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OSCILLATION OF FIRST ORDER LINEAR RETARDED EQUATIONS

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ABSTRACT. This paper deals with oscillatory properties of linear differential equations with distributed delay. Existence of positive solution is also treated.

1. Introduction

We consider first order linear functional equation of the form

\[ \dot{x}(t) + \int_{0}^{\sigma(t)} x(t-s) \, dr(t,s) = 0, \quad t \geq t_0, \]  

where the integral is in the sense of Riemann-Stieltjes, under the standing hypotheses:

(H1) \( \sigma: [t_0, \infty) \to (0, \infty) \) is continuous and \( \lim_{t \to \infty} (t - \sigma(t)) = \infty; \)

(H2) \( r(t,0) = 0 \) for \( t \in [t_0, \infty); \)

(H3) \( r(t, \sigma(t)) : [t_0, \infty) \to \mathbb{R} \) is continuous;

(H4) \( r(t,s) \) is nondecreasing with respect to \( s \) for \( s \in [0, \sigma(t)]. \)

By a solution of equation (1) we mean a continuously differentiable function \( x \) for all \( t \) large enough, which eventually satisfies equation (1). A solution of equation (1) is called oscillatory if it has arbitrarily large zeros and otherwise it is called nonoscillatory. In this paper no solutions are considered for which \( x = 0 \) eventually.

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Our aim is to obtain sufficient conditions for the oscillation of all solutions of equation (1). We also establish conditions under which equation (1) has at least one nonoscillatory solution.

The problem of oscillation and nonoscillation for retarded differential equations has received considerable attention in recent years; see e.g. [4], [5], [8], [11] and the references cited therein. However, most of the works on the subject has been focused on the differential equations with a discrete delay and very little has been published on the equations with a distributed delay. For some results we refer to [1], [2], [5], [6].

The present paper is an attempt to make a study of oscillatory properties of the differential equation of the form (1). Our study here is related to the book of Myskis [6] in which a systematic investigation of the properties of equation (1) is carried out.

2. Main theorems

The following lemma will be useful in the proof of the main result.

**Lemma 1.** Suppose that there exists a continuous nondecreasing function \( \alpha(t) \) such that

\[
0 < \alpha(t) \leq \sigma(t) \quad \text{for} \quad t \geq t_0,
\]

\[
\liminf_{t \to \infty} \int_{t-\alpha(t)}^{t} \left[ r(s, \sigma(s)) - r(s, \alpha(s)) \right] ds > 0
\]

and let \( x(t) \) be a nonoscillatory solution of equation (1). Then there exists a \( T_0 \geq t_0 \) such that \( x(t - \alpha(t))/x(t) \) is bounded on \( [T_0, \infty) \).

**Proof.** Without loss of generality we may assume that \( x(t) \) is positive for all large \( t \). Then in view of conditions \( (H_1)-(H_4) \), equation (1) and inequalities \( t - \sigma(t) \leq t - s \leq t, \ x(t) \) must eventually be monotonically decreasing. We have

\[
-\dot{x}(t) \geq \int_{\alpha(t)}^{\sigma(t)} x(t - s) \, dr(t, s) \geq [r(t, \sigma(t)) - r(t, \alpha(t)) ] x(t - \alpha(t))
\]

for all large \( t \). According to condition (2) there exists a \( T_0 \geq t_0 \) and \( \varepsilon > 0 \) such that

\[
\int_{t-\alpha(t)}^{t} \left[ r(s, \sigma(s)) - r(s, \alpha(s)) \right] ds \geq 2\varepsilon \quad \text{for} \quad t \in [T_0, \infty).
\]
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Then for arbitrary \( t \geq T_0 \) we can find a \( t^* > t \) such that

\[
\int_{t^* - \alpha(t^*)}^{t} [r(s, \sigma(s)) - r(s, \alpha(s))] \, ds \geq \varepsilon
\]

and

\[
\int_{t}^{t^*} [r(s, \sigma(s)) - r(s, \alpha(s))] \, ds \geq \varepsilon.
\]

With regard to inequality (3) and decreasing character of \( x(t) \) we get

\[
x(t^* - \alpha(t^*)) - x(t) \geq x(t - \alpha(t)) \int_{t^* - \alpha(t^*)}^{t} [r(s, \sigma(s)) - r(s, \alpha(s))] \, ds
\]

\[
\geq \varepsilon x(t - \alpha(t)),
\]

\[
x(t) - x(t^*) \geq x(t^* - \alpha(t^*)) \int_{t}^{t^*} [r(s, \sigma(s)) - r(s, \alpha(s))] \, ds
\]

\[
\geq \varepsilon x(t^* - \alpha(t^*)),
\]

\( t \geq T_0 \). Combining the above inequalities we obtain

\[
x(t) \geq \varepsilon^2 x(t - \alpha(t)) \quad \text{for} \quad t \in [T_0, \infty).
\]

The proof is complete. \( \square \)

**Theorem 1.** Suppose that (2) holds and

\[
\limsup_{t \to \infty} \int_{t - \sigma(t)}^{t} r(s, \sigma(s)) \, ds < \infty, \quad (4)
\]

\[
-\lambda + \liminf_{t \to \infty} \frac{\int_{0}^{\sigma(t)} \exp\left(\lambda \int_{s}^{t} r(\xi, \sigma(\xi)) \, d\xi\right) \, dr(t, s)}{r(t, \sigma(t))} > 0 \quad (5)
\]

for all \( \lambda > 0 \). Then all solutions of equation (1) are oscillatory.

**Proof.** Assume, for the sake of contradiction, that equation (1) has a solution \( x \) which is positive for all large \( t \). Set

\[
\Lambda = \{\lambda > 0 : \dot{x}(t) + \lambda r(t, \sigma(t)) x(t) \leq 0 \text{ for all large } t\}.
\]
By using the decreasing nature of $x$ on $[t_1, \infty)$, where $t_1 \geq t_0$ is sufficiently large, it follows from equation (1) that

$$
\dot{x}(t) + x(t) \int_0^{\sigma(t)} dr(t, s) \leq 0,
$$

$$
\dot{x}(t) + r(t, \sigma(t))x(t) \leq 0, \quad t \geq t_1,
$$

which means that $1 \in \Lambda$ and $\Lambda$ is a subinterval of $(0, \infty)$. Next we will prove that $\sup \Lambda < \infty$. By Lemma 1, $x(t - \alpha(t))/x(t)$ is bounded on $[T_0, \infty)$, $T_0 \geq t_1$. So

$$
x(t - \alpha(t))/x(t) \leq c \quad \text{for} \quad t \geq T_0, \quad (6)
$$

where constant $c > 0$ can be chosen so large that $\exp(kc) > c$, where

$$
0 < k \leq \int_{t - \alpha(t)}^t [r(s, \sigma(s)) - r(s, \alpha(s))] \, ds \quad \text{for} \quad t \geq t_2 \geq T_0.
$$

We now claim that $\sup \Lambda \leq c$. Otherwise $c \in \Lambda$ and we obtain

$$
\frac{d}{dt} \left[ x(t) \exp \left( c \int_{t_2}^t r(s, \sigma(s)) \, ds \right) \right]
$$

$$
= [\dot{x}(t) + cr(t, \sigma(t))x(t)] \exp \left( c \int_{t_2}^t r(s, \sigma(s)) \, ds \right) \leq 0,
$$

since $c \in \Lambda$. This implies that the function

$$
x(t) \exp \left( c \int_{t_2}^t r(s, \sigma(s)) \, ds \right)
$$

is decreasing on $[t_2, \infty)$. Hence

$$
x(t - \alpha(t)) \exp \left( c \int_{t_2}^{t-\alpha(t)} r(s, \sigma(s)) \, ds \right) \geq x(t) \exp \left( c \int_{t_2}^t r(s, \sigma(s)) \, ds \right),
$$

$$
x(t - \alpha(t)) \geq x(t) \exp \left( c \int_{t-\alpha(t)}^{t} r(s, \sigma(s)) \, ds \right) \geq x(t) \exp(kc) > cx(t), \quad t \geq t_3 \geq t_2.
$$
Then \( x(t - \alpha(t)) > cx(t) \) for \( t \geq t_3 \), which is a contradiction to (6). Thus \( \sup \Lambda \leq c < \infty \). Set \( \lambda^* = \sup \Lambda \) and let \( \mu \) be an arbitrary number in the interval \( (0, \lambda^*) \). Then

\[
\lambda^* - \mu = \beta \in \Lambda
\]

and there exists a \( T_1 \geq t_2 \) such that

\[
x(t) + \beta r(t, \sigma(t)) x(t) \leq 0 \quad \text{for} \quad t \geq T_1.
\]

So for any \( t, s \) with \( t \geq T_1 \) and \( 0 \leq s \leq \sigma(t) \) we have

\[
\frac{x(t - s) - x(t)}{x(t)} = \exp \left( -\ln \frac{x(t)}{x(t - s)} \right) = \exp \left( - \int_{t-s}^{t} \frac{\dot{x}(\xi)}{x(\xi)} \, d\xi \right)
\]

\[
\geq \exp \left( \beta \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right),
\]

that is

\[
x(t - s) \geq x(t) \exp \left( \beta \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right).
\]

From equation (1) and the above inequality it follows that for \( t \geq T_1 \),

\[
0 \geq \dot{x}(t) + \left[ \int_{0}^{\sigma(t)} \exp \left( \beta \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right) \, dr(t, s) \right] x(t)
\]

\[
= \dot{x}(t) + \frac{\int_{0}^{\sigma(t)} \exp \left( \beta \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right) \, dr(t, s)}{r(t, \sigma(t))} \, r(t, \sigma(t)) x(t).
\]

According to (7) we claim that

\[
\liminf_{t \to \infty} \frac{\int_{0}^{\sigma(t)} \exp \left( \beta \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right) \, dr(t, s)}{r(t, \sigma(t))} \leq \lambda^*.
\]

(8)

Otherwise, there exist a \( \lambda_1^* > \lambda^* \) and \( T_2 \geq T_1 \) such that

\[
\frac{\int_{0}^{\sigma(t)} \exp \left( \beta \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right) \, dr(t, s)}{r(t, \sigma(t))} \geq \lambda_1^* \quad \text{for all} \quad t \geq T_2,
\]

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and therefore (7) yields that

\[ 0 \geq \dot{x}(t) + \lambda_1^* r(t, \sigma(t))x(t) \quad \text{for} \quad t \geq T_2. \]

Hence \( \lambda_1^* \in \Lambda \), which contradicts the hypothesis that \( \lambda_1^* > \lambda^* \). Thus (8) has been established. Finally (8) implies that

\[
-\lambda^* + \liminf_{t \to \infty} \frac{\int_0^{\sigma(t)} \exp \left[ (\lambda^* - \mu) \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right] \, dr(t, s)}{r(t, \sigma(t))} \leq 0.
\]

As \( \mu \to 0 \) and with regard to (4) we obtain

\[
-\lambda^* + \liminf_{t \to \infty} \frac{\int_0^{\sigma(t)} \exp \left( \lambda^* \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right) \, dr(t, s)}{r(t, \sigma(t))} \leq 0,
\]

which contradicts (5) and completes the proof of the theorem.

\[ \square \]

**COROLLARY 1.** Suppose that (2), (4) hold and

\[
\liminf_{t \to \infty} \frac{\int_0^{\sigma(t)} \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \, dr(t, s)}{r(t, \sigma(t))} > \frac{1}{e}.
\]

Then all solutions of equation (1) are oscillatory.

**Proof.** Since

\[
\min_{\lambda > 0} \frac{1}{\lambda} e^{\lambda a} = e a, \quad a > 0,
\]

we have

\[
\min_{\lambda > 0} \frac{1}{\lambda} \exp \left( \lambda \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi \right) = e \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi.
\]
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\[ t \geq T_0, \quad 0 \leq s \leq \sigma(t). \]  

Then for \( \lambda > 0 \) we get

\[
- \lambda + \liminf_{t \to \infty} \frac{\int_0^{\sigma(t)} \exp\left( \int_{t-s}^t r(\xi, \sigma(t)) \, d\xi \right) \, dr(t, s)}{r(t, \sigma(t))} 
\]

\[
= \lambda \left[ -1 + \liminf_{t \to \infty} \frac{\int_0^{\sigma(t)} \int_{t-s}^t r(\xi, \sigma(t)) \, d\xi \, dr(t, s)}{r(t, \sigma(t))} \right] \]

\[
\geq \lambda \left[ -1 + \liminf_{t \to \infty} \frac{\int_0^{\sigma(t)} \int_{t-s}^t r(\xi, \sigma(t)) \, d\xi \, dr(t, s)}{r(t, \sigma(t))} \right] > 0 ,
\]

so the condition (5) is satisfied and we can apply Theorem 1.

**Lemma 2.** If \( v \) is a positive solution of the inequality

\[ \dot{v}(t) + \int_0^{\sigma(t)} v(t - s) \, dr(t, s) \leq 0 , \quad (9) \]

then there exists a positive solution \( x \) of equation (1) with

\[ x(t) \leq v(t) \quad \text{for all large } t . \]

**Proof.** Assume that \( v \) is a positive solution of (9) on \([t_1, \infty)\). Then

\[ \dot{v}(t) \leq - \int_0^{\sigma(t)} v(t - s) \, dr(t, s) \]

and because of \((H_1)-(H_4)\), we have

\[ \dot{v}(t) < 0 \quad \text{for } t \geq T_0 , \]

i.e. \( v \) is strictly decreasing on \([T_0, \infty)\), where \( T_0 \geq t_1 \) is sufficiently large. From (9) we obtain

\[ v(t) \geq \int_0^{\infty} \int_{t-s}^t v(u - s) \, dr(u, s) \, du , \quad t \geq T_0 . \]
We define the set $S \subset C_{\text{loc}}([T_0, \infty), \mathbb{R})$ of functions $x$ which satisfy the inequalities

$$0 \leq x(t) \leq v(t) \quad \text{for} \quad t \geq T_0.$$ 

$F: S \to C_{\text{loc}}([T_0, \infty), \mathbb{R})$ is the operator which is defined by

$$F(x)(t) = \begin{cases} \int_0^\infty x(u-s) \, dr(u,s) \, du & \text{for} \quad t \geq T_1, \\ v(t) - v(T_1) + F(x)(T_1) & \text{for} \quad t \in [T_0, T_1], \end{cases}$$

where $T_1 > T_0$ is such that $t - \sigma(t) \geq T_0$ for $t \geq T_1$.

If $x \in S$, we have

$$0 \leq F(x)(t) = \int_0^\infty x(u-s) \, dr(u,s) \, du \leq v(t), \quad t \geq T_1.$$ 

Thus $F(S) \subset S$. We note that $S$ is a nonempty closed convex subset of $C_{\text{loc}}([T_0, \infty), \mathbb{R})$ and the operator $F$ is continuous. The functions belonging to the set $F(S)$ are equicontinuous on every compact subinterval of $[T_0, \infty)$. Then according to the Schauder-Tychonoff fixed point theorem (cf., e.g. [3; p. 231]), $F$ has an element $x \in S$ such that $x = F(x)$. It is easy to see that $x$ satisfies equation (1) on $[T, \infty)$.

Now we show that $x$ is positive on $[T_1, \infty)$. Obviously $v(t) > v(T_1)$ on $[T_0, T_1)$, $x$ is nonnegative on $[T_1, \infty)$, $x(T_1) = F(x)(T_1) > 0$ and moreover from equation (1) it follows that $x$ is decreasing on $[T_1, \infty)$. Let $T_2 \in (T_1, \infty)$ be the first point in which $x(T_2) = 0$. Then by equation (1) we have

$$\dot{x}(T_2) = - \int_0^{\sigma(T_2)} x(T_2-s) \, dr(T_2,s) < 0.$$ 

By decreasing character of $x$ we always have $x = 0$ on $[T_2, \infty)$, which gives $\dot{x}(T_2) = 0$. This contradiction proves that $x$ has no zeros on $[T_2, \infty)$ and so $x$ is positive on $[T_1, \infty)$. The proof is complete. □

**THEOREM 2.** Suppose that for some $T_0 > t_0$,

$$-\lambda + \sup_{t \geq T_0} \left( \int_0^{\sigma(t)} \exp \left( \lambda \int_{t-s}^t r(\xi, \sigma(\xi)) \, d\xi \right) \, dr(t,s) \right) \leq 0 \quad (10)$$

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for some $\lambda > 0$. Then there exists a positive solution $x$ of equation (1) with
\[ x(t) \leq \exp\left(-\lambda \int_{t_0}^{t} r(s, \sigma(s)) \, ds\right) \quad \text{for all large } t. \]

Proof. Set
\[ v(t) = \exp\left(-\lambda \int_{t_0}^{t} r(s, \sigma(s)) \, ds\right), \quad t \geq t_0. \]
Then by condition (10) we obtain for $t \geq T_0$,
\[
\begin{aligned}
\dot{v}(t) + \int_{0}^{\sigma(t)} v(t-s) \, dr(t,s) &= -\lambda r(t, \sigma(t)) v(t) + \int_{0}^{\sigma(t)} \exp\left(-\lambda \int_{t_0}^{t-s} r(\xi, \sigma(\xi)) \, d\xi\right) \, dr(t,s) \\
&= v(t) \left[-\lambda r(t, \sigma(t)) + \int_{0}^{\sigma(t)} \exp\left(\lambda \int_{t-s}^{t} r(\xi, \sigma(\xi)) \, d\xi\right) \, dr(t,s) \right] \\
& \leq r(t, \sigma(t)) v(t) \left[-\lambda + \sup_{t \geq T_0} \int_{t-s}^{t} \frac{\exp\left(\lambda \int_{t_0}^{\sigma(t)} r(\xi, \sigma(\xi)) \, d\xi\right) \, dr(t,s)}{r(t, \sigma(t))} \right] \leq 0,
\end{aligned}
\]
and consequently $v$ is a positive solution of inequality (9) on $[T_0, \infty)$. The conclusion of the theorem follows by applying Lemma 2. □

**Corollary 2.** Suppose that for some $T_0 > t_0$,
\[
\sup_{t \geq T_0} \int_{t-\sigma(t)}^{t} r(s, \sigma(s)) \, ds \leq \frac{1}{e}. \tag{11}
\]
Then there exists a positive solution $x$ of equation (1) with
\[ x(t) \leq \exp\left(-e \int_{t_0}^{t} r(s, \sigma(s)) \, ds\right) \quad \text{for all large } t. \]

Proof. The condition (11) implies (10) with $\lambda = e$. Hence we can apply Theorem 2. □
COROLLARY 3. Consider the equation
\[ \dot{x}(t) + p \int_0^t x(t-s) \, ds = 0, \quad t \geq t_0, \]  
(12)
where \( p \in (0, \infty) \), \( \sigma \in (0, \infty) \). All solutions of equation (12) are oscillatory if and only if
\[ e^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1 > 0 \quad \text{for all} \quad \lambda > 0. \]  
(13)

Proof. Assume that (13) holds. It is easy to see that conditions (2) and (4) are satisfied. Further for an arbitrary \( \lambda > 0 \) we get
\[ -\lambda + \liminf_{t \to \infty} \frac{\int_0^\sigma \exp(\lambda \int_0^t p\sigma \, d\xi) \, ds}{\int_0^\sigma \exp(p\sigma \lambda) \, ds} = -\lambda + \frac{1}{\sigma} \int_0^\sigma \exp(p\sigma \lambda) \, ds \]
\[ = -\lambda + \frac{1}{p\sigma^2\lambda}(e^{p\sigma^2\lambda} - 1) = \frac{1}{p\sigma^2\lambda}(e^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1) > 0. \]
So condition (5) is fulfilled and we can apply Theorem 1.

Let us now suppose that
\[ e^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1 \leq 0 \quad \text{for some} \quad \lambda > 0. \]
We obtain
\[ -\lambda + \sup_{\sigma \geq \sigma} \frac{\int_0^\sigma \exp(\lambda \int_0^t p\sigma \, d\xi) \, ds}{\int_0^\sigma \exp(p\sigma \lambda) \, ds} = \frac{1}{p\sigma^2\lambda}(e^{p\sigma^2\lambda} - p\sigma^2\lambda^2 - 1) \leq 0. \]
Thus condition (10) is satisfied and Theorem 2 ensures the existence of a nonoscillatory solution of equation (12). This completes the proof.

In the next remark we establish an explicit sufficient condition for all solutions of equation (12) to be oscillatory.

Remark. Suppose that
\[ e^{p\sigma^2} \geq 2. \]  
(14)
Then all solutions of equation (12) are oscillatory.

To prove above assertion, consider function
\[ G(\lambda) = e^{a\lambda} - a\lambda^2 - 1, \quad \text{where} \quad a = p\sigma^2, \quad \lambda \geq 0. \]
We obtain \( G(0) = 0, \ G'(\lambda) = a(e^{a\lambda} - 2\lambda) \). Using the inequality \( e^{a\lambda} \geq a e \lambda \) for \( \lambda \geq 0 \) and (14) we get
\[ G'(\lambda) \geq a\lambda(a e - 2) \geq 0 \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad G'(0) = a > 0. \]
The above relations implicate that
\[ G(\lambda) > 0 \quad \text{for} \quad \lambda > 0, \]
i.e. condition (13) is satisfied.
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EXAMPLE. Consider the equation

$$\dot{x}(t) + \frac{1}{2} \int_{0}^{\pi} x(t-s) \, ds = 0, \quad t \geq \pi. \quad (15)$$

Since condition (14) is fulfilled, all solutions of equation (15) are oscillatory. For example such solutions are functions $\sin t$ and $\cos t$.

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