Young Bae Jun; Hee Sik Kim
On fuzzy topological $d$-algebras


Persistent URL: [http://dml.cz/dmlcz/136802](http://dml.cz/dmlcz/136802)

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 2001

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must contain
these *Terms of use.*
ON FUZZY TOPOLOGICAL $d$-ALGEBRAS

YOUNG BAE JUN* — HEE SIK KIM**

(Communicated by Anatolij Dvurečenskij)

ABSTRACT. In this paper we introduce the concept of fuzzy topological $d$-algebras and apply some of Foster's results on homomorphic images and inverse images to fuzzy topological $d$-algebras.

1. Introduction

Y. Imai and K. Iséki [4] and K. Iséki [5] introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [2], [3], Q. P. Hu and X. Li introduced a wide class of abstract algebras: $BCH$-algebras. They showed that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. J. Neggers and H. S. Kim [11] introduced a new notion, called a $d$-algebra, which is another generalization of $BCK$-algebras, and investigated relations between $d$-algebras and $BCK$-algebras. In [7], Y. B. Jun, J. Neggers and H. S. Kim introduced the notions of fuzzy $d$-subalgebra, fuzzy $d$-ideal, fuzzy $d^2$-ideal and fuzzy $d^*$-ideal, and investigated relations among them. The concept of a fuzzy set, which was introduced in [13], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D. H. Foster (cf. [1]) combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by A. Rosenfeld (cf. [12]), to formulate the elements of a theory of fuzzy topological groups. In 1993, Y. B. Jun [6] combined the structure of a fuzzy topological spaces with that of a fuzzy BCK-algebras to formulate the elements of a theory of fuzzy topological BCK-algebras. In the present paper, we introduce the concept of fuzzy topological $d$-algebras and apply some of Foster’s results on homomorphic images and inverse images to fuzzy topological $d$-algebras.

2000 Mathematics Subject Classification: Primary 06F35, 03G25, 94D05.
Key words: (fuzzy) $d$-algebra, fuzzy topological $d$-algebra.
2. Preliminaries

A \textit{d-algebra} ([11]) is a non-empty set $X$ with a constant $0$ and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0,$
(II) $0 * x = 0,$
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$.

A non-empty subset $N$ of a d-algebra $X$ is called a \textit{d-subalgebra} of $X$ if $x * y \in N$ for any $x, y \in N$.

A mapping $\alpha : X \to Y$ of d-algebras is called a \textit{d-homomorphism} if $\alpha(x * y) = \alpha(x) * \alpha(y)$ for all $x, y \in X$.

We now review some fuzzy logic concepts (see [1] and [13]). Let $X$ be a set.

A \textit{fuzzy set} $A$ in $X$ is characterized by a membership function $\mu_A : X \to [0,1]$. Let $\alpha$ be a mapping from the set $X$ to the set $Y$ and let $B$ be a fuzzy set in $Y$ with membership function $\mu_B$.

The inverse image of $B$, denoted $\alpha^{-1}(B)$, is the fuzzy set in $X$ with membership function $\mu_{\alpha^{-1}(B)}$ defined by $\mu_{\alpha^{-1}(B)}(x) = \mu_B(\alpha(x))$ for all $x \in X$. Conversely, let $A$ be a fuzzy set in $X$ with membership function $\mu_A$. Then the image of $A$, denoted by $\alpha(A)$, is the fuzzy set in $Y$ such that

$$\mu_{\alpha(A)}(y) = \begin{cases} \sup_{z \in \alpha^{-1}(y)} \mu_A(z) & \text{if } \alpha^{-1}(y) = \{x : \alpha(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

A \textit{fuzzy topology} on a set $X$ is a family $\mathcal{T}$ of fuzzy sets in $X$ which satisfies the following conditions:

(i) for all $c \in [0,1]$, $k_c \in \mathcal{T}$, where $k_c$ has a constant membership function,
(ii) if $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
(iii) if $A_j \in \mathcal{T}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a \textit{fuzzy topological space} and members of $\mathcal{T}$ are called \textit{open fuzzy sets}.

Let $A$ be a fuzzy set in $X$ and $\mathcal{T}$ a fuzzy topology on $X$. Then the \textit{induced fuzzy topology} on $A$ is the family of fuzzy subsets of $A$ which are the intersection with $A$ of $\mathcal{T}$-open fuzzy sets in $X$. The induced fuzzy topology is denoted by $\mathcal{T}_A$, and the pair $(A, \mathcal{T}_A)$ is called a \textit{fuzzy subspace} of $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two fuzzy topological spaces. A mapping $\alpha$ of $(X, \mathcal{T})$ into $(Y, \mathcal{U})$ is \textit{fuzzy continuous} if for each open fuzzy set $U$ in $\mathcal{U}$ the inverse image $\alpha^{-1}(U)$ is in $\mathcal{T}$. Conversely, $\alpha$ is \textit{fuzzy open} if for each open fuzzy set $V$ in $\mathcal{T}$, the image $\alpha(V)$ is in $\mathcal{U}$.
ON FUZZY TOPOLOGICAL d-ALGEBRAS

Let \((A, \mathcal{T}_A)\) and \((B, \mathcal{U}_B)\) be fuzzy subspaces of fuzzy topological spaces \((X, \mathcal{T})\) and \((Y, \mathcal{U})\) respectively, and let \(\alpha\) be a mapping from \((X, \mathcal{T})\) to \((Y, \mathcal{U})\). Then \(\alpha\) is a mapping of \((A, \mathcal{T}_A)\) into \((B, \mathcal{U}_B)\) if \(\alpha(A) \subseteq B\). Furthermore \(\alpha\) is relatively fuzzy continuous if for each open fuzzy set \(V'\) in \(\mathcal{U}_B\), the intersection \(\alpha^{-1}(V') \cap A\) is in \(\mathcal{T}_A\). Conversely, \(\alpha\) is relatively fuzzy open if for each open fuzzy set \(U'\) in \(\mathcal{T}_A\), the image \(\alpha(U')\) is in \(\mathcal{U}_B\).

**LEMMA 2.1.** ([1]) Let \((A, \mathcal{T}_A), (B, \mathcal{U}_B)\) be fuzzy subspaces of fuzzy topological spaces \((X, \mathcal{T}), (Y, \mathcal{U})\) respectively, and let \(\alpha\) be a fuzzy continuous mapping of \((X, \mathcal{T})\) into \((Y, \mathcal{U})\) such that \(\alpha(A) \subseteq B\). Then \(\alpha\) is a relatively fuzzy continuous mapping of \((A, \mathcal{T}_A)\) into \((B, \mathcal{U}_B)\).

### 3. Fuzzy topological d-algebras

**DEFINITION 3.1.** ([7]) A fuzzy set \(D\) in a d-algebra \(X\) with membership function \(\mu_D\) is called a fuzzy d-algebra of \(X\) if

\[
\mu_D(x \ast y) \geq \min\{\mu_D(x), \mu_D(y)\} \quad \text{for all } x, y \in X.
\]

**EXAMPLE 3.2.** ([7]) Let \(X = \{0, a, b, c\}\) be a set with the following Cayley table (Table 1) as follows:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & a & 0 \\
\end{array}
\]

Table 1.

Then \((X, \ast, 0)\) is a d-algebra. Define a fuzzy set \(D\) in \(X\) with membership function \(\mu_D\) by \(\mu_D(0) = \mu_D(a) = \mu_D(c) = t_1\) and \(\mu_D(b) = t_2\) for \(t_1 > t_2\). Then \(D\) is a fuzzy d-algebra of \(X\).

**EXAMPLE 3.3.** ([7]) Let \(X = \{0, a, b, c\}\) be a set with the following Cayley table (Table 2) as follows:
Table 2.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X, \ast, 0)\) is a \(d\)-algebra. Define a fuzzy set \(D\) in \(X\) with membership function \(\mu_D\) by \(\mu_D(0) = \mu_D(a) = t_1 > t_2 = \mu_D(b) = \mu_D(c)\), where \(t_1, t_2 \in [0, 1]\). Then \(D\) is a fuzzy \(d\)-algebra of \(X\).

**Proposition 3.4.** Let \(\alpha\) be a \(d\)-homomorphism of a \(d\)-algebra \(X\) into a \(d\)-algebra \(Y\) and \(G\) a fuzzy \(d\)-algebra of \(Y\) with membership function \(\mu_G\). Then the inverse image \(\alpha^{-1}(G)\) of \(G\) is a fuzzy \(d\)-algebra of \(X\).

**Proof.** Let \(x, y \in X\). Then

\[
\mu_{\alpha^{-1}(G)}(x \ast y) = \mu_G(\alpha(x \ast y)) = \mu_G(\alpha(x) \ast \alpha(y)) \\
\geq \min\{\mu_G(\alpha(x)), \mu_G(\alpha(y))\} \\
= \min\{\mu_{\alpha^{-1}(G)}(x), \mu_{\alpha^{-1}(G)}(y)\}.
\]

This completes the proof. \(\square\)

For images, we need the following definition ([12]).

**Definition 3.5.** A fuzzy set \(D\) in a \(d\)-algebra \(X\) with membership function \(\mu_D\) is said to have the sup property if, for any subset \(T \subseteq X\), there exists \(t_0 \in T\) such that

\[
\mu_D(t_0) = \sup_{t \in T} \mu_D(t).
\]

**Proposition 3.6.** Let \(\alpha\) be a \(d\)-homomorphism of a \(d\)-algebra \(X\) onto a \(d\)-algebra \(Y\) and let \(D\) be a fuzzy \(d\)-algebra of \(X\) with the sup property. Then the image \(\alpha(D)\) of \(D\) is a fuzzy \(d\)-algebra of \(Y\).

**Proof.** For \(u, v \in Y\), let \(x_0 \in \alpha^{-1}(u)\), \(y_0 \in \alpha^{-1}(v)\) such that

\[
\mu_D(x_0) = \sup_{t \in \alpha^{-1}(u)} \mu_D(t), \quad \mu_D(y_0) = \sup_{t \in \alpha^{-1}(v)} \mu_D(t).
\]
ON FUZZY TOPOLOGICAL $d$-ALGEBRAS

Then, by the definition of $\mu_{\alpha(D)}$, we have

$$\mu_{\alpha(D)}(u * v) = \sup_{t \in \alpha^{-1}(u * v)} \mu_D(t)$$

$$\geq \mu_D(x_0 * y_0)$$

$$\geq \min\{\mu_D(x_0), \mu_D(y_0)\}$$

$$= \min\{\sup_{t \in \alpha^{-1}(u)} \mu_D(t), \sup_{t \in \alpha^{-1}(v)} \mu_D(t)\}$$

$$= \min\{\mu_{\alpha(D)}(u), \mu_{\alpha(D)}(v)\},$$

completing the proof.

For any $d$-algebra $X$ and any element $a \in X$ we denote by $R_a$ the right translation of $X$ defined by $R_a(x) = x * a$ for all $x \in X$. It is clear that $R_x(0) = 0 = R_x(x)$ for all $x \in X$.

**DEFINITION 3.7.** Let $X$ be a $d$-algebra and $\mathcal{T}$ a fuzzy topology on $X$. Let $D$ be a fuzzy $d$-algebra of $X$ with induced topology $\mathcal{T}_D$. Then $D$ is called a fuzzy topological $d$-algebra of $X$ if for each $a \in X$ the mapping $R_a : (D, \mathcal{T}_D) \rightarrow (D, \mathcal{T}_D)$ is relatively fuzzy continuous.

**THEOREM 3.8.** Given $d$-algebras $X$, $Y$ and a $d$-homomorphism $\alpha : X \rightarrow Y$, let $\mathcal{U}$ and $\mathcal{T}$ be the fuzzy topologies on $Y$ and $X$ respectively, such that $\mathcal{T} = \alpha^{-1}(\mathcal{U})$. Let $G$ be a fuzzy topological $d$-algebra of $Y$ with membership function $\mu_G$. Then $\alpha^{-1}(G)$ is a fuzzy topological $d$-algebra of $X$ with membership function $\mu_{\alpha^{-1}(G)}$.

**Proof.** We have to show that, for each $a \in X$, the mapping

$$R_a : (\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)}) \rightarrow (\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)})$$

is relatively fuzzy continuous. Let $U$ be an open fuzzy set in $\mathcal{T}_{\alpha^{-1}(G)}$ on $\alpha^{-1}(G)$. Since $\alpha$ is a fuzzy continuous mapping of $(X, \mathcal{T})$ into $(Y, \mathcal{U})$, it follows from Lemma 2.1 that $\alpha$ is a relatively fuzzy continuous mapping of $(\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)})$ into $(G, \mathcal{U}_G)$. Note that there exists an open fuzzy set $V \in \mathcal{U}_G$ such that $\alpha^{-1}(V) = U$. The membership function of $R_a^{-1}(U)$ is given by

$$\mu_{R_a^{-1}(U)}(x) = \mu_U(R_a(x)) = \mu_U(x * a) = \mu_{\alpha^{-1}(V)}(x * a)$$

$$= \mu_V(\alpha(x * a)) = \mu_V(\alpha(x) * \alpha(a)).$$

Since $G$ is a fuzzy topological $d$-algebra of $Y$, the mapping

$$R_b : (G, \mathcal{U}_G) \rightarrow (G, \mathcal{U}_G)$$
is relatively fuzzy continuous for each \( b \in Y \). Hence
\[
\mu_{R_a^{-1}(U)}(x) = \mu_V(\alpha(x) * \alpha(a)) = \mu_V(R_{\alpha(a)}(\alpha(x))) \\
= \mu_{R_{\alpha(a)}(V)}(\alpha(x)) = \mu_{\alpha^{-1}(R_{\alpha(a)}^{-1}(V))}(x),
\]
which implies that \( R_a^{-1}(U) = \alpha^{-1}(R_{\alpha(a)}^{-1}(V)) \) so that
\[
R_a^{-1}(U) \cap \alpha^{-1}(G) = \alpha^{-1}(R_{\alpha(a)}^{-1}(V)) \cap \alpha^{-1}(G)
\]
is open in the induced fuzzy topology on \( \alpha^{-1}(G) \). This completes the proof. \( \square \)

The membership function \( \mu_G \) of a fuzzy \( d \)-algebra \( G \) of a \( d \)-algebra \( X \) is said to be \( \alpha \)-invariant \((12)\) if, for all \( x, y \in X \), \( \alpha(x) = \alpha(y) \) implies \( \mu_G(x) = \mu_G(y) \).

**Theorem 3.9.** Given \( d \)-algebras \( X, Y \) and a \( d \)-homomorphism \( \alpha \) of \( X \) onto \( Y \), let \( T \) be the fuzzy topology on \( X \) and let \( U \) be the fuzzy topology on \( Y \) such that \( \alpha(T) = U \). Let \( D \) be a fuzzy topological \( d \)-algebra of \( X \). If the membership function \( \mu_D \) of \( D \) is \( \alpha \)-invariant, then \( \alpha(D) \) is a fuzzy topological \( d \)-algebra of \( Y \).

**Proof.** It is sufficient to show that the mapping
\[
R_b: (\alpha(D), U_{\alpha(D)}) \longrightarrow (\alpha(D), U_{\alpha(D)})
\]
is relatively fuzzy continuous for each \( b \in Y \). Note that \( \alpha \) is relatively fuzzy open; for if \( U' \in T_D \), there exists \( U \in T \) such that \( U' = U \cap D \), and by the \( \alpha \)-invariance of \( \mu_F \),
\[
\alpha(U') = \alpha(U) \cap \alpha(D) \in U_{\alpha(D)}.
\]
Let \( V' \) be an open fuzzy set in \( U_{\alpha(D)} \). Since \( \alpha \) is onto, for each \( b \in Y \) there exists \( a \in X \) such that \( b = \alpha(a) \). Hence
\[
\mu^{-1}_{\alpha^{-1}(R_b^{-1}(V'))}(x) = \mu^{-1}_{\alpha^{-1}(R_{\alpha(a)}^{-1}(V'))}(x) = \mu_{R_{\alpha(a)}^{-1}(V')}(\alpha(x)) \\
= \mu_V(\alpha(x)) = \mu_V(\alpha(x) * \alpha(a)) \\
= \mu_V(\alpha(x) * a) = \mu^{-1}_{\alpha^{-1}(V')}(x * a) \\
= \mu^{-1}_{\alpha^{-1}(V')}(R_a(x)) = \mu_{R_a^{-1}(\alpha^{-1}(V'))}(x),
\]
which implies that \( \alpha^{-1}(R_b^{-1}(V')) = R_a^{-1}(\alpha^{-1}(V')) \). By hypothesis, \( R_a \) is a relatively fuzzy continuous mapping from \( (D, T_D) \) to \( (D, T_D) \) and \( \alpha \) is a relatively fuzzy continuous mapping from \( (D, T_D) \) to \( (\alpha(D), U_{\alpha(D)}) \). Hence
\[
\alpha^{-1}(R_b^{-1}(V')) \cap G = R_a^{-1}(\alpha^{-1}(V')) \cap D
\]
is open in \( T_D \). Since \( \alpha \) is relatively fuzzy open,
\[
\alpha(\alpha^{-1}(R_b^{-1}(V')) \cap D) = R_b^{-1}(V') \cap \alpha(D)
\]
is open in \( U_{\alpha(D)} \). This completes the proof. \( \square \)
ON FUZZY TOPOLOGICAL d-ALGEBRAS

REFERENCES


Received July 21, 1999

* Department of Mathematics Education
Gyeongsang National University
Chinju 660-701
KOREA
E-mail: ybjun@nongae.gsnu.ac.kr

** Department of Mathematics
Hanyang University
Seoul 133-791
KOREA
E-mail: heekim@email.hanyang.ac.kr