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KETTENBRÜCHE ALS SUMMEN EBENSOLCHER

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ABSTRACT. We continue an investigation which showed that for any given positive integer m there exists a simple continued fraction of length m being the sum/difference of two unit fractions.

1. Introduction

In [5] Rieger used Fibonacci numbers in order to show that for any given positive integer m there exists a simple continued fraction of length m which is the sum or difference, respectively, of two unit fractions. We continue this investigation, which may be considered as a particular contribution to the extensive literature on Egyptian fractions (cf. [3; Problem D11]).

For an integer c_0 and positive integers $c_1, \dots, c_n$, we denote by

$$\langle c_0; c_1, \dots, c_m \rangle = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}}$$

the corresponding finite simple continued fraction. We shall be dealing only with continued fractions whose integer part c_0 equals 0. The following properties of continued fractions may all be found in [4]. The Muir symbol corresponding to the sequence $c_i, \dots, c_n$, which we denote by

$$C_i := C_{i,n} := [c_i, c_{i+1}, \dots, c_n]$$

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for $i = 1, 2, \dots, n$, is defined recursively by $C_{n+1,n} := [\,] := 1$, $C_{n,n} := c_n$, and

$$C_{i,n} := c_i C_{i+1,n} + C_{i+2,n}$$

for $i = 1, \dots, n-1$. Then all the $C_{i,n}$ are positive integers with $(C_{i,n}, C_{i+1,n}) = 1$ for $i = 1, \dots, n$ and, most importantly,

$$\langle 0; c_1, \dots, c_n \rangle = \frac{C_{2,n}}{C_{1,n}} .$$

Our first result can be verified immediately by applying the above observations.

THEOREM 1. *For fixed $n \geq 2$ let $c_2, \dots, c_n$ be given positive integers. Then we have for every positive integer k satisfying $(C_2 \mp 1)k \mp C_3 > 0$ that*

$$\langle 0; ((C_2 \mp 1)k \mp C_3), c_2, \dots, c_n \rangle = \frac{1}{C_2 k \mp C_3} \pm \frac{1}{(C_2 k \mp C_3)(C_2 \mp 1)} .$$

Theorem 1 shows that, given an arbitrary sequence $c_2, \dots, c_n$ of positive integers, we can find a continued fraction of length n which contains the given sequence and is the sum/difference of two unit fractions. It is not difficult to impose additional conditions; for example, since C_2 and C_3 are coprime, we can use Dirichlet's theorem to find k such that the denominator of the first unit fraction is a prime. Other properties of the denominators of the unit fractions, e.g. congruence conditions, can be obtained if we consider a few of the c_i's as variables and choose them appropriately. This will be discussed in Theorem 3.

We now give a characterization of continued fractions which are the sum/difference of two unit fractions.

THEOREM 2. *Let n and $c_1, \dots, c_n$ be positive integers. Let $C_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of C_1. Then*

(i) *$\langle 0; c_1, \dots, c_n \rangle$ is the sum of two unit fractions if and only if*

$$C_2 \mid (C_1 + p_1^{\beta_1} \cdots p_r^{\beta_r})$$

for some non-negative integers $\beta_i \leq 2\alpha_i$ $(1 \leq i \leq r)$.

(ii) *$\langle 0; c_1, \dots, c_n \rangle$ is the difference of two unit fractions if and only if*

$$C_2 \mid (C_1 - p_1^{\beta_1} \cdots p_r^{\beta_r})$$

for some non-negative integers $\beta_i \leq 2\alpha_i$ $(1 \leq i \leq r)$ with $C_1 > p_1^{\beta_1} \cdots p_r^{\beta_r}$.

The following result shows that among all continued fractions of prescribed length some can be represented by a sum/difference of unit fractions, where the denominators are restricted to a prime and an almost-prime in given arithmetic progressions. Moreover, with the exception of five numbers, all the partial quotients of the continued fractions can be chosen arbitrarily.

THEOREM 3. *Let $n \geq 5$, s, $c_6, c_7, \dots, c_n$ and u, v with $(u,s) = (v,s) = 1$ denote positive integers. If $n = 5$, the integers $c_6, \dots, c_n$ do not occur. Then there exist positive integers $c_1, \dots, c_5$ and primes p, q satisfying $p \equiv u \mod s$, $q \equiv v \mod s$ and*

$$\langle 0; c_1, c_2, \dots, c_n \rangle = \frac{1}{p} \pm \frac{1}{pq}.$$

The conditions $(u,s) = 1$ and $(v,s) = 1$ in this theorem are obviously necessary. We point out that the continued fraction $\langle 0; c_1, \dots, c_n \rangle$ need not be normalized, i.e. $c_n = 1$ may occur.

Rieger showed in [5] that his explicitly constructed continued fractions of given length are the sum/difference of two unit fractions whose denominators are coprime. By use of Hoheisel's theorem he also proved that there exist arbitrarily long continued fractions which are the sum/difference of two unit fractions with prime denominators. Rieger's approach even implies that for any given sequence $c_2, \dots, c_n$ of positive integers there exist a positive integer c_1 and primes p, q with $\langle 0; c_1, c_2, \dots, c_n, \dots \rangle = \frac{1}{p} \pm \frac{1}{q}$. Rieger used the special case of $c_2 = \dots = c_n = 1$ to prove his statement. The following result indicates the difficulty to find continued fractions of given length which are the sum/difference of two unit fractions with prime denominators.

THEOREM 4. *Let $n \geq 4$ and $c_1, \dots, c_n$ be positive integers, and let $p \leq q$ be primes. Then*

$$\langle 0; c_1, \dots, c_n \rangle = \frac{1}{p} + \frac{1}{q}$$

if and only if

$$p = \frac{1}{2}\left(C_2 - \sqrt{C_2^2 - 4C_1}\right), \qquad q = \frac{1}{2}\left(C_2 + \sqrt{C_2^2 - 4C_1}\right),$$

and

$$\langle 0; c_1, \dots, c_n \rangle = \frac{1}{p} - \frac{1}{q}$$

if and only if

$$p = \frac{1}{2}\left(\sqrt{C_2^2 + 4C_1} - C_2\right), \qquad q = \frac{1}{2}\left(\sqrt{C_2^2 + 4C_1} + C_2\right).$$

Unit fractions are continued fractions of very short length. Rieger [6] asked whether continued fractions of prescribed length can, more generally, be represented by sums/differences of two (shorter) continued fractions with prescribed lengths. The answer is "yes", and in fact we can prove somewhat more.

THEOREM 5. *Let $k \geq 3$, $m \geq 1$, $n \geq 4$, $a_4, \dots, a_k$, $b_2, \dots, b_m$ and $c_5, \dots, c_n$ denote positive integers. If $k = 3$, the integers $a_4, \dots, a_k$ do not occur, and similarly, for $m = 1$ and $n = 4$. Then there exist positive integers $a_1, a_2, a_3; b_1; c_1, c_2, c_3, c_4$ such that*

$$\langle 0; a_1, \dots, a_k \rangle = \langle 0; b_1, \dots, b_m \rangle + \langle 0; c_1, \dots, c_n \rangle .$$

Theorem 5 clearly contains a respective result for differences.

The basic concept in the proofs of Theorems 3 and 5 is to linearize the occurring diophantine equations of degree greater than one. This makes it possible to apply Dirichlet's theorem on primes in arithmetic progressions.

Let $m = 1$ in Theorem 5. Then we have $\langle 0; b_1, \dots, b_m \rangle = 1/b_1$. As it is shown in Section 4, in this case we can weaken the conditions on k and n, and particularly the corresponding proof can be simplified by a certain decomposition into partial fractions.

THEOREM 6. *Let $k \geq 2$, $n \geq 1$, $a_3, \dots, a_k$ and $c_2, \dots, c_n$ denote positive integers. If $k = 2$, the integers $a_3, \dots, a_k$ do not occur, and similarly for $n = 1$. Then there exist positive integers $a_1, a_2; b; c_1$ such that*

$$\langle 0; a_1, \dots, a_k \rangle = \frac{1}{b} + \langle 0; c_1, \dots, c_n \rangle .$$

On putting $n = 1$, $x = b$ and $y = c_1$, Theorem 6 yields the representation $\langle 0; a_1, \dots, a_k \rangle = \frac{1}{x} + \frac{1}{y}$.

In the remainder of the paper, we consider (finite or infinite) generalized continued fractions

$$\frac{\varepsilon_1}{c_1 +} \frac{\varepsilon_2}{c_2 +} \cdots := \cfrac{\varepsilon_1}{c_1 + \cfrac{\varepsilon_2}{c_2 + \ddots}} , \tag{1}$$

where $(\varepsilon_i)_{i \in \mathbb{N}}$ is a given sequence with $\varepsilon_i \in \{-1, +1\}$ (see [4] for details). Whenever the sequence $c_1, c_2, \dots$ satisfies $c_{i-1} + \varepsilon_i \geq 1$ for $i = 2, 3, \dots$, the above fraction represents a real number, and every real number has such an expansion.

First we consider finite continued fractions. Since $\frac{\varepsilon_i}{c_i} = \frac{1}{\varepsilon_i c_i}$, we may extend the Muir symbol to nonzero integers; this leads to

$$\frac{\varepsilon_1}{c_1 +} \cdots + \frac{\varepsilon_n}{c_n} = \frac{[\gamma_2 c_2, \dots, \gamma_n c_n]}{[\gamma_1 c_1, \dots, \gamma_n c_n]} \qquad (n \in \mathbb{N}), \tag{2}$$

where $\gamma_i := \prod_{j=1}^{i} \varepsilon_j$ for $i \in \mathbb{N}$. Now let $\varepsilon_1 = +1$. A continued fraction of length n can be represented in terms of n unit fractions.

THEOREM 7. *Let $\varepsilon_2, \ldots, \varepsilon_n \in \{-1, +1\}$. Then we have for any sequence $c_1, \ldots, c_n \in \mathbb{N}$ which satisfies $c_{i-1} + \varepsilon_i \geq 1$ for $i = 2, 3, \ldots, n$*

$$\frac{1}{c_1} + \frac{\varepsilon_2}{c_2} + \cdots + \frac{\varepsilon_n}{c_n} = \sum_{i=0}^{n-1} \frac{(-1)^i \gamma_{i+1}}{|[\gamma_1 c_1, \ldots, \gamma_i c_i][\gamma_1 c_1, \ldots, \gamma_{i+1} c_{i+1}]|}.$$

Hence, if $\varepsilon_i = +1$ for all i, we get a representation of a simple continued fraction with partial quotients formed by a given sequence of length n as an alternating sum of n unit fractions. If $\varepsilon_i = -1$ for $i = 2, 3, \ldots$, we find a representation of a so called reduced continued fraction (cf. [4; §42]) with partial quotients formed by a given sequence $c_1, \ldots, c_n$ with $c_i \geq 2$ as a sum of n unit fractions. Note that we do not impose any restrictions on the sequence.

Otherwise, we may interpret the formula of Theorem 7 as a representation of a given rational number as a sum of unit fractions; this coincides with the Farey series-algorithm due to Bleicher [1] (but the approach via reduced continued fractions is easier). For example, by division with remainder one gets $\frac{18}{23} = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{3} - \frac{1}{3}$, which leads to $\frac{18}{23} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} + \frac{1}{207}$.

Moreover, we easily see that every rational $\frac{a}{N}$ with integers $a = 4$ or $a = 5$ and $N > a$ can be written as a sum of three unit fractions with distinct denominators, one or three positive. The famous Erdös-Straus conjecture states that there always exists a representation as a sum of three positive unit fractions (see [3]).

Since infinite generalized continued fraction expansions converge, we may also consider infinite sequences.

THEOREM 8. *Let $(\varepsilon_i)_{i \in \mathbb{N}, i \geq 2}$ with $\varepsilon_i \in \{-1, +1\}$. Then we have for any sequence $(c_i)_{i \in \mathbb{N}}$ of positive integers which satisfies $c_{i-1} + \varepsilon_i \geq 1$ for $i = 2, 3, \ldots$*

$$\frac{1}{c_1} + \frac{\varepsilon_2}{c_2} + \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i \gamma_{i+1}}{|[\gamma_1 c_1, \ldots, \gamma_i c_i][\gamma_1 c_1, \ldots, \gamma_{i+1} c_{i+1}]|}.$$

So we get a representation of a simple continued fraction with partial quotients formed by a given infinite sequence as an alternating series of unit fractions.

Moreover, we also have a representation of a reduced continued fraction with partial quotients formed by a given infinite sequence of integers greater than 1 as a series of unit fractions.

A special example of the Muir symbol generates the Fibonacci numbers, defined by $F_1 := F_2 := 1$ and $F_{n+2} := F_{n+1} + F_n$ for $n = 1, 2, \ldots$. Since obviously $[c_1, \ldots, c_i] = F_{i+1}$ if and only if $c_j = 1$ for $j = 1, \ldots, i$, Theorems 7 and 8 yield nice formulas for the Fibonacci numbers, namely

$$\frac{F_n}{F_{n+1}} = \sum_{i=1}^{n} \frac{(-1)^i}{F_i F_{i+1}} \quad (n \in \mathbb{N}) \qquad \text{and} \qquad \frac{\sqrt{5}-1}{2} = \sum_{i=1}^{\infty} \frac{(-1)^i}{F_i F_{i+1}},$$

since $\lim\limits_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2}$.

2. Proof of Theorem 2

LEMMA. *Let the two positive integers a and b be coprime, and let $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of a. Then we have*

(i) *There is an integer $x > a/b$ such that $(bx - a) \mid x^2$ if and only if there exist integers β_i with $0 \le \beta_i \le 2\alpha_i$ $(1 \le i \le r)$ such that $b \mid (a + p_1^{\beta_1} \cdots p_r^{\beta_r})$.*

(ii) *There is an integer $1 \le x < a/b$ such that $(a - bx) \mid x^2$ if and only if there exist integers β_i with $0 \le \beta_i \le 2\alpha_i$ $(1 \le i \le r)$ and $a > p_1^{\beta_1} \cdots p_r^{\beta_r}$ such that $b \mid (a - p_1^{\beta_1} \cdots p_r^{\beta_r})$.*

Proof. We only show (i); the proof of (ii) is similar. We first assume that $(bx - a) \mid x^2$ holds for some $x > a/b$. For $bx - a = 1$ we have $b \mid (a + 1)$, and the desired property follows trivially. Therefore we may assume $bx - a \ge 2$, and there exists a prime p satisfying $p \mid (bx - a)$. For each such p we have $p \mid a$, since $p \mid x^2$. Hence $bx - a = p_1^{\beta_1} \cdots p_r^{\beta_r}$ for some non-negative β_i $(1 \le i \le r)$. It follows that $b \,\big|\, \big(a + p_1^{\beta_1} \cdots p_r^{\beta_r}\big)$ and

$$x = \frac{1}{b}\big(a + p_1^{\beta_1} \cdots p_r^{\beta_r}\big).$$

By assumption

$$p_1^{\beta_1} \cdots p_r^{\beta_r} \,\Big|\, \frac{1}{b^2}\big(a + p_1^{\beta_1} \cdots p_r^{\beta_r}\big)^2.$$

This implies $p_i^{\beta_i} \mid a^2 = p_1^{2\alpha_1} \cdots p_r^{2\alpha_r}$, and therefore $\beta_i \le 2\alpha_i$ for $1 \le i \le r$.

Now let $b \,\big|\, \big(a + p_1^{\beta_1} \cdots p_r^{\beta_r}\big)$ for some $0 \le \beta_i \le 2\alpha_i$ $(1 \le i \le r)$. We put

$$x = \frac{1}{b}\big(a + p_1^{\beta_1} \cdots p_r^{\beta_r}\big),$$

which is greater than a/b. Consequently $bx - a = p_1^{\beta_1} \cdots p_r^{\beta_r}$. Since $(a, b) = 1$, we obtain

$$(bx - a) \mid x^2 = p_1^{\beta_1} \cdots p_r^{\beta_r} \frac{1}{b^2}\big(p_1^{2\alpha_1 - \beta_1} \cdots p_r^{2\alpha_r - \beta_r} + 2p_1^{\alpha_1} \cdots p_r^{\alpha_r} + p_1^{\beta_1} \cdots p_r^{\beta_r}\big).$$

□

P r o o f o f T h e o r e m 2 . We only show (i); (ii) follows similarly by using Lemma (ii).

We have

$$\langle 0; c_1, \dots, c_m \rangle = \frac{1}{x} + \frac{1}{y} \iff C_2 xy = C_1(x+y),$$

which, by substituting $x+y=t$, is the same as $C_2 x(t-x) = C_1 t$. This implies $t = C_2 s$ for some s, and thus our initial identity turns out to be equivalent with

$$\big(C_2 x - C_1\big)s = x^2 .$$

Part (i) of the preceding lemma completes the proof of Theorem 2. □

3. Proofs of Theorems 3 and 4

P r o o f o f T h e o r e m 3 . In what follows we shall use the abbreviation $C_i := C_{i,n}$. First let $n \geq 6$. The integers C_6 and C_7 are coprime, where $n = 6$ implies that $C_7 = 1$. Hence, by Dirichlet's theorem, there exists some positive integer c_5 such that d defined by $d := C_5 = c_5 C_6 + C_7$ is a prime greater than s. Particularly we have $(d,s) = 1$. Therefore there exists some integer c_4 satisfying

$$b := C_4 = c_4 d + C_6 \equiv \pm 1 \mod s\,, \tag{3}$$

where the upper sign corresponds to the assertion of the theorem involving a sum of unit fractions, the lower one to a difference. It is clear that

$$(b,s) = 1 \qquad \text{and} \qquad (b,d) = 1\,. \tag{4}$$

By (3), (4), and the hypothesis on u and s, one gets

$$1 = (s, -ub) = \big(s, -ub - (\pm b - 1)d\big) = \big(s, (-u \mp d)b + d\big)\,.$$

Then it follows from (4) that the integers bs and $(-u \mp d)b + d$ are coprime. Hence there exists some sufficiently large positive integer m such that $mbs + (-u \mp d)b + d$ is prime, where $c_3 := ms + (-u \mp d) > 0$ holds. This yields

$$a := C_3 = c_3 b + d = mbs + (-u \mp d)b + d \in \mathbb{P}\,. \tag{5}$$

It follows by (3) that

$$a \equiv \pm(-u \mp d) + d \equiv \mp u \mod s\,. \tag{6}$$

From $d \geq 2$, $c_3 \geq 1$ and (5), we have $a > b + 1$. Since a is prime and $b \geq 2$, one gets

$$(a, b \mp 1) = 1\,. \tag{7}$$

By (3) and (7) it can easily be seen that $(a,s) = 1$. Thus, the diophantine equation $ax - sy = v - b \pm 1$ is solvable; we denote some solution by x_0, y_0.

Let us assume that x_0 and s are not coprime. Then, by (3), we have

$$1 < (s, v - b \pm 1) = (s, v).$$

This contradicts the hypothesis on v and s. Thus it is proved that $(ax_0, s) = 1$, which yields $\big(as,\ ax_0 + (b \mp 1)\big) = 1$ by (3) and (7). From Dirichlet's theorem one gets some sufficiently large integer t such that

$$ast + (ax_0 + b \mp 1) \in \mathbb{P}, \tag{8}$$

where $c_2 := x_0 + st > 0$, $k := y_0 + at > 0$, and $ac_2 - ks = ax_0 - sy_0 = v - b \pm 1$ holds. Let

$$A := ac_2 + b = ax_0 + ast + b = v \pm 1 + ks. \tag{9}$$

By (8) and (9) we have $q := A \mp 1 \in \mathbb{P}$ and $q \equiv v \mod s$.

From (6) it is clear that s divides $u \pm a$, in particular one gets

$$(A, s) \mid (u \pm a). \tag{10}$$

Here a and b are coprime, since both integers represent subsequent Muir symbols. Hence, by (9), the numbers a and A are coprime, too. By (10) and the Chinese remainder theorem, the system of congruences $x \equiv \mp a \mod A$, $x \equiv u \mod s$ is solvable; the general solution is given by $x \equiv r \mod [A, s]$ with some specific integer r. From $r \equiv \mp a \mod A$ and $(a, A) = 1$ we conclude that $(r, A) = 1$. Similarly it follows from $r \equiv u \mod s$ and $(u, s) = 1$ that $(r, s) = 1$. Collecting together, we obtain $(r, As) = 1$, and particularly $\big(r, [A, s]\big) = 1$. Let $p > 3a$ denote some prime such that $p \equiv r \mod [A, s]$. Obviously the diophantine equation

$$(A \mp 1)w - Az = a \tag{11}$$

has the general solution

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \mp a \\ \mp a \end{pmatrix} + \alpha \cdot \begin{pmatrix} A \\ A \mp 1 \end{pmatrix} \qquad (\alpha \in \mathbb{Z}).$$

Let $\alpha := (p \pm a)/A > 0$. Then one gets $w = p$, and the corresponding integer z is positive, since

$$z = \mp a + \frac{p \pm a}{A}(A \mp 1) > -a + 2a\Big(1 - \frac{1}{A}\Big) > 0 \qquad (\text{by } A \geq 2).$$

Put $y := (A \mp 1)p = pq$. Since $p \equiv u \mod s$ and $q \equiv v \mod s$, we have already checked the arithmetic properties of p and q in the theorem. In order to deduce the desired identity, we first note that

$$(Az + a)(w \pm y) \overset{(11)}{=} (A \mp 1)w(w \pm y) = y(w \pm y) = \pm y \cdot Aw.$$

This yields

$$\frac{A}{Az+a} = \frac{w \pm y}{\pm wy} = \frac{1}{w} \pm \frac{1}{y} = \frac{1}{p} \pm \frac{1}{pq}.$$

Put $c_1 := z > 0$. We have $C_1 = c_1 C_2 + C_3 = c_1(ac_2 + b) + a$, and consequently

$$\langle 0; c_1, \ldots, c_n \rangle = \frac{C_2}{C_1} = \frac{ac_2 + b}{c_1(ac_2+b)+a} \overset{(9)}{=} \frac{A}{c_1 A + a} = \frac{1}{p} \pm \frac{1}{pq}.$$

Finally let $n = 5$. It is already proved that there exist positive integers $c_1, \ldots, c_5$ such that $\langle 0; c_1, \ldots, c_5, 1 \rangle$ can be written as a sum/difference of certain unit fractions. But the continued fraction equals $\langle 0; c_1, c_2, c_3, c_4, 1 + c_5 \rangle$. This finishes the proof of Theorem 3. □

P r o o f o f T h e o r e m 4 . We provide a proof for the sum part of the theorem. For differences a similar argument works. It is easy to see that primes p and q of given type satisfy the equation. It remains to show that these are the only prime solutions.

The given equation is equivalent with

$$C_1(p+q) = C_2 pq. \tag{12}$$

Clearly $p \neq q$ in (12), because we otherwise had $2C_1 = C_2 p$ and thus $C_2 \mid 2$, which would imply $m \leq 3$. Therefore $(p+q, pq) = 1$. With $(C_1, C_2) = 1$ we obtain $C_1 = pq$ and $C_2 = p + q$.

Multiplication of (12) with C_2 yields

$$(C_2 p - C_1)(C_2 q - C_1) = C_1^2 = (pq)^2.$$

We have $(C_2 q - C_1, p) = (C_2 q - pq, p) = (C_2 q, p) = 1$ and similarly $(C_2 p - C_1, q) = 1$. Hence

$$C_2 p - C_1 = p^2 \qquad \text{and} \qquad C_2 q - C_1 = q^2.$$

This implies

$$\left(p - \frac{C_2}{2}\right)^2 = \left(\frac{C_2}{2}\right)^2 - C_1 = \left(q - \frac{C_2}{2}\right)^2, \tag{13}$$

and so we obtain the desired values for p and q. □

Remark. It is not difficult to show that (12) implies for $n \geq 7$ that $p \neq 2$ and $q \neq 2$. Consequently C_2 is even, and the number $C_2/2$ in (13) is an integer. For $n \leq 6$ this may be false: let

$$C_1 = \left[1, 1, \tfrac{p-5}{4}, 1, 2, 1\right]$$

for a prime $p > 5$ with $p \equiv 1 \mod 4$. Then (12) holds with p and $q = 2$, but C_2 is odd.

4. Proofs of Theorems 5 and 6

Proof of Theorem 5. First we prove the assertion of the theorem for $k \geq 4$, $m \geq 2$ and $n \geq 5$. The notations $A_i := A_{i,k}$, $B_i := B_{i,m}$ and $C_i := C_{i,n}$ are used to denote the Muir symbols. In particular we have $A_5 = 1$ (if $k = 4$), $B_3 = 1$ (if $m = 2$) and $C_6 = 1$ (if $n = 5$). The number B_2 will play an important role; for the sake of brevity we denote it by s. Since A_4 and A_5 are coprime, there exists some positive integer a_3 such that $s < A_3 = a_3 A_4 + A_5 \in \mathbb{P}$, hence we have $(A_3, s) = 1$. By A_3^{-1} we denote the uniquely determined integer such that $A_3 A_3^{-1} \equiv 1 \mod s^2$ and $1 \leq A_3^{-1} \leq s^2$. By the same argument as used for the prime A_3 we also can find some positive integer c_4 satisfying $C_4 = c_4 C_5 + C_6 \in \mathbb{P}$ and $(C_4, s) = 1$. This guarantees the existence of some positive integer c_3 such that

$$A_3 < C_3 = c_3 C_4 + C_5 \equiv A_3^{-1} B_3 \mod s^2. \tag{14}$$

Repeating our argument a third time, we have some positive integer a_2 with

$$s^2 C_3 + C_4 < A_2 = a_2 A_3 + A_4 \in \mathbb{P}. \tag{15}$$

From $(A_3^{-1} B_3, s) = 1$ and (14) we conclude that C_3 and s are coprime. Thus the linear equation

$$g s^2 - h C_3 = C_4 + A_3^{-1} s - A_2 A_3^{-1} C_3 \tag{16}$$

with unknowns g, h is solvable. Its general solution is given by $g = g_0 + \alpha C_3$, $h = h_0 + \alpha s^2$, where g_0, h_0 are some specific numbers and α runs through all integers. There is exactly one integer α such that the inequalities

$$C_4 < \alpha s^2 C_3 + h_0 C_3 + C_4 \leq C_4 + s^2 C_3 \overset{(15)}{<} A_2 \tag{17}$$

hold simultaneously. Using this α, put $c_2 := h_0 + \alpha s^2$. From $C_3 > 0$ and the left-hand inequality in (17) we obtain $c_2 > 0$, and

$$C_2 = c_2 C_3 + C_4. \tag{18}$$

In (16) we may replace h by c_2 and g by the corresponding number $g_0 + \alpha C_3$. This yields

$$C_2 = g s^2 + A_2 A_3^{-1} C_3 - A_3^{-1} s \equiv A_2 A_3^{-1} C_3 - A_3^{-1} s \mod s^2,$$

and consequently $A_2 C_3 - A_3 C_2 \equiv s \mod s^2$. Therefore one gets

$$t := \frac{A_2 C_3 - A_3 C_2}{s} \equiv 1 \mod s, \qquad \text{where} \quad (s,t) = 1. \tag{19}$$

It follows from (17) and (18) that $C_2 < A_2$; using (15) we obtain $(A_2, C_2) = 1$. Obviously, A_2 and A_3 are coprime; thus we have $(A_2, A_2 C_3 - A_3 C_2) = 1$.

Applying $(s,t) = 1$ from (19), it follows that $(A_2 s, t) = 1$. Thus some positive integer z exists satisfying

$$A_2 sz + A_3 \equiv 0 \mod t. \tag{20}$$

Furthermore, by (14), we get

$$(A_2 sz + A_3) \cdot (C_2 sz + C_3) \equiv A_3 C_3 \equiv B_3 \mod s.$$

By (20), the left-hand side of this congruence is divisible by t. Using the congruence in (19), we find some integer x such that the identity

$$\frac{(A_2 sz + A_3) \cdot (C_2 sz + C_3)}{t} = B_3 + sx \tag{21}$$

holds. In order to show $x > 0$ we first apply $C_2 < A_2$ and $A_3 < C_3$ (from (14)) to realize that $A_2 C_3 - A_3 C_2 > 0$. This even proves $A_2 C_3 - A_3 C_2 \geq s$, and consequently $t \geq 1$. Obviously, one has $A_2 C_3 > 0$, $A_3 C_2 > 0$, $C_2 \geq C_3$ and $s = B_2 \geq B_3$. Altogether we find that

$$\frac{(A_2 sz + A_3)(C_2 sz + C_3)}{t} \geq s \cdot \frac{(A_2 sz + A_3)(C_2 sz + C_3)}{A_2 C_3} > \frac{s^3 z^2 C_2}{C_3} \geq s \geq B_3,$$

and this implies $x > 0$ in (21). By the definition of t in (19), the identity in (21) takes the form

$$s(A_2 sz + A_3)(C_2 sz + C_3) - B_3(A_2 C_3 - A_3 C_2) = s(A_2 C_3 - A_3 C_2)x, \tag{22}$$

or, as can be derived by straightforward computations,

$$\begin{aligned} s(A_2 sz + A_3)(C_2 sz + C_3) + C_2(sx + B_3)(A_2 sz + A_3) \\ = A_2(sx + B_3)(C_2 sz + C_3), \\ \frac{s}{sx + B_3} + \frac{C_2}{C_2 sz + C_3} = \frac{A_2}{A_2 sz + A_3}. \end{aligned} \tag{23}$$

Finally, put $b_1 := x$ and $a_1 := c_1 := sz$. Then we obtain, using $s = B_2$,

$$sx + B_3 = B_1, \qquad C_2 sz + C_3 = C_1, \qquad A_2 sz + A_3 = A_1.$$

Putting these terms into (23), we have proved the theorem for $k \geq 4$, $m \geq 2$ and $n \geq 5$.

It remains to consider the cases when $k = 3$ or $m = 1$ or $n = 4$. Putting $a_4 := 1$ or $b_2 := 1$ or $c_5 := 1$, respectively, the assertion of the theorem follows by normalization of the corresponding continued fractions, as indicated in the proof of Theorem 3. □

Proof of Theorem 6. It suffices to consider the case when $k \geq 3$ and $n \geq 2$. By Dirichlet's theorem, an integer a_2 exists such that $A_2 = a_2A_3 + A_4 \in \mathbb{P}$ and $A_2 > \max\{C_2; A_3C_2/C_3\}$. In the proof of Theorem 5 we have solved the diophantine equation (22) with unknowns x and z. Now we consider the simpler equation

$$(A_2z + A_3)(C_2z + C_3) = (A_2C_3 - A_3C_2)x. \tag{24}$$

By $(A_2, A_2C_3 - A_3C_2) = 1$ and $A_2C_3 - A_3C_2 > 0$, the linear equation $(A_2C_3 - A_3C_2)y - A_2z = A_3$ is solvable with positive integers y, z. Hence, a solution of (24) is given by z and $x := (C_2z + C_3)y$. To finish the proof we still have to rearrange the equation in (24):

$$\frac{A_2}{A_2z + A_3} = \frac{1}{x} + \frac{C_2}{C_2z + C_3}.$$

With $a_1 := c_1 := z$ and $A_1 = a_1A_2 + A_3$, $C_1 = c_1C_2 + C_3$, the theorem is proved. □

5. Proofs of Theorems 7 and 8

Proof of Theorem 7. Every real number $\xi_0 \in [0,1)$ has a representation (1), which can be computed by

$$\xi_{i-1} = c_{i-1} + \frac{\varepsilon_i}{\xi_i} \qquad \text{with} \quad \xi_i > 1 \ \ (i = 1, 2, \dots). \tag{25}$$

This stops if some ξ_j is an integer, then $c_j = \xi_j$ and ξ_0 is a rational. Otherwise the expansion is infinite. But since $1 = 2 - \frac{1}{2}{}_{-}\frac{1}{2}{}_{-}\dots$ (where we may write ${}_{-}\frac{1}{c_i}$ instead of ${}_{+}\frac{-1}{c_i}$) an infinite continued fraction represents not necessarily an irrational. Moreover, every rational can be written as a certain infinite continued fraction. So, in general, the expansion is not uniquely determined. By (25) we have $c_{i-1} + \varepsilon_i \geq 1$ for $i = 2, 3, \dots$. The convergents to ξ are given by (2) (note, that the condition $c_{i-1} + \varepsilon_i \geq 1$ of the continued fraction expansion ensures that $[\gamma_1c_1, \dots, \gamma_ic_i]$ never vanishes). Moreover, we have for two consecutive convergents

$$\frac{[\gamma_2c_2, \dots, \gamma_{i+1}c_{i+1}]}{[\gamma_1c_1, \dots, \gamma_{i+1}c_{i+1}]} = \frac{[\gamma_2c_2, \dots, \gamma_ic_i]}{[\gamma_1c_1, \dots, \gamma_ic_i]} + \frac{(-1)^i\gamma_{i+1}}{|[\gamma_1c_1, \dots, \gamma_ic_i][\gamma_1c_1, \dots, \gamma_{i+1}c_{i+1}]|}$$
$$(i \in \mathbb{N}).$$

So the distance between two consecutive convergents is a unit fraction. Inductively we get the representation of Theorem 7 via the sequence of convergents. □

Proof of Theorem 8. If $\varepsilon_i = \varepsilon \in \{-1,+1\}$ for all $i = 2,3,\dots$, then the denominators of the convergents form a strictly increasing sequence of integers. This is trivial for simple continued fractions. For reduced continued fractions induction on k proves this by use of

$$\begin{aligned} &\big|[c_1,-c_2,\dots,(-1)^i c_{i+1}]\big| \\ &= \big|(-1)^i c_{i+1}[c_1,-c_2,\dots,(-1)^{i-1}c_i] + [c_1,-c_2,\dots,(-1)^{i-2}c_{i-1}]\big| \\ &\geq 2\big|[c_1,-c_2,\dots,(-1)^{i-1}c_i]\big| - \big|[c_1,-c_2,\dots,(-1)^{i-2}c_{i-1}]\big|. \end{aligned}$$

Since $[\,] = 1$, $[c_1] = c_1 \geq 2$ it follows that $\big|[c_1,-c_2,\dots,(-1)^{i-1}c_i]\big| \geq i+1$. Hence the resulting series converges. Otherwise, when the ε_i take on both values for $i \to \infty$, $[\gamma_1 c_1,\dots,\gamma_i c_i]$ can be very small for some i. But Blumer [2] showed that

$$[\gamma_1 c_1,\dots,\gamma_i c_i] \xrightarrow{i\to\infty} \infty$$

and even, that no value can be taken twice. Thus, also in the general case the series converges, which proves Theorem 8. □

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