m$ is a direct factor of $\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ is archimedean and let $Y \in \mathcal{C}_m$. Let Z be as above. According to [12], the lattice ordered group G is archimedean as well. Hence, in view of [9; Theorem 1'], Z is a direct factor of G. Now, by applying the same method as in the proof of 2.4 we obtain that Y is a direct factor of $\mathcal{A}$. □

2.7. Lemma. *Let Y and Y' be distinct elements of $\mathcal{C}_m$. Then $Y \cap Y' = \{0\}$.*

Proof. Let Z be as above and let Z' be defined analogously as Z (with Y replaced by Y'). In view of [5; Lemma 6] we have $Z \cap Z' = \{0\}$. Therefore $Y \cap Y' = \{0\}$. □

Let $T \subseteq A$, $t > 0$ for each $t \in T$. The set T will be called *disjoint* if $t_1 \wedge t_2 = 0$ whenever t_1 and t_2 are distinct elements of T.

2.8. THEOREM. *Let $\mathcal{C}_m = \{Y_i\}_{i\in I}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a direct product of linearly ordered MV-algebras.*

(ii) *$\mathcal{A}$ is a direct product $\prod\limits_{i\in I} Y_i$.*

(iii)

a) *Each Y_i has a greatest element;*

b) *if $\{y^i\}_{i\in I} \subseteq A$ with $y^i \in Y_i$ for each $i \in I$, then $\bigvee\limits_{i\in I} y^i$ does exist in A;*

c) *if, moreover, $0 < y^i$ for each $i \in I$, then $\{y^i\}_{i\in I}$ is a maximal disjoint set in A.*

Proof.

a$_1$) Suppose that (i) holds. Hence there exists a system $S = \{T_j\}_{j\in J}$ of linearly ordered MV-algebras T_j such that $\mathcal{A}$ is a direct product of linearly ordered MV-algebras T_j ($j \in J$). Without loss of generality we can suppose that $T_j \neq \{0\}$ for each $j \in J$. It is obvious that each T_j belongs to $\mathcal{C}_m$. Thus $S \subseteq \mathcal{C}_m$. We want to verify that $\mathcal{C}_m \subseteq S$. By way of contradiction, suppose that there exists $Y \in \mathcal{C}_m$ such that $Y \neq T_j$ for each $j \in J$. There exists $y \in Y$ with $y > 0$. For each $j \in J$ let y_j be the component of y in T_j. Hence there exists $j \in J$ such that $y_j > 0$. We have $y_j \in T_j$ and, moreover, $y_j \in [0, y]$ whence $y_j \in Y$. In view of 2.7 we arrived at a contradiction. Hence (ii) holds.

a$_2$) Let (ii) be valid. Since A has a greatest element, each Y_i must have a greatest element. For each $i \in I$ let $y^i \in Y_i$. By way of contradiction, suppose that $\{y^i\}_{i\in I}$ fails to be a maximal disjoint subset of A. Thus there exists $a \in A$ with $a > 0$ such that $a \wedge y_i = 0$ for each $i \in I$. There exists $i \in I$ such that $a_i > 0$, where a_i is the component of a in Y_i. Then, since Y_i is linearly ordered, we have $a_i \wedge y_i > 0$, which is a contradiction. Next, from (ii) we infer that there exists $y \in A$ such that for each $i \in I$, y^i is the component of y in Y_i. Hence we obtain $y = \bigvee\limits_{i\in I} y^i$.

a$_3$) Let (iii) hold. If $i \in I$, then according to 2.4, Y_i is a direct factor of $\mathcal{A}$. For $a \in A$ we denote by a_i the component of a in Y_i. Consider the mapping $\varphi\colon A \longrightarrow \prod\limits_{i\in I} Y_i$ such that $\varphi(a) = (a_i)_{i\in I}$ for each $a \in A$. If $i(1) \in I$ and $y^i \in Y_i$

for each $i \in I$, $y^i = 0$ whenever $i \neq i(1)$, then $(y^i)_{i\in I}$ will be identified with $y^{i(1)}$. The mapping φ is a homomorphism of $\mathcal{A}$ into $\prod_{i\in I} Y_i$. Let $(y^i)_{i\in I} \in \prod_{i\in I} Y_i$. In view of the assumption, there exists $y = \bigvee_{i\in I} y^i$ in A. It is easy to verify that $y_i = y^i$ for each $i \in I$, hence $\varphi(y) = (y^i)_{i\in I}$. Therefore φ is a surjection.

It remains to verify that φ is a monomorphism. By way of contradiction, suppose that there are distinct elements a and a' in A such that $\varphi(a) = \varphi(a')$. Put $a \wedge a' = u_0$, $a \vee a' = v$. Hence $u_0 < v$ and $\varphi(u_0) = \varphi(v)$. There exists $t \in A$ such that $u_0 \oplus t = v$. Thus $t > 0$. Choose, for each $i \in I$, a strictly positive element y^i in Y_i. Then $\{y^i\}_{i\in I}$ is a maximal disjoint subset of A. Hence there exists $i(1) \in I$ such that $t \wedge y^{i(1)} > 0$. We have $t_{i(1)} \geqq t \wedge y^{i(1)}$, whence $t_{i(1)} > 0$. On the other hand, the relation $\varphi(u_0) = \varphi(v)$ yields that $\varphi(t) = 0$ and so we arrived at a contradiction. □

3. Maximal chains

In this section we deal with maximal chains in an interval $[u, v]$ of a distributive lattice L with applications to the case when $L = \ell(\mathcal{A})$, where $\mathcal{A}$ is an MV-algebra; we also obtain some results of [14] and [6] as corollaries.

For a lattice L we denote by $\mathcal{C}^0(L)$ the system of all chains in L; this system is partially ordered by the set-theoretical inclusion. Let $\mathcal{C}^0_m(L)$ be the set of all maximal elements of $\mathcal{C}^0(L)$. The elements of $\mathcal{C}^0_m(L)$ are called *maximal chains* in L.

A linearly ordered set X is called *dense* if for each $x_1, x_2 \in X$ with $x_1 < x_2$ there exists $x_3 \in X$ such that $x_1 < x_3 < x_2$.

A sublattice L_1 of a lattice L is said to be *strongly dense* in L if, whenever $a, b \in L$ and $a < b$, then either both a and b belong to L_1 or there exists $x \in L_1$ with $a < x < b$.

Let condition (JD) be as above. It is well known that each finite modular lattice satisfies condition (JD). If $\mathcal{A}$ is an MV-algebra, then the lattice $\ell(\mathcal{A})$ is distributive. Hence, if $\mathcal{A}$ is finite, then the lattice $\ell(\mathcal{A})$ satisfies condition (JD).

Let φ be a mapping of a linearly ordered set L_1 into a linearly ordered set L_2 such that for each $x, y \in L_1$ the relation

$$x \leqq y \iff \varphi(x) \leqq \varphi(y)$$

is valid. Then we say that φ is an *isomorphism* of L_1 into L_2.

3.1. Lemma. *Let L be a distributive lattice and let a, b be elements of L which are incomparable, $a \wedge b = u$, $a \vee b = v$. Suppose that*

(i) $C_1 \in \mathcal{C}^0_m([u, a])$, $C_2 \in \mathcal{C}^0_m([u, b])$;

(ii) *the chain* C_1 *is dense;*

(iii) *there exists an isomorphism* φ *of* C_1 *into* C_2 *such that* $\varphi(C_1)$ *is a strongly dense sublattice of* C_2 *and* $u, b \in \varphi(C_1)$.

Then the set $\{x \vee \varphi(x) :\ x \in C_1\}$ *is an element of* $\mathcal{C}_m^0([u,v])$.

Proof. Denote

$$R = \{x \vee \varphi(x) :\ x \in C_1\}.$$

It is obvious that R is a chain in the lattice $[u,v]$. By way of contradiction, assume that R fails to be an element of $\mathcal{C}_m^0([u,v])$. Hence there exists $z \in [u,v]$ such that z is comparable with each element of R and $z \notin R$.

We put

$$R_1 = \{r \in R :\ r < z\}, \qquad R_2 = \{r \in R :\ r > z\}.$$

We have $u, v \in R$, whence $R_1 \neq \emptyset \neq R_2$.

Let $x \in C_1$. Then $a \wedge \varphi(x) = u$, thus

$$a \wedge (x \vee \varphi(x)) = (a \wedge x) \vee (a \wedge \varphi(x)) = a \wedge x = x. \tag{1}$$

From (1) we conclude that if $x \vee \varphi(x) \in R_1$, then $x \leqq a \wedge z$. Analogously, if $x \vee \varphi(x) \in R_2$, then $a \wedge z \leqq x$. Hence $a \wedge z$ is comparable with all elements of C_1. Therefore $a \wedge z$ belongs to C_1. We denote $x_1 = a \wedge z$.

a) We have either

$$x_1 \vee \varphi(x_1) < z, \tag{a$_1$}$$

or

$$x_1 \vee \varphi(x_1) > z. \tag{a$_2$}$$

First assume that (a$_1$) holds. Let $x_2 \in C_1$, $x_2 \vee \varphi(x_2) \in R_1$. Thus

$$x_2 = a \wedge (x_2 \vee \varphi(x_2)) \leqq a \wedge z = x_1,$$

whence $\varphi(x_2) \leqq \varphi(x_1)$ and then $x_2 \vee \varphi(x_2) \leqq x_1 \vee \varphi(x_1)$. Therefore $x_1 \vee \varphi(x_1)$ is the greatest element of R_1.

Analogously we verify: the relation (a$_2$) implies that $x_1 \vee \varphi(x_1)$ is the least element of R_2.

b) Assume that there exists $x_2 \in C_1$ such that $b \wedge z = \varphi(x_2)$.

The case $x_2 = x_1$ is impossible, since then we would have

$$z = (a \wedge z) \vee (b \wedge z) = x_1 \vee \varphi(x_1) \in R.$$

Suppose that $x_1 < x_2$. In view of (ii) there exists $x_3 \in C_1$ with $x_1 < x_3 < x_2$. Then $\varphi(x_1) < \varphi(x_3) < \varphi(x_2)$ and $x_3 \vee \varphi(x_3) \in R$. A simple calculation (using the distributivity of L) shows that the elements

$$x_3 \vee \varphi(x_3), \qquad x_1 \vee \varphi(x_2) = z$$

are incomparable, which is a contradiction.

Similarly, the assumption $x_2 < x_1$ leads to a contradiction.

c) In view of b) we conclude that the element $z \wedge b$ does not belong to $\varphi(C_1)$. Suppose that $z \wedge b$ is an element of C_2.

If (a_1) is valid, then in view of (iii) there exists $x_2 \in C_1$ such that $\varphi(x_1) < \varphi(x_2) < z \wedge b$. Then $x_1 < x_2$ and

$$x_1 \vee \varphi(x_1) < x_2 \vee \varphi(x_2) \leqq z,$$

which contradicts a). Similarly we verify that from (a_2) we obtain a contradiction. Hence the element $z \wedge b$ does not belong to C_2.

d) We have $z \wedge b \in [u, v]$, thus according to c), there exists $y \in C_2$ such that the elements y and $z \wedge b$ are incomparable.

Assume that (a_1) is valid. Then

$$\varphi(x_1) < z \wedge b.$$

Since $\varphi(x_1) \in C_2$, the elements $\varphi(x_1)$ and y are comparable. If $y \leqq \varphi(x_1)$, then $y < z \wedge b$, which is impossible. Hence $\varphi(x_1) < y$. The element y cannot belong to $\varphi(C_1)$, since each element of $\varphi(C_1)$ is comparable with $z \wedge b$. Therefore in view of (iii), there exists $x_3 \in C_1$ such that

$$\varphi(x_1) < \varphi(x_3) < y.$$

Then we have $x_1 < x_3$ and $x_1 \vee \varphi(x_1) < x_3 \vee \varphi(x_3)$. According to a) we conclude that $x_3 \vee \varphi(x_3) > z$, whence $\varphi(x_3) \geqq z \wedge b$ and thus $y > z \wedge b$, which is a contradiction.

Similarly we can verify that by using (a_2) we arrive at a contradiction. Thus the element z must belong to R. □

3.2. PROPOSITION. *Let the assumptions of* 3.1 *be satisfied. Suppose that* $\operatorname{card} C_1 \neq \operatorname{card} C_2$. *Then* L *does not satisfy condition* (JD).

P r o o f. In view of the isomorphism φ we conclude that

$$\operatorname{card} C_1 < \operatorname{card} C_2.$$

Let R be as in 3.1. Then $\operatorname{card} R = \operatorname{card} C_1$. Denote

$$C_3 = \{a \vee y : y \in C_2\}, \qquad R' = C_1 \cup C_3.$$

From the fact that L is distributive we infer that $C_3 \in \mathcal{C}_m^0([a, v])$ and that $R' \in \mathcal{C}_m^0([u, v])$. Condition (ii) of 3.1 yields that C_1 is infinite, whence $\operatorname{card} R' = \operatorname{card} C_3 > \operatorname{card} R$. Therefore condition (JD) fails to be valid for the lattice L. □

Let L be a lattice, $u \in L$. A nonempty subset $\{a_i\}_{i \in I}$ of L is called *u-orthogonal* if $a_i \geqq u$ for each $i \in I$ and $a_{i(1)} \wedge a_{i(2)} = u$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

3.3. LEMMA. *Let L be an infinitely distributive lattice. Let $u \in L$ and let $\{a_i\}_{i\in I}$ be an u-orthogonal subset of L such that $u < a_i$ for each $i \in I$, and $\bigvee\limits_{i\in I} a_i = v$. Suppose that* $\operatorname{card} I > 1$ *and*

(i) *whenever $\{x_i\}_{i\in I} \subseteq L$ such that for each $i \in I$ the relation $u \leqq x_i \leqq a_i$ is valid, then $\bigvee\limits_{i\in I} x_i$ exists in L;*

(ii) *for each $i \in I$, $C_i \in \mathcal{C}_m^0([u, a_i])$ and the chain C_i is dense;*

(iii) *there exists $i(0) \in I$ such that for each $i \in I \setminus \{i(0)\}$ there exists an isomorphism φ_i of $C_{i(0)}$ into C_i such that $\varphi_i(C_{i(0)})$ is a strongly dense sublattice of C_i and $u, a_i \in \varphi_i(C_{i(0)})$.*

Then the set

$$R = \Big\{x \vee \bigvee_{i\in I\setminus\{i(0)\}} \varphi_i(x) : \ x \in C_{i(0)}\Big\}$$

is an element of $\mathcal{C}_m^0([u, v])$.

P r o o f. First we verify that if $x \in C_{i(0)}$, then the element

$$\bigvee_{i\in I\setminus\{i(0)\}} \varphi_i(x)$$

exists in L. For each $i \in I$ we put

$$y_i = \begin{cases} \varphi_i(x) & \text{if } i \neq i(0), \\ u & \text{if } i = i(0). \end{cases}$$

According to (i), $\bigvee\limits_{i\in I} y_i$ exists in L. Since $y_i \geqq u$ for each $i \in I$, we have

$$\bigvee_{i\in I} y_i = \bigvee_{i\in I\setminus\{i(0)\}} y_i = \bigvee_{i\in I\setminus\{i(0)\}} \varphi_i(x).$$

It is clear that R is a chain in $[u, v]$. Let z be an element of $[u, v]$ such that z is comparable with each element of R. We have to show that z belongs to R. If I is a one-element set, then the assertion holds trivially. Suppose that $\operatorname{card} I > 1$.

Let i be a fixed element of I, $i \neq i(0)$. Put $v_i = a_{i(0)} \vee a_i$,

$$R^i = \{x \vee \varphi_i(x) : \ x \in C_{i(0)}\}.$$

In view of 3.1 we have

$$R^i \in \mathcal{C}_m^i([u, v_i]). \tag{$*$}$$

Further, analogously as in the proof of 3.1 we verify that $z \wedge v_i$ is comparable with each element of R^i. Hence according to $(*)$, $z \wedge v_i$ must belong to R^i. Thus there exists $x_1 \in C_{i(0)}$ such that

$$z \wedge v_i = x_1 \vee \varphi_i(x_1).$$

Moreover, we have

$$x_1 = (z \wedge v_i) \wedge a_{i(0)} = z \wedge a_{i(0)},$$

and similarly

$$\varphi_i(x_1) = z \wedge a_i .$$

The infinite distributivity of L yields

$$\begin{aligned} z = z \wedge v = z \wedge \Big(a_{i(0)} \vee \bigvee_{i \in I \setminus \{i_0\}} a_i\Big) \\ = (z \wedge a_{i(0)}) \vee \bigvee_{i \in I \setminus \{i_0\}} (z \wedge a_i) = x_1 \vee \bigvee_{i \in I \setminus \{i_0\}} \varphi_i(x_1) . \end{aligned}$$

Therefore $z \in R$. □

3.4. Lemma. *Let the assumptions of* 3.3 *be satisfied. Let* R *and* $i(0)$ *be as in* 3.3. *Then* $\operatorname{card} R = \operatorname{card} C_{i(0)}$.

Proof. For each $x \in C_{i(0)}$ we put

$$\psi(x) = x \vee \bigvee_{i \in I \setminus \{i(0)\}} \varphi_i(x) .$$

Let $x_1, x_2 \in C_{i(0)}$, $x_1 < x_2$. Then clearly $\psi(x_1) \leqq \psi(x_2)$. It suffices to show that $\psi(x_1) < \psi(x_2)$. By way of contradiction, assume that $\psi(x_1) = \psi(x_2)$. Then in view of infinite distributivity we have

$$x_2 = x_2 \wedge \psi(x_2) = x_2 \wedge \psi(x_1) = (x_2 \wedge x_1) \vee \bigvee_{i \in I \setminus \{i(0)\}} (x_2 \wedge \varphi_i(x_1)) = x_2 \wedge x_1 = x_1 ,$$

since $x_2 \wedge \varphi_i(x_1) = u$ for each $i \in I \setminus \{i(0)\}$. Thus we arrived at a contradiction. □

3.5. Proposition. *Let the assumptions of* 3.3 *be valid and let* $i \in I$. *There exists* $T^{[i]} \in C_m^0([u,v])$ *such that* $\operatorname{card} T^{[i]} = \operatorname{card} C_i$.

Proof.

a) Let $i = i(0)$. Then it suffices to put $T^{[i]} = R$ and to apply Lemma 3.4.

b) Let $i \neq i(0)$. Put $J_i = \{i_1 \in I : i_1 \neq i\}$. Similarly as in the proof of 3.3 (i.e., by using condition (i) from 3.3) we verify that the element

$$\bigvee_{j \in J_i} a_j$$

exists in L. Denote

$$\begin{aligned} v^{[i]} &= \bigvee_{j \in J_i} a_j , \\ R^{[i]} &= \Big\{x \vee \bigvee_{j \in J(i)} \varphi_j(x) : \ x \in C_{i(0)}\Big\} , \\ Q^{[i]} &= \{v^{[i]} \vee y_i : \ y_i \in C_i\} , \\ T^{[i]} &= R^{[i]} \cup Q^{[i]} . \end{aligned}$$

We have

$$\operatorname{card} R^{[i]} = \operatorname{card} C_{i(0)} \leqq \operatorname{card} C_i = \operatorname{card} Q^{[i]};$$

since $C_{i(0)}$ is infinite, we conclude that $\operatorname{card} T^{[i]} = \operatorname{card} C_i$. □

3.6. Lemma. *Let the assumptions of* 3.3 *be valid. Put* $\operatorname{card} I = \alpha$ *and suppose that* $\operatorname{card} C_i \leqq \alpha$ *for each* $i \in I$. *Then there exists* $T \in \mathcal{C}_m^0([u,v])$ *such that* $\operatorname{card} T = \alpha$.

Proof. We apply the Axiom of Choice; then we can assume that the set I is well-ordered and that I has a greatest element.

Since I is well-ordered, it has the least element which will be denoted by i_1. We put $b_{i_1} = a_{i_1}$ and $C'_{i_1} = C_{i_1}$. Suppose that $i \in I$, $i > i_1$ and that we have defined b_j and C'_j for each $j \in I$ with $j < i$. We put

$$b_i^0 = \bigvee_{j \in I,\, j<i} a_j .$$

For proving the existence of this element in the lattice L we use an analogous method as in the proof of 3.3. Namely, for each $j \in J$ we denote

$$y_j = \begin{cases} a_j & \text{if } j < i, \\ u & \text{otherwise.} \end{cases}$$

In view of condition (i) from 3.3, $\bigvee_{j \in I} y_j$ exists in L. We have $y_j \geqq u$ for each $j \in J$, thus

$$\bigvee_{j \in I} y_j = \bigvee_{j \in I,\, j<i} y_j = b_i^0 .$$

Now we set

$$b_i = b_i^0 \vee a_i ,$$
$$C'_i = \{ b_i^0 \vee x : \ x \in C_i \} .$$

Further, let us denote

$$R = \bigcup_{i \in I} C'_i .$$

Then R is a chain in $[u,v]$. Since I has a least element and a greatest element we conclude that $u, v \in R$. Next, because $\operatorname{card} C'_i = \operatorname{card} C_i \leqq \alpha$ for each $i \in I$, we get $\operatorname{card} R = \alpha$.

Let $z \in [u,v]$ and assume that z is comparable with each element of R. Put

$$R_1 = \{ r \in R : \ r \leqq z \}, \qquad R_2 = \{ r \in R : \ r > z \} .$$

We have to prove that z belongs to R. If $R_2 = \emptyset$, then $z = v \in R$. Consider the case when $R_2 \neq \emptyset$.

We distinguish two cases.

a) Assume that there exists $i \in I$ and $x, y \in C_i'$ such that $x \leqq z \leqq y$. The definition of C_i' yields that C_i' is a maximal chain in the interval $[b_i^0, b_i]$ of L. Since $C_i' \subseteq R$ and $z \in [b_i^0, b_i]$ we conclude that $z \in C_i'$ and thus $z \in R$.

b) Suppose that the assumption from a) is not satisfied. Let us denote by I_1 the set of all $i \in I$ such that there exists $x \in C_i'$ with $x \leqq z$. Further, put $I_2 = I \setminus I_1$. Similarly as we did for b_i^0 we can prove that the element

$$\bigvee_{i \in I_1} a_i$$

exists in L.

If $I_2 = \emptyset$, then we would have $z = v$, which is a contradiction. Thus $I_2 \neq \emptyset$. Hence I_2 has a least element which will be denoted by i_2. Then $b_{i_2}^0 > z$.

From the definition of $b_{i_2}^0$ we get

$$b_{i_2}^0 = \bigvee_{i \in I_1} a_i .$$

For $i \in I_1$ we have

$$a_i \leqq b_i \leqq z ,$$

thus $b_{i_2}^0 \leqq z$, which is a contradiction. Hence z is an element of R. Therefore $R \in C_m^0([u, v])$. □

3.7. Proposition. *Let the assumptions of* 3.6 *be valid. Then for each cardinal* β *such that* $\beta \leqq \alpha$ *and* $\beta \geqq \operatorname{card} C_i$ *for each* $i \in I$ *there exists* $Q \in C_m^0([u, v])$ *such that* $\operatorname{card} Q = \beta$.

Proof. Assume that β is a cardinal with the mentioned properties. If $\beta = \alpha$, then the assertion concerning β is a consequence of 3.6.

Let $\beta < \alpha$. Similarly as in the proof of 3.6 we suppose that I is a well-ordered set. Without loss of generality we can also suppose that $i(0)$ is the greatest element of I. There exists i_1 in I such that i_1 is the first element of I with respect to the property that the set $I_1 = \{i \in I : \ i < i_1\}$ has the cardinality β. Put $I_2 = I \setminus I_1$. Then $I_2 \neq \emptyset$. Denote

$$v_1 = \bigvee_{i \in I_1} a_i , \qquad v_2 = \bigvee_{i \in I_2} a_i .$$

The existence of these elements in the lattice L can be proved by a method analogous to that used in the proof of 3.6. We have $v_1 \wedge v_2 = u$ and $v_1 \vee v_2 = v$.

We construct a chain R in $[u, v_1]$ in the same way as in 3.6 with the distinction that instead of v we now have the element v_1. Then $\operatorname{card} R = \beta$.

Next, we construct a chain Q in $[u, v_2]$ in an analogous manner as we constructed R in 3.3 with the distinction that we have v_2 instead of v. According to 3.4 we have $\operatorname{card} Q = \operatorname{card} C_{i(0)} \leqq \beta$.

Put

$$Q' = \{v_1 \vee q :\ q \in Q\}, \qquad T = Q \cup R.$$

Similarly as in the proof of 3.2 we can verify that T is an element of $\mathcal{C}_m^0([u,v])$. We obviously have $\operatorname{card} T = \beta$. □

In view of 3.7 we introduce the following definition.

Let α_1, α_2 be cardinals with $\alpha_1 < \alpha_2$ and let L be a lattice. We say that L *satisfies condition* $c(\alpha_1, \alpha_2)$ if there are elements $u, v \in L$, $u < v$ such that for each cardinal β with $\alpha_1 \leqq \beta \leqq \alpha_2$ there exists a chain $C_\beta \in \mathcal{C}_m^0([u,v])$ whose cardinality is β.

Let L_0 be a lattice. Let I be a set of indices and for each $i \in I$ let $L_i = L_0$. Then the direct product

$$\prod_{i \in I} L_i$$

will be called a *direct power* of the lattice L_0 and it will be denoted by L_0^α, where $\alpha = \operatorname{card} I$. An analogous notation will be applied for MV-algebras.

Suppose that $u_0, v_0 \in L_0$, $u_0 < v_0$. We define u, v and a_i $(i \in I)$ in L_0^α as follows:

$$u(i) = u_0, \quad v(i) = v_0 \qquad \text{for each} \quad i \in I;$$

$$a_i(j) = \begin{cases} u_0 & \text{if } j \neq i, \\ v_0 & \text{if } j = i, \end{cases}$$

where j runs over the set I.

3.8. Lemma. *Suppose that there exists $C_0 \in \mathcal{C}_m^0([u_0, v_0])$ such that the chain C_0 is dense. Further, suppose that the lattice L_0 is infinitely distributive. Then the lattice L_0^α is infinitely distributive and* (*under the notation as above*) *the assumptions of* 3.6 *are satisfied.*

P r o o f. The assertion is an immediate consequence of the definition of the elements u, v and a_i $(i \in I)$. □

From 3.7 and 3.8 we obtain the following proposition:

3.9. Proposition. *Let L_0 be an infinitely distributive lattice. Further, suppose that there are $u_0, v_0 \in L_0$ with $u_0 < v_0$ and $C_0 \in \mathcal{C}_m^0([u_0, v_0])$ such that C_0 is dense and* $\operatorname{card} C_0 = \alpha_1$. *Let α_2 be a cardinal with $\alpha_1 < \alpha_2$. Then the lattice $L_0^{\alpha_2}$ satisfies condition $c(\alpha_1, \alpha_2)$.*

[14; Theorem 3] is a corollary of 3.9. Further, let L_0 be the interval $[0,1]$ of reals. Then L_0 is complete and completely distributive. Hence for each cardinal α, L_0^α is complete and completely distributive. This yields that the following result is also a corollary of 3.9.

3.9.1. COROLLARY. ([7]) *Let α be a cardinal, $\alpha \geqq c$. There exists a complete and completely distributive lattice with the last element f_0 and the greatest element f_1 which has the following property: for any cardinal number β with $c \leqq \beta \leqq \alpha$ there exists in S_α a maximal chain R_β the length of which is β.*

If $\mathcal{A}$ is an MV-algebra, then the lattice $\ell(\mathcal{A})$ is infinitely distributive. Thus we have:

3.10. COROLLARY. *Let $\mathcal{A}$ be an MV-algebra. Suppose that there are $u_0, v_0 \in A$ with $u_0 < v_0$ and $C_0 \in \mathcal{C}_m^0([u_0, v_0])$ such that C_0 is dense and* $\operatorname{card} C_0 = \alpha_1$. *Let α_2 be a cardinal with $\alpha_1 < \alpha_2$. Then the lattice $(\ell(\mathcal{A}))^{\alpha_2}$ satisfies condition $c(\alpha_1, \alpha_2)$.*

Now let (A) be as in Introduction.

Proof of (A).

It is well known that there exists an MV-algebra $\mathcal{A}$ such that $\ell(A)$ is the set of all rational numbers x with $0 \leqq x \leqq 1$ (under the natural linear order). Put $\alpha_1 = \aleph_0$ and let α_2 be a cardinal with $\alpha_2 > \alpha_1$. In view of 3.10, the lattice $\ell(\mathcal{A}^{\alpha_2})$ satisfies condition $c(\alpha_1, \alpha_2)$. Further, $\ell(\mathcal{A}^{\alpha_2})$ is completely distributive. Let $0 < y \in \mathcal{A}^{\alpha_2}$. Then there exist $z_1, z_2 \in \mathcal{A}^{\alpha_2}$ such that $0 < z_1 < z_2$ and the interval $[0, z_2]$ of $\mathcal{A}^{\alpha_2}$ is linearly ordered. Hence z_1 has no complement in $[0, z_2]$. Thus the element y fails to be boolean in $\mathcal{A}^{\alpha_2}$. □

Similarly, there exists an MV-algebra $\mathcal{A}_2$ such that the lattice $\ell(\mathcal{A}_2)$ is the interval $[0, 1]$ of reals. By using $\mathcal{A}_2$ we can obtain an analogous result to (A) with the distinction that:

(i) instead of $\aleph_0$ we have the power of the continuum;
(ii) the lattice $\ell(\mathcal{A}_2^{\alpha_2})$ turns out to be complete.

In view of condition (iii) from (A) let us conclude this section by some remarks concerning the question what is the situation in the case when there exists a boolean element in $\mathcal{A}$.

It is well known that for each Boolean algebra B there exists an MV-algebra $\mathcal{A}$ such that $\ell(\mathcal{A}) = B$; then each element of $\mathcal{A}$ is boolean.

3.11. PROPOSITION. ([7a]) *Let S be an infinite Boolean algebra which is complete and completely distributive. Then S does not satisfy condition* (JD).

Further, there exist infinite Boolean algebras B having no atom; if B has this property and $u = 0$, $v \in B$, $v > 0$ and if $C \in \mathcal{C}_m^0([u, v])$, then C must be dense. If α_2 is a cardinal with $\alpha_2 > \alpha_1$, then according to 3.10 we get that $\mathcal{A}^{\alpha_2}$ is an MV-algebra such that $\ell(\mathcal{A}^{\alpha_2})$ satisfies condition $c(\alpha_1, \alpha_2)$ and each element of $\mathcal{A}^{\alpha_2}$ is boolean.

4. Examples

The examples given in this section concern the investigation performed in Section 2.

4.1. Let G be the lattice ordered group of all bounded continuous real functions defined on the set of all reals (the group operation is the addition, lattice operations are defined component-wise). Let $u \in G$ such that u is identically equal to 1. If $\mathcal{A} = \mathcal{A}_0(G; u)$, then $\mathcal{C} = \mathcal{C}_m = \emptyset$.

The following examples 4.2, 4.3 and 4.4 show that conditions a), b) and c) from 2.8(iii) are independent.

4.2. Let Z be the additive group of all integers with the natural linear order. Put $G = Z \circ (Z \times Z)$, where $\circ$ denotes the operation of lexicographic product and $\times$ is the symbol of the operation of the direct product. Put $u = (1, 0, 0)$ and let $\mathcal{A} = \mathcal{A}_0(G; u)$, $Y_1 = \{(0, z, 0)\}_{z \in Z}$, $Y_2 = \{(0, 0, z)\}_{z \in Z}$. Then $\mathcal{C}_m = \{Y_1, Y_2\}$. Neither Y_1 nor Y_2 has a greatest element. Conditions b) and c) from 2.8(iii) are satisfied.

4.3. Let G and u be as in 2.1. Put $G' = G \times Z \times Z$, $u' = (u, 1, 1)$, $\mathcal{A}' = \mathcal{A}_0(G', u')$. Denote $Y_1 = \{(0, z, 0)\}_{z \in \{0,1\}}$, $Y_2 = \{(0, 0, z)\}_{z \in \{0,1\}}$. Then $\mathcal{C}_m = \mathcal{C} = \{Y_1, Y_2\}$. Conditions a) and b) from 2.8(iii) are satisfied, but condition c) fails to hold.

4.4. Let $I = Z$ and for each $i \in I$ let $G_i = Z$, $G = \prod_{i \in I} G_i$. Let H be the subgroup of G consisting of all $g \in G$ which satisfy the following condition: there exists a finite subset $I(g)$ of I such that, whenever $i(1)$ and $i(2)$ belong to $I \setminus I(g)$, then $g(i(1)) = g(i(2))$. Then H is an ℓ-subgroup of G. Let $u \in G$, $u(i) = 1$ for each $i \in I$. We have $u \in H$; moreover, u is a strong unit of H. Denote $\mathcal{A} = \mathcal{A}_0(H, u)$. For each $i(1) \in I$ let $g^{i(1)} \in G$ such that $g^{i(1)}(i(1)) = 1$ and $g^{i(1)}(i) = 0$ if $i \neq i(1)$; next, let $Y_{i(1)} = \{0, g^{i(1)}\}$. Then $\mathcal{C} = \mathcal{C}_m = \{Y_{i(1)}\}_{i(1) \in I}$. Conditions a) and c) from 2.8(iii) are satisfied. Let $I(1)$ be an infinite subset of I such that $I \setminus I(1)$ is infinite as well. Put $y^{i(1)} = g^{i(1)}$ if $i(1) \in I(1)$ and $y^{i(1)} = 0$ otherwise. The element $\bigvee_{i(1) \in I} y^{i(1)}$ does not exist in $\mathcal{A}$, hence condition b) is not satisfied.

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