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CLOSURE OPERATORS AND GENERATING SETS

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ABSTRACT. For given set-theoretic operations there are considered all closure operators on a fixed finite set such that it holds: For each subset the system of all its generating sets is closed under the set-theoretic operations. To characterize the closure operators there are determined semigroups such that each closure operator is the meet of the ideal closure operators on some semigroups, exactly.

1. Introduction and summary

The geometric notion of convexity is an important one. It influences and suggests research in various branches of mathematics. Therefore, it is of interest to investigate properties of the structure of the convex subsets of a vector space in the abstract. In study the lattice theoretic point of view as well as the closure theoretic point of view play essential roles. There are discussed connections between lattice theoretic or closure theoretic results and concepts belonging to geometry (cf. [7]). For this convexity lattices (cf. [6]) or convexity closure operators (cf. [5]) are considered, especially. These connections have suggested new areas of research in closure theory, too. In a series of papers finite closure spaces are investigated which are related with certain of the properties of the convexity closure operators. In particular, closure operators (on finite sets) are introduced for which the intersection of two generating sets is also a generating set, i.e., each set has the least generating set with respect to inclusion (cf. [8], [9], [10]). On the other hand closure theoretic results which are related with convexity have influenced concepts not belonging to geometry. In several papers (finite) filtered semigroups are considered (cf. [15], [16], [17]), i.e., semigroups for which any subsemigroup has the least generating set with respect to inclusion.

Moreover, the results of this paper show that the ideal closure operator on a finite semigroup has the property that the intersection of two generating sets

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is also a generating if each J-class has the cardinality one (where two elements belong to the same J-class if they generate the same principal ideal, cf. [2], [14]). Generally, any system of generating sets is closed under given set-theoretic operations (inductively defined by union and intersection) if and only if the cardinalities of the J-classes are restricted by some suitable natural number.

The aim of the present paper is to bring together all closure operators on finite sets which satisfy the intersection property discussed above with semigroups in such a way that the closure operators are represented by ideal closure operators on the semigroups (cf. [1], [2], [3]).

More generally, for given set-theoretic operations (inductively defined by union and intersection) there are considered all closure operators on finite sets such that for each set the system of all generating sets is closed under the operations. There are determined some natural number (which restricts the cardinalities of the J-classes) and finite semigroups such that each closure operator is the meet of ideal closure operators on some of the semigroups, exactly.

2. Preliminaries and basic concepts

At first, there are recalled some basic notions of set theory and universal algebra (cf. [8], [12]), respectively. For a set A let $P(A)$ be the set of all subsets of A and C be a function from $P(A)$ into $P(A)$. Then C is called to be a *closure operator on A* if and only if it satisfies the following properties for $X, Y \in P(A)$:

$X \subseteqq C(X)$,
if $X \subseteqq Y$, then $C(X) \subseteqq C(Y)$,
$C(C(X)) = C(X)$.

Any set T is called to be a *C-set of A* if and only if $T \subseteqq A$ and $C(T) = T$. For the purpose of simplicity it is assumed that the empty set is always a C-set.

Let $T \subseteqq A$. Then a set U is called to be a *generating set of T* if and only if $U \subseteqq T$ and $T \subseteqq C(U)$. Let $\mathrm{GEN}(T)$ be the system of all generating sets of T.

For a natural number n let $\mathrm{OP}(n)$ be the system of all $(n+1)$-ary set-theoretic operations defined inductively as follows:

The projections π_i^n with

$$\pi_i^n(X_0, \ldots, X_n) := X_i$$

for $0 \leqq i \leqq n$ are $(n+1)$-ary operations;
if α and β are $(n+1)$-ary operations, then so are $\alpha \cap \beta$ and $\alpha \cup \beta$ defined by

$$(\alpha \cap \beta)(X_0, \ldots, X_n) := \alpha(X_0, \ldots, X_n) \cap \beta(X_0, \ldots, X_n),$$
$$(\alpha \cup \beta)(X_0, \ldots, X_n) := \alpha(X_0, \ldots, X_n) \cup \beta(X_0, \ldots, X_n)$$

for sets $X_0,\dots,X_n$, respectively.

Let
$$\mathrm{OP} := \bigcup\{\mathrm{OP}(n):\ n \in \mathbb{N}\}$$
(where $\mathbb{N}$ is the set of the natural numbers).

In the present paper, for $\Omega \subseteqq \mathrm{OP}$ there is considered the class $\mathrm{FCL}(\Omega)$ of all closure operators on any finite sets A such that for each subset $T \subseteqq A$ the system $\mathrm{GEN}(T)$ of all generating sets of T is closed under each $\omega \in \Omega$, i.e., if ω is an $(n+1)$-ary operation, then the following implication holds:
$$(\forall X_0,\dots,X_n)\big(X_0,\dots,X_n \in \mathrm{GEN}(T) \implies \omega(X_0,\dots,X_n) \in \mathrm{GEN}(T)\big).$$

Throughout this paper the following basic concepts are used. For a natural number n let ω_n be the $(n+1)$-ary set-theoretic operation with
$$\omega_n(X_0,\dots,X_n) := \bigcup\{X_i \cap X_j:\ 0 \leqq i < j \leqq n\}$$
for $n \geqq 1$ and $\omega_0(X_0) := X_0$.

Let C be a closure operator on a set A and $T \subseteqq A$. Then a set I is called to be *isolated in* T if and only if $\emptyset \neq I \subseteqq T$ and $I \cap C(T \setminus I) = \emptyset$. For each natural number $n \geqq 1$ let $\mathrm{ISO}_n(T)$ be the system of all sets I such that I is isolated in T and $|I| \leqq n$.

Lemma 2.1. *For each closure operator C on a finite set A, for each subset $T \subseteqq A$ and for each natural number $n \geqq 1$ the following statements are pairwise equivalent:*

(i) *If $X_0,\dots,X_n \in \mathrm{GEN}(T)$, then $\omega_n(X_0,\dots,X_n) \in \mathrm{GEN}(T)$.*

(ii) *$|T \setminus C(W)| \leqq n$ for each maximal set $W \subseteqq T$ with $T \setminus C(W) \neq \emptyset$.*

(iii) *$\mathrm{GEN}(T)$ is equal to the system of all $U \subseteqq T$ with $U \cap I \neq \emptyset$ for each $I \in \mathrm{ISO}_n(T)$.*

(iv) *If $S_0 \subseteqq A$ is a C-set, then $|S_0 \setminus S_1| \leqq n$ for each maximal C-set $S_1 \subseteqq S_0$ with $S_0 \setminus S_1 \neq \emptyset$.*

Proof.

(i) $\implies$ (ii): Let W be a maximal subset of T such that $T \setminus C(W) \neq \emptyset$. Since W is maximal, $T \subseteqq C(W \cup \{t\})$ for $t \in T \setminus C(W)$. This implies $|T \setminus C(W)| \leqq n$. Otherwise, there exists some set $\{a_0,\dots,a_n\} \subseteqq T \setminus C(W)$ of the cardinality $n+1$. Let $X_i := W \cup \{a_i\}$ for $0 \leqq i \leqq n$. Obviously, $X_0,\dots,X_n \in \mathrm{GEN}(T)$. From (i) it follows that $\omega_n(X_0,\dots,X_n) = W \in \mathrm{GEN}(T)$, contradicting $T \setminus C(W) \neq \emptyset$.

(ii) $\implies$ (iii): Let $U \subseteqq T$ and $U \cap I \neq \emptyset$ for each $I \in \mathrm{ISO}_n(T)$. Then $U \in \mathrm{GEN}(T)$. Otherwise, $U \notin \mathrm{GEN}(T)$, i.e., $T \setminus C(U) \neq \emptyset$. Because T is finite, there exists a maximal set W with $U \subseteqq W \subseteqq T$, $T \setminus C(W) \neq \emptyset$ and

$|T \backslash C(W)| \leqq n$ by (ii), consequently. Obviously, $T \backslash C(W) \in \mathrm{ISO}_n(T)$. Therefore, $U \cap (T \setminus C(W)) \neq \emptyset$, i.e., $U \cap (T \setminus U) \neq \emptyset$, a contradiction. Similarly, if $U \in \mathrm{GEN}(T)$, then $U \cap I \neq \emptyset$ for each $I \in \mathrm{ISO}_n(T)$.

(iii) $\Longrightarrow$ (i): Let the statement (iii) be fulfilled. Then, from $X_0, \dots, X_n \in \mathrm{GEN}(T)$ it follows that $I \cap X_i \neq \emptyset$ for each $I \in \mathrm{ISO}_n(T)$, $0 \leqq i \leqq n$. Since $|I| \leqq n$, $\omega_n(X_0, \dots, X_n) \cap I \neq \emptyset$, too. Therefore, $\omega_n(X_0, \dots, X_n) \in \mathrm{GEN}(T)$ and this implies (i).

(iv) $\Longrightarrow$ (i): By (iv) and the implications above, from $X_0, \dots, X_n \in \mathrm{GEN}\big(C(T)\big)$ it follows that $\omega_n(X_0, \dots, X_n) \in \mathrm{GEN}\big(C(T)\big)$. Clearly,

$$\mathrm{GEN}(T) = \mathrm{GEN}\big(C(T)\big) \cap \{U : \ U \subseteqq T\}.$$

Consequently, from $X_0, \dots, X_n \in \mathrm{GEN}(T)$ it follows that $\omega_n(X_0, \dots, X_n) \in \mathrm{GEN}(T)$.

Obviously, (iv) follows from (ii). □

3. Ideal closure operators on semigroups

Let S be a (finite) semigroup. It is well known that there exists *the ideal closure operator on* S (cf. [14]), in symbol: J, such that for each $U \subseteqq S$, $J(U)$ is the *ideal of* S *generated by* U, i.e., $J(U) = U \cup S \cdot U \cup U \cdot S \cup S \cdot U \cdot S$.

Elements $a, b \in S$ belong to the same J*-class* if and only if $J\big(\{a\}\big) = J\big(\{b\}\big)$ (cf. [2], [14]). By Lemma 2.1, it is easy to check that $J \in FCL\big(\{\omega_n\}\big)$ if and only if for each J-class D, $|D| \leqq n$.

For each natural number $n \geqq 1$ there are characterized all finite semigroups such that their ideal closure operators belong to $\mathrm{FCL}\big(\{\omega_n\}\big)$.

For this there are considered disjunctions of equations which define classes of semigroups S (cf. [11], [13]). Let $\mathbb{X}$ be a (countable) set of variables, $\mathbb{X}^+$ be the free semigroup on $\mathbb{X}$ and $\mathcal{D} \subseteqq \mathbb{X}^+ \times \mathbb{X}^+$, i.e., $\mathcal{D}$ is a set of equations. For a semigroup S it is said that $\mathcal{D}$ *holds in* S *disjunctively*, in symbols: $S \models \mathcal{D}$, if and only if for each homomorphism φ from $\mathbb{X}^+$ into S there exists an equation $(p, q) \in \mathcal{D}$ such that the equality $\varphi(p) = \varphi(q)$ is fulfilled.

Any class $\mathfrak{V}$ of semigroups S is called to be *disjunctively defined* if and only if there exists a system $\mathfrak{A}$ of sets $\mathcal{D} \subseteqq \mathbb{X}^+ \times \mathbb{X}^+$ such that $\mathfrak{V}$ is equal to the class of all semigroups S where $S \models \mathcal{D}$ for each $\mathcal{D} \in \mathfrak{A}$, in symbols: $\mathfrak{V} = \mathrm{MOD}(\mathfrak{A})$.

Let $\mathbb{X} := \{z\} \cup \{x_i : \ i \in \mathbb{N}\} \cup \{y_i : \ i \in \mathbb{N}\}$ be a (countable) set of variables and $\mathbb{X}^+$ be the free semigroup on $\mathbb{X}$.

For natural numbers $n, k \geqq 1$ and $i = 1, 2, 3$ let

$$F_1(n,k) := \{(x_m \cdots x_0)(x_n \cdots x_0)^k z : \ 0 \leqq m \leqq n\},$$
$$F_2(n,k) := \{z(y_0 \cdots y_n)^k (y_0 \cdots y_m) : \ 0 \leqq m \leqq n\},$$
$$F_3(n,k) := \{(x_m \cdots x_0)(x_n \cdots x_0)^k z (y_0 \cdots y_n)^k (y_0 \cdots y_m) : \ 0 \leqq m \leqq n\},$$

$$\mathcal{D}_i(n,k) := \{(p,q) : \ p, q \in F_i(n,k), \ p \neq q\},$$
$$\mathcal{D}_i(n) := \bigcup \{\mathcal{D}_i(n,k) : \ 1 \leqq k \in \mathbb{N}\},$$
$$\mathfrak{A}(n) := \{\mathcal{D}_1(n), \mathcal{D}_2(n), \mathcal{D}_3(n)\}.$$

PROPOSITION 3.1. *Let S be a finite semigroup. Then for each natural number $n \geqq 1$ the following statements are equivalent:*

(i) *The ideal closure operator J on S belongs to* $\mathrm{FCL}(\{\omega_n\})$.

(ii) $S \in \mathrm{MOD}(\mathfrak{A}(n))$.

Proof.

(i) $\Longrightarrow$ (ii): Let (i) be fulfilled. Then $S \in \mathrm{MOD}(\mathfrak{A}(n))$. Otherwise, for $i = 1$ or 2 or 3 there exists some homomorphism φ from $\mathbb{X}^+$ into S such that for each $(p,q) \in \mathcal{D}_i(n)$ there holds $\varphi(p) \neq \varphi(q)$.

Let $\varphi(x_i) = s_i \in S$, $\varphi(y_i) = t_i \in S$ for $i \in \mathbb{N}$ and $w_1 = s_n \cdots s_0 \in S$, $w_2 = t_0 \cdots t_n \in S$. Because S is finite, $w_1^a = w_1^b$ with $1 \leqq a < b \in \mathbb{N}$ and $w_2^c = w_2^d$ with $1 \leqq c < d \in \mathbb{N}$. Let $k = a + c$. Consequently,

$$w_1^k = w_1^{b+c}, \qquad k < b + c, \tag{1}$$
$$w_2^k = w_2^{a+d}, \qquad k < a + d. \tag{2}$$

Let

$$T_i := \{\varphi(p) : \ p \in F_i(n,k)\} \subseteqq S.$$

From the assumptions on φ it follows that

$$|T_i| = \left|\{\varphi(p) : \ p \in F_i(n,k)\}\right| = n + 1. \tag{3}$$

It is easy to check that from (1), (2) and (3) it follows that $\mathrm{ISO}_n(T_i) = \emptyset$ with respect to the ideal closure operator of S. Therefore $\emptyset \in \mathrm{GEN}(T_i)$ and $T_i = \emptyset$ by (i) and Lemma 2.1, contradicting $T_i \neq \emptyset$.

(ii) $\Longrightarrow$ (i): Let $S \in \mathrm{MOD}(\mathfrak{A}(n))$. Furthermore, let T be a (finite) subset of S and W be a maximal set such that $W \subseteqq T$ and $T \setminus J(W) \neq \emptyset$. Then there holds $|T \setminus J(W)| \leqq n$: Otherwise, there exists some set $I = \{i_0, \ldots, i_n\} \subseteqq T$ with $|I| = n + 1$, $I \cap J(W) = \emptyset$ and $i \in J(W \cup \{j\})$ for $i, j \in I$. Then exactly three cases are possible.

Case 1:
There hold the following decompositions:

$$i_0 = s_n i_n \qquad \text{with} \quad s_n \in S\,,$$
$$i_m = s_{m-1} i_{m-1} \qquad \text{with} \quad s_{m-1} \in S \quad \text{for } 1 \leqq m \leqq n.$$

For each natural number $k \geqq 1$ one gets by repeatedly replacing i_0 by $s_n i_n$ or i_g by $s_{g-1} i_{g-1}$, $1 \leqq g \leqq n$, in the right hand sides of the above equations

$$i_0 = (s_n \cdots s_0)(s_n \cdots s_0)^k i_0\,,$$
$$i_m = (s_{m-1} \cdots s_0)(s_n \cdots s_0)^k i_0 \qquad \text{for} \quad 1 \leqq m \leqq n\,.$$

Because $S \models \mathcal{D}_1(n)$, $|I| \leqq n$, contradicting $|I| = n+1$.
Case 2:
There hold the following decompositions:

$$i_0 = i_n r_n \qquad \text{with} \quad r_n \in S\,,$$
$$i_m = i_{m-1} r_{m-1} \qquad \text{with} \quad r_{m-1} \in S \quad \text{for } 1 \leqq m \leqq n\,.$$

For each natural number $k \geqq 1$ one gets by repeatedly replacing i_0 by $i_n r_n$ or i_g by $i_{g-1} r_{g-1}$, $1 \leqq g \leqq n$, in the right hand sides of the above equations

$$i_0 = i_0 (r_0 \cdots r_n)^k (r_0 \cdots r_n)\,,$$
$$i_m = i_0 (r_0 \cdots r_n)^k (r_0 \cdots r_{m-1}) \qquad \text{for} \quad 1 \leqq m \leqq n\,.$$

Because $S \models \mathcal{D}_2(n)$, $|I| \leqq n$, contradicting $|I| = n+1$.
Case 3:
If Cases 1 and 2 do not hold, then there exist $k_0, k_1, l_0, l_1 \in I$ such that $k_0 = a_0 k_1$ or $k_0 = a_0 k_1 c_0$ with $a_0, c_0 \in S$ and $l_0 = l_1 b_0$ or $l_0 = d_0 l_1 b_0$ with $b_0, d_0 \in S$. Now let $i, j \in I$. Then $i \in J(W \cup \{k_0\})$, $k_1 \in J(W \cup \{l_0\})$, $l_1 \in J(W \cup \{j\})$. Therefore, $i = ajb$ with $a, b \in S$ and the following decompositions hold:

$$i_0 = s_n i_n r_n \qquad \text{with} \quad s_n, r_n \in S\,,$$
$$i_m = s_{m-1} i_{m-1} r_{m-1} \qquad \text{with} \quad s_{m-1}, r_{m-1} \in S \quad \text{for } 1 \leqq m \leqq n\,.$$

For each natural number $k \geqq 1$ one gets by repeatedly replacing i_0 by $s_n i_n r_n$ or i_g by $s_{g-1} i_{g-1} r_{g-1}$, $1 \leqq g \leqq n$, in the right hand sides of the above equations

$$i_0 = (s_n \cdots s_0)(s_n \cdots s_0)^k i_0 (r_0 \cdots r_n)^k (r_0 \cdots r_n)\,,$$
$$i_m = (s_{m-1} \ldots s_0)(s_n \ldots s_0)^k i_0 (r_0 \ldots r_n)^k (r_0 \ldots r_{m-1}) \qquad \text{for} \quad 1 \leqq m \leqq n\,.$$

Because $S \models \mathcal{D}_3(n)$, $|I| \leqq n$, contradicting $|I| = n+1$.
Cases 1, 2, 3 imply $J \in \mathrm{FCL}(\{\omega_n\})$ by Lemma 2.1. $\square$

4. The Class $\mathrm{FCL}(\Omega)$

For any finite set A it is proved that each closure operator C on A belongs to $\mathrm{FCL}(\Omega)$ if and only if C is the meet of suitable ideal closure operators on semigroups with the carrier A.

Using Lemma 2.1 we have the following lemma:

LEMMA 4.1. *For each natural number $n \geqq 1$ we have the following inclusion*

$$\mathrm{FCL}(\{\omega_n\}) \subsetneqq \mathrm{FCL}(\{\omega_{n+1}\}).$$

Proof. For $1 \leqq n \in \mathbb{N}$ let $A_n := \{0,\dots,n\}$ and C_n be that closure operator on A_n with $C_n(\emptyset) = \emptyset$ and $C_n(X) = A_n$ for $\emptyset \neq X \subseteqq A_n$. Then there hold $C_n \in \mathrm{FCL}(\{\omega_{n+1}\})$ and $C_n \notin \mathrm{FCL}(\{\omega_n\})$.

$C_n \in \mathrm{FCL}(\{\omega_{n+1}\})$:
For this, let T be a subset of A_n and $W \subseteqq T$ be a maximal set with $T \setminus C_n(W) \neq \emptyset$. Obviously, $|T \setminus C_n(W)| \leqq n+1$.

$C_n \notin \mathrm{FCL}(\{\omega_n\})$:
We have $\{\{0\},\dots,\{n\}\} \subseteqq \mathrm{GEN}(A_n)$ and $\omega_n(\{0\},\dots,\{n\}) = \emptyset \notin \mathrm{GEN}(A_n)$. □

PROPOSITION 4.2. *For each set $\Omega \subseteqq \mathrm{OP}$ there exists a natural number n such that*

$$\mathrm{FCL}(\Omega) = \mathrm{FCL}(\{\omega_n\}).$$

Proof.

(a) For each $\omega \in \mathrm{OP}$ there exists a natural number r such that

$$\mathrm{FCL}(\{\omega\}) = \mathrm{FCL}(\{\omega_r\}).$$

For this, let $\omega \in \mathrm{OP}(n)$, $n \in \mathbb{N}$. If $\mathrm{FCL}(\{\omega\}) = \mathrm{FCL}(\{\omega_0\})$, then there is nothing to prove. Otherwise $n \geqq 1$ and, without loss of generality, ω can be represented in the form

$$\omega(X_0,\dots,X_n) = P_0 \cup \dots \cup P_p, \qquad p \in \mathbb{N},$$

where $P_i = \bigcap\{X_q : q \in Q_i\}$ with $Q_i \subseteqq \{0,\dots,n\}$ and $|Q_i| \geqq 2$, $0 \leqq i \leqq p$. Therefore, $\omega(X_0,\dots,X_n) \subseteqq \omega_n(X_0,\dots,X_n)$ for sets $X_0,\dots,X_n$ and one gets $\mathrm{FCL}(\{\omega\}) \subseteqq \mathrm{FCL}(\{\omega_n\})$. Since $n \geqq 1$, $\mathrm{FCL}(\{\omega_n\}) \subsetneqq \mathrm{FCL}(\{\omega_{n+1}\})$ by Lemma 4.1, consequently. Then let m be the least natural number such that $\mathrm{FCL}(\{\omega\}) \subsetneqq \mathrm{FCL}(\{\omega_m\})$. Obviously, $m \geqq 2$ and there is a closure operator C on a finite set A such that $C \in \mathrm{FCL}(\{\omega_m\})$ with $C \notin \mathrm{FCL}(\{\omega\})$. Consequently, there exist a set $T \subseteqq A$ and sets $U_0,\dots,U_n \in \mathrm{GEN}(T)$ such that $\omega(U_0,\dots,U_n) \notin \mathrm{GEN}(T)$. Then by Lemma 2.1 there exists a set $I \in \mathrm{ISO}_m(T)$

such that $U_i \cap I \neq \emptyset$ for $0 \leqq i \leqq n$, $I \cap \omega(U_0, \dots, U_n) = \emptyset$. Since $U_i \cap I \neq \emptyset$ for $0 \leqq i \leqq n$, there exist a function k from $\{0, \dots, n\}$ into $\{0, \dots, m-1\}$ and a set $\{t_{k(i)} : 0 \leqq i \leqq n\} \subseteqq I$ with $t_{k(i)} \in U_i$.

Let $\omega^* \in \mathrm{OP}(m-1)$ with $\omega^*(X_0, \dots, X_{m-1}) := \omega(X_{k(0)}, \dots, X_{k(n)})$, i.e.,

$$\omega^*(X_0, \dots, X_{m-1}) = P_0{}^* \cup \dots \cup P_p{}^*$$

with $P_i^* = \bigcap\{X_{k(q)} : q \in Q_i\}$, $0 \leqq i \leqq p$. Since $I \cap \omega(U_0, \dots, U_n) = \emptyset$, $I \cap \bigcap\{U_q : q \in Q_i\} = \emptyset$ and $|\{k(q) : q \in Q_i\}| \geqq 2$, $0 \leqq i \leqq p$. Therefore, $\omega^*(X_0, \dots, X_{m-1}) \subseteqq \omega_{m-1}(X_0, \dots, X_{m-1})$ for sets $X_0, \dots, X_{m-1}$ and one gets $\mathrm{FCL}(\{\omega^*\}) \subseteqq \mathrm{FCL}(\{\omega_{m-1}\})$. Obviously, $\mathrm{FCL}(\{\omega\}) \subseteqq \mathrm{FCL}(\{\omega^*\})$ and by Lemma 4.1, it holds that $\mathrm{FCL}(\{\omega_{m-1}\}) \subsetneqq \mathrm{FCL}(\{\omega_m\})$. From the above facts it follows that

$$\mathrm{FCL}(\{\omega\}) \subseteqq \mathrm{FCL}(\{\omega^*\}) \subseteqq \mathrm{FCL}(\{\omega_{m-1}\}) \subsetneqq \mathrm{FCL}(\{\omega_m\}).$$

The assumption on m implies $\mathrm{FCL}(\{\omega\}) = \mathrm{FCL}(\{\omega_r\})$ with $r = m-1$.

(b) Let $\Omega \subseteqq \mathrm{OP}$. If $\mathrm{FCL}(\Omega) = \mathrm{FCL}(\{\omega_0\})$, then there is nothing to prove. Let $\mathrm{FCL}(\Omega) \neq \mathrm{FCL}(\{\omega_0\})$. Then by (a) there exists a set $\mathbb{N}^* \subseteqq \mathbb{N}$ with $0 \notin \mathbb{N}^* \neq \emptyset$ such that $\mathrm{FCL}(\Omega) = \mathrm{FCL}(\{\omega_r : r \in \mathbb{N}^*\})$. Let n be the least number of $\mathbb{N}^*$. Then $\mathrm{FCL}(\Omega) = \mathrm{FCL}(\{\omega_n\})$. □

Let $\mathcal{C}$ be a (nonempty) set of closure operators on A. Then there exists the closure operator C on A such that for $X \in P(A)$

$$C(X) = \bigcap\{C'(X) : C' \in \mathcal{C}\}.$$

C is called to be the *meet of* the $C' \in \mathcal{C}$, in symbols: $C = \bigwedge \mathcal{C}$.

LEMMA 4.3. *Let A be a finite set and $\mathcal{C}$ be a (nonempty) set of closure operators on A. Then for each natural number $n \geqq 1$, from $\mathcal{C} \subseteqq \mathrm{FCL}(\{\omega_n\})$ it follows that $\bigwedge \mathcal{C} \in \mathrm{FCL}(\{\omega_n\})$.*

Proof. Because A is finite it is enough to prove that from $\{C', C''\} \subseteqq \mathrm{FCL}(\{\omega_n\})$ it follows that $C := \bigwedge\{C', C''\} \in \mathrm{FCL}(\{\omega_n\})$.

Let $S_0 \subseteqq A$ be a C-set and S_1 be maximal C-set with $S_1 \subseteqq S_0$ and $S_0 \setminus S_1 \neq \emptyset$. By definition on C, $S_0 = C'(S_0) \cap C''(S_0)$. Moreover, there exist a maximal C'-set S_1' and a maximal C''-set S_1'' with $S_1' \subseteqq C'(S_0)$, $S_1'' \subseteqq C''(S_0)$ and $S_1 = S_1' \cap S_1''$.

Therefore, from $S_1' \subsetneqq T' \subseteqq C'(S_0)$ it follows that $S_0 = T' \cap S_1''$, i.e., $S_0 \subseteqq S_1''$ and from $S_1'' \subsetneqq T'' \subseteqq C''(S_0)$ it follows that $S_0 = T'' \cap S_1'$, i.e., $S_0 \subseteqq S_1'$.

Consequently, $S_1' = C'(S_0)$ or $S_1'' = C''(S_0)$. Otherwise, $S_0 = S_1$, contradicting the assumption on S_1.

Without loss of generality let $S_1' = C'(S_0)$ and $S_1 = C'(S_0) \cap S_1''$.

Then S_1'' is a maximal C''-set with $S_1'' \subsetneqq C''(S_0)$ and $C''(S_0) \setminus S_1'' \neq \emptyset$, clearly.

By Lemma 2.1, we have $|C''(S_0) \setminus S_1''| \leqq n$. Therefore, $|S_0 \setminus S_1| \leqq n$ and $C = \bigwedge\{C', C''\} \in \mathrm{FCL}(\{\omega_n\})$. □

Finally, we have the following proposition.

Proposition 4.4. *Let $1 \leqq n \in \mathbb{N}$, A be a finite set and $\mathcal{J}_n$ be the set of the ideal closure operators on all semigroups S with the carrier A such that $S \in \mathrm{MOD}(\mathfrak{A}(n))$. Then for each closure operator C on A the following statements are equivalent:*

(i) $C \in \mathrm{FCL}(\{\omega_n\})$.

(ii) *Either $C \in \mathcal{J}_n$ or C is not an ideal closure operator on any semigroup S with the carrier A, but $C = \bigwedge \mathcal{C}$ for some $\mathcal{C} \subseteqq \mathcal{J}_n$.*

Proof.

(i) $\Longrightarrow$ (ii): Let $C \in \mathrm{FCL}(\{\omega_n\})$ and $\mathfrak{H}$ be the system of all C-sets $T \subseteqq A$. Obviously, $\mathfrak{H}$ is the union of chains $\mathfrak{K} = \{A_0, \dots, A_k\}$, $0 \leqq k \in \mathbb{N}$, where

$$\emptyset = A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_k = A,$$

and by Lemma 2.1,

$$0 \neq |A_{i+1} \setminus A_i| \leqq n \qquad \text{for} \quad 0 \leqq i \leqq k-1.$$

Consequently, there exists a set $\mathcal{C} \subseteqq \mathrm{FCL}(\{\omega_n\})$ such that each $C' \in \mathcal{C}$ is a closure operator on A, where the C'-sets form a chain $\mathfrak{K}$ and $C = \bigwedge \mathcal{C}$.

It is easy to check that for each C' there exists a semigroup S with the carrier A such that

$$a_i \cdot a_j = a_j \cdot a_i = a_i \qquad \text{for} \quad a_i \in A_i,\ a_j \in A_j \setminus A_i,\ 0 \leqq i < j \leqq k$$

and

$$A_{i+1} \setminus A_i = \langle a_{i+1} \rangle \qquad \text{with} \quad {a_{i+1}}^{e_{i+1}} = a_{i+1},\ 2 \leqq e_{i+1} \leqq n+1,$$

for $0 \leqq i \leqq k-1$, where $\langle a_{i+1} \rangle$ is the subsemigroup of S generated by a_{i+1}.

Therefore, C' is the ideal closure operator on S. This fact implies (ii) by the Proposition 3.1.

By Proposition 3.1 and Lemma 4.3, (i) follows directly from (ii). □

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