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A NOTE ON TWO CIRCUMFERENCE GENERALIZATIONS OF CHVÁTAL'S HAMILTONICITY CONDITION

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ABSTRACT. In his book [BOLLOBÁS, B.: *Extremal graph theory*, Academic Press, London-New York-San Francisco, 1978] the author asks about possible generalizations of Chvátal's well-known hamiltonicity condition [CHVÁTAL, V.: *On hamilton's ideals*, J. Combin. Theory Ser. B **12** (1972), 163–168]. For $c = 3$ and 4 this follows directly from 2-connectivity. However, Häggkvist [Personal communication with J. A. Bondy] found counterexamples for any $c \geq 7$. In this paper we treat the remaining cases and show that for $c = 5$ such generalization is possible while for $c = 6$ we give counterexamples. Moreover, we show that some circumference generalization of Chvátal's condition for any c is even possible.

1. Introduction

Several hamiltonicity conditions were actually proved in terms of circumference (see [BChS], [Bon] for example). The *circumference* $c(G)$ of a graph G is the length of its longest cycle. The next result is one of the well-known hamiltonicity conditions.

THEOREM 1. (Chvátal [Ch]) *Let G be a graph of order $n \geq 3$ with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. If $d_i \leq i < n/2$ implies $d_{n-i} \geq n-i$, then G is hamiltonian.*

In his book "Extremal graph theory", Bollobás asks about the following circumference generalization of the previous theorem.

PROBLEM 1. (Bollobás [Bol]) Let G be a 2-connected graph of order n with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose $3 \leq c \leq n$ and $d_i \leq i < c/2$ implies $d_{n-i} \geq c - i$. Does it follow that $c(G) \geq c$?

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For $c = 3$ and 4 the positive answer to the above problem follows directly from the 2-connectivity, and R. Häggkvist [H] found counterexamples for any c, $7 \le c < n$. In this paper we treat the remaining cases and show that for $c = 5$ the problem has a positive answer, while for $c = 6$ we give counterexamples. Thus there are only three values of c for which Theorem 1 extends to a circumference condition. However, we show that some circumference generalization of Chvátal's condition for any c is even possible.

2. Results

First, let us investigate the case $c = 5$ in Problem 1. In what follows we characterize all 2-connected graphs with the circumference less than 5. The following result plays an important role in this task.

LEMMA 1. (Bondy-Lovász [BL]) *Let $S = \{x, y\}$ be a set of two vertices in a 2-connected graph G. Then exactly one of the following two statements is true:*

(i) *The cycles through S generate the cycle space of G.*

(ii) *G contains a connected subgraph H which is disjoint from S, and two subgraphs H_i $(i = x, y)$ such that H_i contains i and has exactly two vertices, say u_i and v_i, in H. Moreover, $\{u_x, u_y, v_x, v_y\}$ is a vertex cut separating x and y in G, see Figure 1a).*

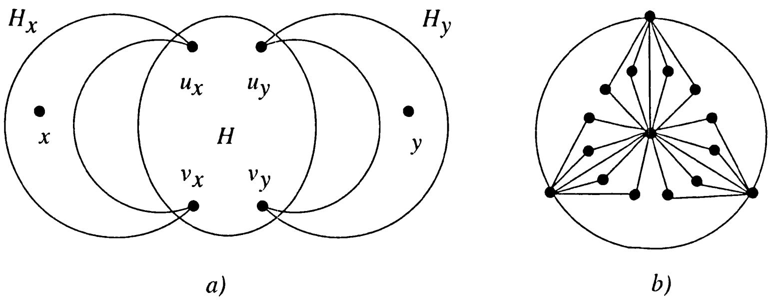

FIGURE 1.

For each $n \ge 5$, the graph $K_{2,n-2}$ has order n, is 2-connected, and $c(K_{2,n-2}) = 4$. The following lemma shows that for every $n \ge 5$, there is exactly one further 2-connected graph of order n with circumference less than 5 the graph $K_{1,1,n-2}$.

LEMMA 2. *Every 2-connected graph G of order $n \ge 5$ with $c(G) < 5$ is isomorphic either to the graph $K_{2,n-2}$ or to the graph $K_{1,1,n-2}$.*

Proof. Let G be a 2-connected graph of order $n \geq 5$ with $c(G) < 5$. We will use the following well-known fact, which follows e.g. from the generalized Menger's theorem:

(Φ) Any 2-connected graph contains a cycle through any two vertices, two edges, or a vertex and an edge.

We distinguish two cases.

(i) G is bipartite. If the graph G contains an induced P_4 (a path on four vertices), by (Φ), it must contain a cycle of length at least 6, a contradiction. So we may assume G does not contain any induced P_4. If G contains an edge xy with $d(x), d(y) \geq 3$, then the vertex x has at least two neighbours, say a, b and similarly y has at least two neighbours, say c, d. Since G is bipartite and it does not contain any induced P_4, it follows that all the edges ac, ad, bc and bd are in G. But now the cycle (a, x, y, d, b, c, a) has length 6. Thus we may assume that each edge of G has at least one end-vertex of degree two. If G has all vertices of degree two, then it is a cycle on $n \geq 6$ vertices, again a contradiction. Hence let u be a vertex of degree at least 3 in G. We have proved that all its neighbours, say $a_1, a_2, \dots, a_l$, must be of degree two (they cannot be of degree one). Since G does not contain any induced P_4, all these l vertices are adjacent to another vertex, say v. It follows that $d(v) \geq 3$. By the same arguments as above, all neighbours of v are of degree two and are neighbours of u. Hence $l = n - 2$ and G is isomorphic to $K_{2,n-2}$.

(ii) G is not bipartite. If G does not contain two non-adjacent vertices, then it is a complete graph with $c(G) = n \geq 5$, a contradiction. Thus let x and y be two non-adjacent vertices of G. By Lemma 1, either the cycles through $\{x, y\}$ generate the cycle space of G or G contains a connected subgraph H which is disjoint from $\{x, y\}$, and two subgraphs H_i $(i = x, y)$ such that H_i contains i and has exactly two vertices, say u_i and v_i, in H. Moreover, $\{u_x, u_y, v_x, v_y\}$ is a vertex cut separating x and y in G.

In the former case (recall G is not bipartite) there must exist at least one odd cycle through the vertices x and y. Since x and y are non-adjacent, the length of the cycle is at least 5, again a contradiction.

Let us consider the latter case. By (Φ), G contains a cycle through x and y. Since any such cycle contains all the vertices x, y, u_x, u_y, v_x and v_y and since $c(G) < 5$, we must have $u_x = u_y = u$ and $v_x = v_y = v$. If H or H_x or H_y contains an edge with both end-vertices different from u and v, then, using (Φ), G would contain a cycle of length at least 5, a contradiction. Hence each edge of G has at least one end-vertex from $\{u, v\}$, i.e., $\{u, v\}$ is a dominating set of G. Since G is 2-connected and non-bipartite, one can observe that uv is the edge of G and G is isomorphic to the graph $K_{1,1,n-2}$. □

THEOREM 2. *With $c = 5$, Problem 1 has an affirmative solution.*

P r o o f . It follows from Lemma 2 that every 2-connected graph G of order $n \geq 5$ and $c(G) < 5$ is isomorphic either to the graph $K_{2,n-2}$ or to $K_{1,1,n-2}$. One can observe that these graphs do not satisfy the assumptions of Problem 1 with $c = 5$. □

Second, let us investigate the case $c = 6$. In the proof of the following theorem we give counterexamples to Problem 1, which works for all $c \geq 6$ and all $n > \lfloor \frac{c-1}{2} \rfloor (c-3)$ (if $c = 6$, then $n \geq 7$).

THEOREM 3. *For any $n \geq 7$ there is a graph G of order n which satisfies assumptions of Problem 1 with $c = 6$, but $c(G) < 6$.*

P r o o f . Let $c \geq 6$ and $n > \lfloor \frac{c-1}{2} \rfloor (c-3)$ be given integers. Let $k = \lfloor \frac{c-1}{2} \rfloor$. Choose $m_i \geq c-4$ $(i = 1, 2, \dots, k)$ such that $\sum_{i=1}^{k} m_i = n - k - 1$. Consider the graph $G = G(k; m_1, m_2, \dots, m_k)$ consisting of a cycle $C_k = (v_1, v_2, \dots, v_k, v_1)$ and one extra vertex joined by m_i internally disjoint paths of length two and one edge to v_i for $i = 1, 2, \dots, k$. Note that in the case when $k = 2$, C_2 is an edge. The graph $G(3; 4, 4, 4)$ is depicted in Figure 1b).

The graph G is obviously 2-connected of order n. Since its minimum degree is 2 and since it has $\lceil c/2 \rceil$ vertices of degree at least $c - 2$, it satisfies the assumptions of Problem 1. But, obviously, $c(G) = k + 3 < c$.

It follows from the previous that there are only three values of c for which Chvátal's hamiltonicity condition yields a circumference condition by replacing n by c and requiring 2-connectivity. Our next result shows that some circumference generalization of Chvátal's condition for any c is even possible.

THEOREM 4. *Let G be a graph with vertices ordered according to their degrees $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. Let $W = \{v_{n-w+1}, v_{n-w+2}, \dots, v_n\}$, $w \geq 3$. If $d(v_{n-w+1}) > n - w$, and for any $i > n - w$, $d(v_i) \leq i < \frac{n}{2}$ implies $d(v_{n-i}) \geq n - i$, then G contains a cycle through all vertices of W.*

P r o o f . The proof is similar to Chvátal's original one. Let G be a graph of order n satisfying the conditions of the theorem. First of all note that the set W has the property, say (P), that if it contains a vertex of degree l, then it must contain all vertices of degree $> l$.

Suppose by way of contradiction that there is no cycle through all the vertices in W. Obviously, adding any new edge between two non-adjacent vertices from W results in a graph satisfying the assumptions of the theorem.

Hence we may assume that there are as many edges as possible in W (such that G does not contain any cycle through all vertices of W). Now, any pair of non-adjacent vertices in W is connected by a path that contains all the vertices of W.

Since at least one edge is missing in W, we can find two vertices, say x and y, such that

$$d(x) = i\,, \tag{1}$$
$$d(x) \le d(y)\,, \tag{2}$$
$$xy \notin E(G)\,, \tag{3}$$
$$d(x) + d(y) \qquad \text{is as large as possible.} \tag{4}$$

Let $P = (x = x_0, x_1, \dots, x_l = y)$ be a $x-y$ path that contains all the vertices of W. Let x_i be a neighbour of x on P. It holds that its predecessor x_{i-1} is not adjacent to y. Because otherwise $(x, x_1, \dots, x_{i-1}, y, x_{l-1}, \dots, x_i, x)$ would be a cycle containing all vertices of W, a contradiction. From (1) and (4), it follows that the degree of x_{i-1} is at most i. Similarly, it holds that any neighbour of x not on P is not adjacent to y. Moreover, since this vertex is not in W, by (1) and from the property (P), its degree is at most i. By the arguments above, if x has degree l, then there must be at least l edges missing at y, thus we have

$$d(x) + d(y) < n\,. \tag{5}$$

It follows from (2) that $i < n/2$. Moreover, there are at least i vertices of degree at most i in G, hence $d(v_i) \le i < n/2$. Since $x \in W$, it holds that $d(x) \ge d(v_{n-w+1}) > n - w$, hence it follows that $i > n - w$. By the assumptions of the theorem, we must have $d(v_{n-i}) \ge n - i$. Thus there are at least $i+1$ vertices, each of degree at least $n-i$. We claim, that all these vertices are in W. Indeed, since $i < n - i$, all these vertices have degree greater than the vertex x which is from W. The claim follows from the fact that W has the property (P).

At least one of these $i+1$ vertices is non-adjacent to x, say z. But $d(x) + d(z) \ge n$, a contradiction with (4) and (5). We conclude that G contains a cycle through all vertices of W. □

COROLLARY 1. *Let G be a graph of order $n \ge 3$ with degrees $d_1 \le d_2 \le \dots \le d_n$ and let $3 \le c \le n$. If $d_{n-c+1} > n - c$, and for any $i > n - c$, $d_i \le i < \frac{n}{2}$ implies $d_{n-i} \ge n - i$, then $c(G) \ge c$.*

3. Concluding remarks

Let us note that if $w = n$, then $d(v_{n-w+1}) > n-w$ follows from the Chvátal's part of the condition in Theorem 4, thus Theorem 4 generalizes Chvátal's hamiltonicity condition. If $w < n$, then the following examples show that the condition is necessary. Indeed, for $w \le n/2 + 1$ take any tree of order n. The following examples show the necessity of the condition also for several $w \ge n/2 + 5$.

Let G be given graph with two distinguished vertices, say u and v. Define the new graph $H = G(u,v)$ as follows. $V(H) = V(G) + \{x,y\}$ and $E(H) - E(G) + \{xu, xv, yu, yv\}$. We say that H arises from G by *downing* vertices u and v and that x and y constitute a *nodal* pair.

Let l and n be integers such that n is even and $2 \leq l < n/2$. Construct the graph $G(l,n)$ as follows. Take the complete graph on $n + 2l - 2$ vertices and pick up $n/2 - l + 1$ pairs of its vertices. Now apply the operation downing to all the distinguished pairs of vertices. Finally, in the present graph choose one vertex of degree two (this will be a vertex of a nodal pair) and connect it to all $4l - 4$ unused vertices of the complete graph. The degree sequence of $G(l,n)$ is the following:

$$\underbrace{2,\dots,2}_{n-2l+1}, 4l-2, \underbrace{n+2l-2,\dots,n+2l-2}_{4l-4}, \underbrace{n+2l-1,\dots,n+2l-1}_{n-2l+2}.$$

Now, let $w = n + 2l + 1$. Then W is the set of all vertices of degree at least $4l - 2$ plus two vertices of degree two. Obviously, the ordering of vertices can be chosen so that these two vertices constitute a nodal pair. Because of the nodal pair, there is no cycle through all the vertices of W in $G(l,n)$. However, one can find an ordering of vertices of G such that the degree condition of Theorem 4 is satisfied. Fortunately, the condition $d(v_{2n-w+1}) > 2n - w$ is not for $w \leq 2n - 2$.

Note that Theorem 4 is of similar nature as recent results of Shi [S] and Bollobás and Brightwell [BB]. However, it is not their extention in general.

THEOREM 5. (Bollobás—Brightwell [BB]; $d \geq n/2$, Shi [S]) *If G is a graph of order n and W is a set of w vertices of degree at least $d \geq 2$. If $s = \left\lceil \frac{w}{\lceil \frac{n}{d} \rceil - 1} \right\rceil \geq 3$, then there is a cycle through at least s vertices of W.*

The previous theorem guarantees cycles through all the vertices of W only if $d \geq n/2$.

REFERENCES

[BChS] BEDROSSIAN, P.—CHEN, G. SCHELP, R. H.: *A generalization of Fan's cond tion for hamiltonicity, pancyclicity, and hamiltonian connectedness*, Discrete Math. **115** (1993), 39 50.

[Bol] BOLLOBÁS, B.: *Extremal Graph Theory*, Acad. Press, London-New York-San Francisco, 1978.

[BB] BOLLOBÁS, B.—BRIGHTWELL, G.: *Cycles through specified vertices*, Combinatorica **13** (1993), 147 155.

[Bon] BONDY, J. A.: *Large cycles in graphs*, Discrete Math. **1** (1971), 121 132.

[BL] BONDY, J. A. LOVÁSZ, L.: *Cycles through specified vertices of a graph*, Combinatorica **1** (1981), 117 140.

[H] HÄGGKVIST, R.: (Personal communication with J. A. Bondy).

[Ch] CHVÁTAL, V.: *On hamilton's ideals*, J. Combin. Theory Ser. B **12** (1972), 163–168.

[S] SHI, R.: 2-*Neighborhoods and hamiltonian conditions*, J. Graph Theory **16** (1992), 267–271.

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