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ZEROS OF CONTINUOUS FUNCTIONS
AND THE COMPACT-OPEN TOPOLOGY

PETER VADOVIČ

(Communicated by Lubica Holá)

ABSTRACT. We consider the space of all continuous real-valued functions equipped with the compact-open topology. The principal aim of this paper is the generalization of Theorem 2.1 from the paper [BALÁZ, V.—ŠALÁT, T.: Zeros of continuous functions and the structure of two function spaces, Math. Slovaca 52 (2002), 397-408].

DEFINITIONS AND NOTATION. Let \( X \) be a Tychonoff (completely regular \( T_1 \) ) space, \( C(X, \mathbb{R}) \) or simply \( C(X) \) be the set of all continuous functions on \( X \) to the set of all real numbers \( \mathbb{R} \), let \( \tau_{co} \) be the compact-open topology on \( C(X) \). We define \( C_0(X) = \{ f \in C(X) : f^{-1}(\{0\}) \neq \emptyset \} \) and investigate the sets

\[
\begin{align*}
H &= \{ f \in C(X) : f^{-1}(\{0\}) \text{ is perfect and nowhere dense} \}, \\
A &= \{ f \in C(X) : f^{-1}(\{0\}) \text{ is not nowhere dense} \}, \\
D &= \{ f \in C(X) : f^{-1}(\{0\}) \text{ is not perfect} \}.
\end{align*}
\]

If moreover \( X \) is second countable and \( \mathfrak{B} = \{ I_n : n \in \mathbb{N} \} \) is a countable base for the topology for \( X \), then for each \( n \in \mathbb{N} \) we put \( A_n = \{ f \in C(X) : I_n \subseteq f^{-1}(\{0\}) \} \) and \( D_n = \{ f \in C(X) : (\exists x_0 \in I_n)(f(x_0) = 0) \} \).

The definitions of all other terms are taken from Kelley [4] and Engelking [3].

Note 1.

(a) Observe that \( A = \bigcup_{n=1}^{\infty} A_n \) and \( D = \bigcup_{n=1}^{\infty} D_n \) if \( X \) is a second countable Tychonoff space.

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(b) If $f \in A_n^c$ (the complement of $A_n$), then there is $x \in I_n$ such that $f(x) \neq 0$, which implies that $f \in W(\{x\}, \mathbb{R}\setminus\{0\}) \subseteq A_n^c$, where $W(K,U) = \{g \in C(X) : g[K] \subseteq U\}$ for some $K \subseteq X$ compact and $U \subseteq \mathbb{R}$ open. Hence $A_n^c$ is $\tau_{co}$-open. Moreover the set $W(\{x\}, \mathbb{R}\setminus\{0\})$ is the preimage $P_x^{-1}[\mathbb{R}\setminus\{0\}]$ of the open set $\mathbb{R}\setminus\{0\}$ in the projection on the $x$th coordinate and so $A_n^c$ is even open in the topology $\tau_p$ of pointwise convergence. Consequently each $A_n$ is $\tau_p$-closed and also $\tau_{co}$-closed.

**Theorem 1.** If $X$ is a second countable Tychonoff space, then $A$ is of first category in $(C(X), \tau_{co})$.

**Proof.** By Note 1(a) and 1(b) it is sufficient to show that the interior of each $A_n$ is void. We show an equivalent statement that $A_n^c$ is dense in $C(X)$. Let $B$ be a $\tau_{co}$-basic set and let $f \in B$. If $f \notin A_n$, then $f \in B \cap A_n^c \neq \emptyset$ and we are done. So suppose that $f \in A_n$. We know that on $C(X)$ the compact-open topology is the topology of uniform convergence on compacta ([4; Theorem 7.11]) and hence there is $\epsilon > 0$ and a compact set $K \subseteq X$ such that $W(f,K,\epsilon) \subseteq B$, where $W(f,K,\epsilon) = \{g \in C(X) : (\forall x \in K)(|f(x) - g(x)| < \epsilon)\}$. Define a function $g$ on $X$ by:

$$g(x) = f(x) + \epsilon/2 \quad \text{for each} \quad x \in X.$$ 

Clearly $g \in C(X, \mathbb{R})$ and $|g(x) - f(x)| = \epsilon/2$ for each $x \in X$, which implies $g \in W(f,K,\epsilon) \subseteq B$. On the other hand $f \in A_n$ and so for each $x \in I_n$ we have $g(x) = 0 + \epsilon/2 = \epsilon/2$, which means that $g \notin A_n$. Consequently $B \cap A_n^c \neq \emptyset$ and hence $A_n^c$ is dense. \hfill $\Box$

**Note 2.**

(a) The construction of $g$ used in the proof of Theorem 1 doesn’t even need the space $X$ to be Tychonoff, however, the complete regularity of $X$ is required to obtain a “reasonable” class of continuous functions.

(b) From the construction of $g$ we do not know whether $g$ has a zero point at all, however, if $f \in A_n$, then we may choose two distinct points $x_1 \neq x_2$ in $I_n$ (this can be done if $X$ is without isolated points). Since $X$ is a Tychonoff space, there is $h \in C(X,[0,1])$ such that $h(x_1) = 1$ and $h(x_2) = 0$. Let $g(x) = f(x) + \epsilon/2 \cdot h(x)$ for each $x \in X$, i.e. clearly $g \in C(X)$. Moreover $0 \leq h(x) \leq 1$, that is $|g(x) - f(x)| = \epsilon/2 \cdot h(x) \leq \epsilon/2$ for each $x \in X$ and so $g \in W(f,K,\epsilon) \subseteq B$. We also see that $g(x_1) = 0 + \epsilon/2 = \epsilon/2$, and so $g \notin A_n$. On the other hand $g(x_2) = 0 + 0 = 0$, and so $g \in C_0(X)$. Thus $g \in C_0(X) \cap B \cap A_n^c$. Such a function will be used in the later course of the paper.

Before investigating the situation for the set $D$ we state two auxiliary facts.

**Proposition 1.** In a second countable locally connected space each open set has countably many components.
Proof. Let \( \mathcal{B} = \{I_n : n \in \mathbb{N}\} \) be a countable base for the topology, \( U \) be an open set and let \( \{C_\alpha : \alpha \in M\} \) be the family of all components of \( U \). We know (see [4; Exercise 1.8]) that in a locally connected space every component of an open set is open. Therefore each \( C_\alpha \) is open and hence for each \( \alpha \in M \) we have \( C_\alpha = \bigcup\{I_n : n \in N_\alpha\} \), where \( N_\alpha \) is a nonempty subset of \( \mathbb{N} \). Observe that if \( n \in N_\alpha \cap N_\beta \), then \( I_n \subseteq C_\alpha \cap C_\beta \), which implies that the set \( C_\alpha \cup C_\beta \) is a connected set in \( U \) containing both \( C_\alpha \) and \( C_\beta \). Thus \( N_\alpha \cap N_\beta = \emptyset \) whenever \( \alpha \neq \beta \), i.e. the family \( \{N_\alpha : \alpha \in M\} \) is a decomposition of \( \mathbb{N} \) into \( |M| \)-many pairwise disjoint nonempty subsets. Since \( \mathbb{N} \) is countable it is clear that \( M \) cannot be uncountable. Consequently \( U \) has countably many components.

**Corollary 2.** If \( X \) is a locally compact, locally connected separable metric space, then there is a countable base for the topology consisting of connected sets with compact closures.

**Proof.** Every separable metric space is a second countable Tychonoff space and vice versa (see [4; Theorem 4.17]). Using Proposition 1 we construct a countable base consisting of connected sets. Finally, with local compactness at hand it is not hard to show that the subfamily consisting of connected sets with compact closures is still a countable base and we are done.

**Theorem 2.** Let \( X \) be a locally compact, locally connected separable metric space without isolated points. Then \( D \) is of first category in \((C(X), \tau_{co})\).

**Proof.** By the preceding corollary, let \( \mathcal{B} = \{I_n : n \in \mathbb{N}\} \) be a countable base where each \( I_n \) is connected and \( \overline{I_n} \) is compact. By Note 1(a) it is sufficient to show that each \( D_n \) is nowhere dense in \((C(X), \tau_{co})\). Thus let \( n \in \mathbb{N} \) be fixed, let \( B \) be an arbitrary \( \tau_{co} \)-basic set and consider \( f \in B \). If the function \( f \) has no zero point in \( \overline{I_n} \), then \( f[\overline{I_n}] \subseteq \mathbb{R} \setminus \{0\} \) and hence the nonempty open set \( B \cap W(\overline{I_n}, \mathbb{R} \setminus \{0\}) \) is contained in \( B \setminus D_n \), which means that \( D_n \) is nowhere dense. So we can suppose that \( f \) has a zero point \( x_0 \in \overline{I_n} \). Again, on \( C(X) \) the compact-open topology is the topology of uniform convergence on compacta ([4; Theorem 7.11]), so there is \( \varepsilon > 0 \) and a compact \( K \) such that \( W(f, K, \varepsilon) \subseteq B \). The continuity of \( f \) at \( x_0 \) implies that there is \( \delta > 0 \) such that \( |f(x)| < \varepsilon/4 \) whenever \( x \in B(x_0, \delta) \) (the open ball with the center \( x_0 \) and the radius \( \delta \)). From \( x_0 \in \overline{I_n} \) it follows that \( V = B(x_0, \delta) \cap I_n \) is a nonempty open set and so we may choose \( x_1 \in V \) and \( x_2 \in V \) with \( x_1 \neq x_2 \) (this can be done because \( X \) is without isolated points). In the Tychonoff space \( X \) there are two disjoint sets \( I_{n_1} \) and \( I_{n_2} \) in \( \mathcal{B} \) such that \( x_i \in I_{n_i} \subseteq V \) for \( i = 1, 2 \). Finally choose \( x_3, x_4 \) such that \( x_3 \in I_{n_1} \), \( x_3 \neq x_1 \), and \( x_4 \in I_{n_2}, x_4 \neq x_2 \). Let \( A = \{x_1, x_2, x_3, x_4\} \). If we consider \( A \) with the discrete topology (which is the relative topology from \( X \) since \( A \) consists of isolated points), then the function \( h' : A \to [-1, 1] \) defined by \( h'(x_1) = h'(x_2) = 1 \) and \( h'(x_3) = h'(x_4) = -1 \) is
continuous on $A$. By the Tietze extension theorem there is an extension $h \in C(X, [-1, 1])$ of the function $h'$. Now define the function $g$ on $X$ by:

$$g(x) = f(x) + \varepsilon/2 \cdot h(x) \quad \text{for each} \quad x \in X,$$

thus $g \in C(X, \mathbb{R})$. Put $K' = K \cup I_n$, then $K'$ is compact. We want to show that $W(g, K', \varepsilon/4)$ is the desired open set. Let $s \in W(g, K', \varepsilon/4)$. From $K \subseteq K'$ we have $|s(x) - g(x)| < \varepsilon/4$ and $|g(x) - f(x)| = \varepsilon/2 \cdot |h(x)| \leq \varepsilon/2$ for each $x \in K$ and hence $s \in W(f, K, \varepsilon) \subseteq B$. On the other hand $A \subseteq B(x_0, \delta)$, so for each $i = 1, \ldots, 4$ we have $-\varepsilon/4 < f(x_i) < \varepsilon/4$. Moreover $h$ restricted to $A$ equals $h'$, which implies that for $i = 1, 2$

$$g(x_i) = \varepsilon/2 \cdot h(x_i) + f(x_i) > \varepsilon/2 - \varepsilon/4 = \varepsilon/4$$

and for $i = 3, 4$

$$g(x_i) = f(x_i) + \varepsilon/2 \cdot h(x_i) < \varepsilon/4 - \varepsilon/2 = -\varepsilon/4.$$

Since $A \subseteq I_n \subseteq K'$, the value of every $s \in W(g, K', \varepsilon/4)$ is positive at $x_1$ and $x_2$ and negative at $x_3$ and $x_4$. But $x_1$ and $x_3$ are elements of a connected set $I_{n_1}$ and $s$ is continuous, i.e. the image $s[I_{n_1}]$ is connected, so with $s(x_1)$ and $s(x_3)$ it must contain zero. Thus $s$ has a zero point in $I_{n_1}$ and similarly it has a zero point in $I_{n_2}$, which are two disjoint subsets of $I_n$. We conclude that $s$ has at least two zero points in $I_n$, that is $s \notin D_n$. This shows that $W(g, K', \varepsilon/4) \subseteq B \setminus D_n$ and hence $D_n$ is nowhere dense. \qed

**Theorem 3.** Let $X$ be a locally compact, locally connected second countable Tychonoff space without isolated points. Then the set $H$ is residual in $(C(X), \tau_{co})$.

**Proof.** We know that a second countable locally compact Tychonoff space is a hemicompact $k$-space (see [3; Exercise 3.4.E]). Thus [5; Corollary 5.2.2] implies that $(C(X), \tau_{co})$ is completely metrizable and hence it is a Baire space. Therefore it suffices to show that the complement of $H$ is of first category. But $H^c = A \cup D$, so Theorem 1 and Theorem 2 yield the desired statement. \qed

Now, consider the space $C_0(X)$ with the relativized compact-open topology from $C(X)$. Define the set $H_0 = H \cap C_0(X)$. Since all the other considered sets $(A, D, A_n$ and $D_n$) are subsets of $C_0(X)$, they do not change if we restrict ourselves to the space $C_0(X)$. In particular $H = C(X) \setminus (A \cup D)$ and $H_0 = C_0(X) \setminus (A \cup D)$. The first question of course is: which of the foregoing results hold if we replace $C(X)$ with $C_0(X)$?

Clearly, in the space $(C_0(X), \tau_{co})$ both statements of Note 1 remain valid. Note 2(b) shows that if $X$ is without isolated points, then an analogy of Theorem 1 is also true in $C_0(X)$. A thorough look on the proof of Theorem 2 reveals
that it will work in $C_0(X)$, too, since the constructed function $g$ belongs to $C_0(X)$. We conclude that the set $A \cup D$, which equals $C_0(X) \setminus H_0$, is of first category in $(C_0(X), \tau_{co})$. To complete our effort we will need some additional results.

**Proposition 3.** If $T$ is a Baire space and $S$ is a subset such that $\text{int} S$ is dense in $T$, then $S$ is a Baire space with respect to the relative topology from $T$.

**Proof.** Consider a subset $E'$ of $S$. We know that, without any assumption on the subspace $S$, if $E'$ is nowhere dense in $S$, then it is nowhere dense in $T$. Therefore a set of first category in $S$ is clearly of first category in $T$ or in other words, if $E' \subseteq S$ is of second category in $T$, then it is of second category in $S$. So let $U'$ be a nonempty open set in $S$, hence $U' = U \cap S$ for some nonempty open set $U$ in $T$. Since $\text{int} S$ is dense, the set $U \cap \text{int} S$ is nonempty open and thus of second category in $T$. As $U'$ contains $U \cap \text{int} S$, it must also be of second category in $T$ and by our initial remarks $U'$ is of second category in $S$. Therefore $S$ is a Baire space. \(\square\)

**Proposition 4.** For a non-compact Tychonoff space $X$ the set $C_0(X)$ is dense in $(C(X), \tau_{co})$.

**Proof.** If $B = \bigcap_{i=1}^{m} W(K_i, U_i)$ is a $\tau_{co}$-basic neighborhood of a function $f \in C(X)$, then the set $K = \bigcup_{i=1}^{m} K_i$ is a compact (and closed) subset of the non-compact space $X$ and so there is a point $x_0 \in X \setminus K$ and a function $h \in C(X, [0,1])$ with $h(x_0) = 0$ and $h[K] = \{1\}$. Hence the function $g$ defined by $g(x) = f(x) \cdot h(x)$ coincides with $f$ on $K$ (thus $g \in B$) and assumes zero at $x_0$ (thus $g \in C_0(X)$), which shows that $C_0(X)$ is dense. \(\square\)

However, in the sequel we will need a somehow stronger statement composed in the following proposition.

**Proposition 5.** If $X$ is a non-compact locally connected Tychonoff space without isolated points, then $\text{int} C_0(X)$ is dense in $(C(X), \tau_{co})$.

**Proof.** Let $B = \bigcap_{i=1}^{m} W(K_i, U_i)$ be a $\tau_{co}$-basic set and $f \in B$. Then $K = \bigcup_{i=1}^{m} K_i$ is a compact (and hence closed) subset of $X$. Since $X$ is not compact, $X \setminus K$ is a nonempty open set, so let $x_1 \in X \setminus K$. Then there is an open connected set $V$ with $x_1 \in V \subseteq K^c$. Choose a point $x_2 \in V$, $x_2 \neq x_1$ (this can be done because $X$ is without isolated points). Having in mind that $X$ is a Tychonoff space we know that there are continuous functions $h_i \in C(X, [0,1])$ for $i = 1, \ldots, 4$ with the following properties: since $\{x_1\}$ and $K$ are disjoint
closed subsets we choose $h_1$ such that $h_1[K] = \{1\}$ and $h_1(x_1) = 0$. Similarly for $x_2$ we put $h_2[K] = \{1\}$ and $h_2(x_2) = 0$. Furthermore $x_1$ and $K \cup \{x_2\}$ are disjoint closed subsets and so we can choose $h_3$ in order that $h_3[K \cup \{x_2\}] = 0$ and $h_3(x_1) = 1$. In the same manner we put $h_4[K \cup \{x_1\}] = 0$ and $h_4(x_2) = 1$. Now we define the function $g$ on $X$ by:

$$g(x) = f(x) - h_1(x) + h_2(x) + h_3(x) - h_4(x)$$

for each $x \in X$.

Clearly $g \in C(X, \mathbb{R})$. Moreover if $x \in K$, then $g(x) = f(x) \cdot 1 \cdot 1 + 0 - 0 = f(x)$ and so $g \in B$. Next, $g(x_1) = f(x_1) \cdot 0 \cdot h_2(x_1) + 1 - 0 = 1$ and $g(x_2) = f(x_2) \cdot h_1(x_2) \cdot 0 + 0 - 1 = -1$. Thus $g \in W(\{x_1\}, \mathbb{R}^+) \cap W(\{x_2\}, \mathbb{R}^-) = U$ which is a $T_{co}$-basic set ($\mathbb{R}^+$ and $\mathbb{R}^-$ denote all positive and negative real numbers respectively). Finally whenever $h \in U$, then $h(x_1) > 0 > h(x_2)$ where $x_1$ and $x_2$ are elements of the connected set $V$, and $h$ is continuous, which implies that $h[V]$ is connected and with $h(x_1)$ and $h(x_2)$ it must contain zero. Hence $h^{-1}(\{0\}) \neq \emptyset$ and $h \in C_0(X)$. This shows that $U$ is a neighborhood of $g$ contained in $C_0(X)$, i.e. $g \in \text{int} C_0(X)$. Consequently $B \cap \text{int} C_0(X) \neq \emptyset$ and therefore $\text{int} C_0(X)$ is dense.

**Corollary 6.** If $X$ is a non-compact locally compact, locally connected paracompact Tychonoff space without isolated points, then $(C_0(X), \tau_{co})$ is a Baire space.

**Proof.** By [5; Theorem 5.3.1], the space $(C(X), \tau_{co})$ is a Baire space. The rest is supplied by Proposition 3 and Proposition 5.

**Theorem 4.** Let $X$ be a locally compact, locally connected second countable Tychonoff space without isolated points. Then the set $H_0$ is residual in $(C_0(X), \tau_{co})$.

**Proof.** As in the proof of Theorem 3, $(C(X), \tau_{co})$ is completely metrizable. It is not difficult to show that if $X$ is compact, then $C_0(X)$ is closed in $(C(X), \tau_{co})$, that is, $(C_0(X), \tau_{co})$ is completely metrizable and hence it is a Baire space. On the other hand, if $X$ is not compact, then Corollary 6 implies that $(C_0(X), \tau_{co})$ is a Baire space, too. Therefore we only have to show that the set $C_0(X) \setminus H_0$ is of first category in $C_0(X)$, but that is true by the remarks preceding Proposition 3.

To highlight the resemblances and differences, compared to our Theorem 3 and Theorem 4, we now state the original assertion of Bãlãzã and Sãlãt ([1; Theorem 2.1]). Herein $C(a, b)$ denotes the complete metric space of all continuous real-valued functions on a compact interval $[a, b]$ of the reals equipped with the uniform metric. The definitions of all other occurring sets, namely $C_0(a, b)$, $H(a, b)$, $H_0(a, b)$, $A(a, b)$ and $D(a, b)$, are totally analogous to the definitions at the beginning of our paper.

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**Peter Vadović**
THEOREM. ([1; Theorem 2.1])

(i) The set \( H(a, b) \) is residual in \( C(a, b) \).
(ii) The set \( H_0(a, b) \) is residual in \( C_0(a, b) \).

The paper [1] also contains propositions revealing other properties of \( A \) and \( D \). We shall now investigate the general situation.

PROPOSITION 7. If \( X \) is normal, then \( A \) is dense in \( (C_0(X), \tau_{co}) \). If moreover every singleton in \( X \) is a \( G^s \) set, then \( D \) is dense, too.

Proof. Let \( B \) be an arbitrary \( \tau_{co} \)-basic set in \( C_0(X) \) and let \( f \in B \). Hence there is \( x_0 \in X \) with \( f(x_0) = 0 \). On \( C_0(X) \) the topology of uniform convergence on compacta and the compact-open topology coincide ([4; Theorem 7.11]), and so there is a compact \( K \subseteq X \) and \( \varepsilon > 0 \) such that \( W(f, K, \varepsilon) \subseteq B \). Since \( f \) is continuous at \( x_0 \), there is \( U \subseteq X \) open such that \( |f(x)| < \varepsilon/2 \) whenever \( x \in U \). Furthermore there is an open set \( V \) such that \( x_0 \in V \subseteq \overline{V} \subseteq U \) and hence \( \overline{V} \) and \( U^c \) are two disjoint closed sets, so by the Urysohn lemma there is a continuous function \( h_1 \in C(X,[0,1]) \) such that \( h_1[\overline{V}] = \{0\} \) and \( h_1[U^c] = \{1\} \). Define the function \( g_1 \) on \( X \) by: \( g_1(x) = h_1(x) \cdot f(x) \) for each \( x \in X \). Clearly \( g_1 \in C_0(X) \) and we also see that \( V \subseteq g_1^{-1}(\{0\}) \). Thus \( g_1^{-1}(\{0\}) \) is not nowhere dense, so \( g_1 \in A \). Moreover, for each \( x \in X \) we have \( 0 \leq h_1(x) \leq 1 \), which implies \( 0 \leq 1 - h_1(x) \leq 1 \). Hence if \( x \in U \), then

\[
|g_1(x) - f(x)| = |f(x)| \cdot |1 - h_1(x)| \leq |f(x)| < \varepsilon/2.
\]

On the other hand, if \( x \in U^c \), then \( g_1(x) = 1 \cdot f(x) = f(x) \). Consequently, for each \( x \in X \) we have \( |g_1(x) - f(x)| < \varepsilon/2 \), which implies \( g_1 \in W(f, K, \varepsilon) \subseteq B \). Thus \( A \cap B \neq \emptyset \) and \( A \) is therefore dense.

Concerning the set \( D \): since the singleton \( \{x_0\} \) is a closed \( G^s \) set, [3; Corollary 1.5.11] implies the existence of a continuous function \( h_2 \in C(X,[0,1]) \) for which \( \{x_0\} = h_2^{-1}(\{0\}) \). Define the function \( g_2 \) on \( X \) by: \( g_2(x) = g_1(x) + \varepsilon/2 \cdot h_2(x) \) for each \( x \in X \). Clearly \( g_2 \in C_0(X) \). Next, for each \( x \in X \) we have

\[
|g_2(x) - f(x)| \leq |g_1(x) - f(x)| + \varepsilon/2 \cdot |h_2(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Thus \( g_2 \in W(f, K, \varepsilon) \subseteq B \). Since \( g_1 \) is constantly zero on \( \overline{V} \), we see that \( g_2(x) = \varepsilon/2 \cdot h_2(x) \) for each \( x \in \overline{V} \), and therefore \( x_0 \) is the only zero point of \( g_2 \) in \( V \). Consequently \( g_2 \in D \), which shows that \( D \) intersects \( B \) and we are done. \( \square \)

In view of Proposition 4 and Proposition 7 the next two statements are clear, provided we realize that if \( X \) is compact, then \( C(X) \setminus C_0(X) = W(X, R \setminus \{0\}) \) is a nonempty open set disjoint from \( C_0(X) \) and hence no subset of \( C_0(X) \) (neither \( A \) nor \( D \)) can be dense in \( C(X) \).
COROLLARY 8(a). Let $X$ be a normal space. The set $A$ is dense in $(C(X), \tau_{co})$ if and only if $X$ is not compact.

COROLLARY 8(b). Let $X$ be a normal space and let every singleton in $X$ be a $G_\delta$ set. The set $D$ is dense in $(C(X), \tau_{co})$ if and only if $X$ is not compact.

PROPOSITION 9. Let $X$ be a locally connected normal space without isolated points. Then $A$ is not nowhere dense in $(C(X), \tau_{co})$. If moreover every singleton in $X$ is a $G_\delta$ set, then $D$ is not nowhere dense, too.

Proof. To show a set is not nowhere dense it suffices to find a nonempty open set in which the set is dense. Let $V \subseteq X$ be a nonempty open connected set (by local connectedness such set exists). Since $X$ is without isolated points, we can choose two distinct points $x_1 \neq x_2$ in $V$. Put $B = W(\{x_1\}, \mathbb{R}^+) \cap W(\{x_2\}, \mathbb{R}^-)$, where $\mathbb{R}^+$ and $\mathbb{R}^-$ denote all positive and negative real numbers respectively. Clearly $B$ is a $\tau_{co}$-basic set. Again, there is a continuous function $h \in C(X,[0,1])$ with $h(x_1) = 1$ and $h(x_2) = 0$. Define a function by $f(x) = h(x) - 1/2$ for each $x \in X$. Immediately we see that $f \in C(X)$ and $f \in B$, hence $B$ is a nonempty open set in $(C(X), \tau_{co})$. Furthermore every function in $B$ is positive and negative at $x_1$ and $x_2$ respectively, which are elements of a connected set $V$. Therefore every function in $B$ has a zero point in $X$ (this argument has been used several times before), that is $B \subseteq C_0(X)$. According to Proposition 7, under their respective assumptions the sets $A$ and $D$ are dense in $B$ which proves the proposition. □

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