Mathematica Slovaca

Sylvia Pulmannová

Blocks in homogeneous effect algebras and MV-algebras

Mathematica Slovaca, Vol. 53 (2003), No. 5, 525--539

Persistent URL: http://dml.cz/dmlcz/136895

Math. Slovaca, 53 (2003), No. 5, 525–539

BLOCKS IN HOMOGENEOUS EFFECT ALGEBRAS AND MV-ALGEBRAS

SYLVIA PULMANNOVÁ

(Communicated by Anatolij Dvurečenskij)

ABSTRACT. Some conditions under which a homogeneous effect algebra can be covered by MV-algebras are shown. Relations between completeness of homogeneous effect algebras and that of their blocks are studied.

1. Introduction

Effect algebras (or D-posets) have been introduced for the study of foundations of quantum mechanics (see [9], [18], [11], [8]). The prototype effect algebra is $(\mathcal{E}(H); \oplus, 0, I)$, where H is a Hilbert space and $\mathcal{E}(H)$ consists of all self-adjoint operators A on H such that $0 \leq A \leq I$. For $A, B \in \mathcal{E}(H)$, $A \oplus B$ is defined if and only if $A + B \leq I$ and then $A \oplus B = A + B$. Elements of $\mathcal{E}(H)$ are called *effects* and they play an important role in the theory of quantum measurements ([2], [3]).

The class of effect algebras includes orthoalgebras [10] and a subclass (called MV-effect algebras or Boolean D-posets) which are essentially equivalent to MV-algebras introduced by Chang in [4] (cf. e.g. [6], [9], [8] for relations between effect algebras and MV-algebras). The class of orthoalgebras includes further well-known structures that were considered as quantum logics, like orthomodular posets ([20]) and orthomodular lattices ([1], [16]).

A very important relation from the point of view of physical applications is the compatibility relation. It is well known that maximal sets of pairwise compatible elements in an orthomodular lattice L form maximal Boolean subalgebras (so called blocks) of L ([1], [16]). A similar result was obtained in orthomodular

2000 Mathematics Subject Classification: Primary 81P10; Secondary 03G12.
Keywords: effect algebra, homogeneous effect algebra, compatibility, Riesz decomposition property, block.

This research is supported by grant VEGA 2/3163/23.

posets, where a stronger relation of so called f-compatibility has to be used instead of the pairwise compatibility ([20]).

Recently, these results have been extended to lattice ordered effect algebras ([25]), where it was shown that maximal pairwise compatible sets of elements form MV-algebras; to effect algebras with the Riesz interpolation property, where it was proved that, provided the latter effect algebras satisfy an additional natural property, then maximal sets of pairwise strongly compatible elements form MV-algebras; and to homogeneous effect algebras ([12]), where pairwise compatibility was replaced by the existence of orthogonal covers (an analogue of f-compatibility from orthomodular posets), and it was proved that homogeneous effect algebras can be covered by blocks which are effect subalgebras with the Riesz decomposition property.

In the present paper, we will further study compatibility properties in homogeneous effect algebras and find conditions under which they can be covered by MV-algebras.

2. Definitions and basic results

An *effect algebra* is a partial algebra $(E;\oplus,0,1)$ with a partial binary operation $\oplus$ and two nullary operations 0, 1 satisfying the following conditions.

(E1) If $a\oplus b$ is defined, then $b\oplus a$ is defined and $a\oplus b=b\oplus a$.
(E2) If $a\oplus b$ and $(a\oplus b)\oplus c$ are defined, then $b\oplus c$ and $a\oplus(b\oplus c)$ are defined and $(a\oplus b)\oplus c=a\oplus(b\oplus c)$.
(E3) For every $a\in E$ there is a unique $a'\in E$ such that $a\oplus a'=1$.
(E4) If $a\oplus 1$ exists, then $a=0$.

Effect algebras in the latter form were introduced by Foulis and Bennett in [9]. Independently, Kôpka and Chovanec introduced an essentially equivalent structure called *D-poset*, [18]. Another equivalent structure, called *weak orthoalgebra*, was introduced by Giuntini and Greuling in [11].

For brevity, we denote the effect algebra $(E;\oplus,0,1)$ by E. For $a,b\in E$, we write $a\le b$ if there is $c\in E$ such that $a\oplus c=b$. It turns out that $\le$ is a partial order on E such that for every $a\in E$, $0\le a\le 1$. Moreover, it is possible to introduce a new partial binary operation $\ominus$ such that $b\ominus a$ is defined if and only if $a\le b$ and then $a\oplus(b\ominus a)=b$. It can be proved that $a\oplus b$ is defined if and only if $a\le b'$ if and only if $b\le a'$. In analogy with orthomodular posets, we say that a and b are *orthogonal* and write $a\perp b$ if $a\oplus b$ exists. Let $E_0\subset E$ be such that $1\in E_0$ and $a,b\in E_0$ with $a\le b$ implies $b\ominus a\in E_0$. Since $a'=1\ominus a$, and $a\oplus b=(a'\ominus b)'$, E_0 is closed with respect to $'$ and $\oplus$. We then call E_0 a *sub-effect algebra* of E.

Remark. For our purposes, it is natural to consider orthomodular lattices, orthomodular posets, MV-algebras and Boolean algebras as special types of effect algebras. In this paper, we will write briefly "orthomodular lattice" instead of "effect algebra associated with an orthomodular lattice" and similarly for orthomodular posets, MV-algebras and Boolean algebras.

For $a \in E$ we define $0 \cdot a = 0$, and $(n+1)a = na \oplus a$ if all involved elements exist. The greatest n such that na exists is called the *isotropic index* of a, denoted $\iota(a)$.

Two elements a, b in an effect algebra E are called (*Mackey*) *compatible* (written $a \leftrightarrow b$) if there are elements $a_1, b_1, c \in E$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \oplus b_1 \oplus c \in E$. The triple (a_1, b_1, c) is called a *Mackey decomposition* of the ordered pair (a, b).

An effect algebra E is

- an *orthoalgebra* if $a \perp a \implies a = 0$ (cf. [10]); equivalently, if $\iota(a) = 1$ whenever $a \neq 0$;
- an *orthomodular poset* if and only if, for all $a, b, c \in E$, $a \perp b$, $b \perp c$, $c \perp a$ implies that $a \oplus b \perp c$; observe that an orthomodular poset is an orthoalgebra. Indeed, if $a \perp a$, then together with $a \perp a'$ it gives $a \perp 1$, which entails $a = 0$;
- an *orthomodular lattice* if it is a lattice ordered orthomodular poset;
- an *MV-algebra* if E is lattice ordered and for every $a, b \in E$, $a \leftrightarrow b$ holds;
- a *Boolean algebra* if E is an MV-algebra and an orthoalgebra in the same time, equivalently, if E is an orthomodular lattice and for every $a, b \in E$, $a \leftrightarrow b$ holds.

According to [24], an effect algebra E satisfies the *Riesz decomposition property* ((RDP), for short) if one (and hence both) of the following equivalent properties is satisfied.

(R1) $a, b, c \in E$, $b \perp c$ and $a \leq b \oplus c$ implies $a = a_1 \oplus a_2$ with $a_1 \leq b$, $a_2 \leq c$.

(R2) $a \oplus b = c \oplus d$ implies that there are orthogonal elements $w_{11}, w_{12}, w_{21}, w_{22}$ such that $a = w_{11} \oplus w_{12}$, $b = w_{21} \oplus w_{22}$, $c = w_{11} \oplus w_{21}$, $d = w_{12} \oplus w_{22}$.

A partially ordered set M satisfies the *Riesz interpolation property* ((RIP), for short) if for every $a, b, c, d \in M$ such that $a, b \leq c, d$[1], then there is $x \in M$ such that $a, b \leq x \leq c, d$.

If M is a lattice, then it satisfies (RIP). It was proved in [24] that (RDP) implies (RIP), but the converse implication need not hold. Indeed, in a lattice ordered effect algebra, (RIP) always holds, but there are examples of lattice ordered effect algebras not satisfying (RDP) (see, e.g., so called diamond in [8]).

[1] $a, b \leq c, d$ is an abbreviation of $a \leq c$, $a \leq d$ and $b \leq c$, $b \leq d$.

By the results of [5], [6], [21] a lattice ordered effect algebra is an MV-algebra if and only if every two elements in E are compatible. Equivalently, a lattice ordered effect algebra is an MV-algebra if and only if E satisfies (RDP).

3. Compatibility and blocks

In the previous section, we already introduced the notion of Mackey compatibility: an ordered pair (a,b) in an effect algebra E is compatible ($a \leftrightarrow b$) if there is a triple (a_1, b_1, c) in E such that $a_1 \oplus b_1 \oplus c$ is defined and $a = a_1 \oplus c$, $b = b_1 \oplus c$. Simple basic properties of Mackey compatibility are included in the following lemma.

LEMMA 3.1. ([17], [8]) *Let E be an effect algebra, $a,b \in E$, and (a_1, b_1, c) be a Mackey decomposition of (a,b).*

(i) *$a \leftrightarrow b$ if and only if $b \leftrightarrow a$, and the Mackey triple corresponding to (b,a) is (b_1, a_1, c).*
(ii) *$a \perp b$ implies $a \leftrightarrow b$ with $a = a \oplus 0$, $b = b \oplus 0$.*
(iii) *$a \le b$ implies $a \leftrightarrow b$ with $a = 0 \oplus a$, $b = b \ominus a \oplus a$.*
(iv) *$a \leftrightarrow b$ implies $a' \leftrightarrow b$ with $a' = d' \oplus b_1$, $b = c \oplus b_1$, where $d := a_1 \oplus b_1 \oplus c$.*
(v) *$a \leftrightarrow b$ if and only if there are elements $c, d \in E$ such that $c \le a, b \le d$ with $a \ominus c = d \ominus b$ (equivalently, $b \ominus c = d \ominus a$).*
(vi) *If E is an orthomodular poset, and $a \leftrightarrow b$ with Mackey decomposition (a_1, b_1, c), then $a \vee b$ and $a \wedge b$ exist, and $c = a \wedge b$, $a_1 \oplus b_1 \oplus c = a \vee b$.*

Moreover, in every effect algebra E, the following holds.

LEMMA 3.2. *Let E be an effect algebra. Let $a,b,c \in E$ be such that $b \le c$, $a \vee b \in E$, $a \wedge b \in E$ and $b \ominus (a \wedge b) = (a \vee b) \ominus a$, $a \ominus (a \wedge c) = (a \vee c) \ominus c$. Then $a \leftrightarrow b$, $a \leftrightarrow c$ and $a \leftrightarrow (c \ominus b)$.*

Proof. Lemma 3.1(v) implies that $a \leftrightarrow b$ and $a \leftrightarrow c$. To show that $a \leftrightarrow (c \ominus b)$, we follow the pattern of [7; Proposition 3.1(ii)]. Since $a \wedge b \le a \wedge c$, there is an element $w \in E$ such that $(a \wedge b) \oplus w = a \wedge c$. Therefore $a \vee b = \big(b \ominus (a \wedge b)\big) \oplus \big(a \ominus (a \wedge b)\big) \oplus (a \wedge b) \le a \vee c = \big(a \ominus (a \wedge c)\big) \oplus \big(c \ominus (a \wedge c)\big) \oplus (a \wedge c)$. From $a = (a \wedge b) \oplus \big(a \ominus (a \wedge b)\big) = (a \wedge c) \oplus \big(a \ominus (a \wedge c)\big)$, we have $b \ominus (a \wedge b) \le c \ominus (a \wedge c)$. There exists an element $v \in E$ such that $\big(b \ominus (a \wedge b)\big) \oplus v = c \ominus (a \wedge c)$. Then $c = \big(c \ominus (a \wedge c)\big) \oplus a \wedge c = (a \wedge b) \oplus w \oplus \big(b \ominus (a \wedge b)\big) \oplus v$ and $c \oplus \big(a \ominus (a \wedge c)\big) = (a \wedge b) \oplus w \oplus v \oplus \big(b \ominus (a \wedge b)\big) \oplus \big(a \ominus (a \wedge c)\big) \in E$. Hence $c \ominus b = w \oplus v$ and $a = w \oplus \big[(a \wedge b) \oplus \big(a \ominus (a \wedge c)\big)\big]$, which concludes $a \leftrightarrow (c \ominus b)$. □

Observe that, owing to (vi) of Lemma 3.1, Mackey decompositions in orthomodular posets are uniquely defined, but in a general situation, there may exist several Mackey decompositions for a pair of compatible elements (cf. e.g. [19]). The following lemma was proved in [22].

LEMMA 3.3. *Let (a_1, b_1, c) be a Mackey decomposition of (a, b). Let $c \leq \bar{c} \leq a, b$. Then $(a \ominus \bar{c}, b \ominus \bar{c}, \bar{c})$ is a Mackey decomposition of (a, b).*

Proof. We can write $a = \bar{c} \oplus (a \ominus \bar{c})$, $b = \bar{c} \oplus (b \ominus \bar{c})$. Then $a \oplus (b \ominus \bar{c}) \leq a \oplus (b \ominus c) = a_1 \oplus b_1 \oplus c$. It follows that $(a \ominus \bar{c}, b \ominus \bar{c}, \bar{c})$ is a Mackey decomposition of (a, b). □

In [21], the following strengthening of Mackey compatibility was introduced. We say that a, b are *strongly compatible* (written $a \overset{c}{\leftrightarrow} b$) and that (a_1, b_1, c) is a *strong Mackey decomposition* of (a, b), if $a \leftrightarrow b$ with $a = a_1 \oplus c$, $b = b_1 \oplus c$, $d := a_1 \oplus b_1 \oplus c$ exists and $a_1 \wedge b_1 = 0$.

Clearly, $a \overset{c}{\leftrightarrow} b$ implies $a \leftrightarrow b$, the converse need not hold, in general. The converse implication holds trivially in all orthoalgebras, since there $a_1 \perp b_1$ implies $a_1 \wedge b_1 = 0$. In lattice ordered effect algebras, Lemma 3.3 and Proposition 3.4 below imply that pairwise compatible elements are also strongly compatible (see also [21]). In [22], the following characterizations of strong compatibility were found. For the convenience of readers, we include the proofs.

PROPOSITION 3.4. *Let a, b be elements of an effect algebra E. Let (a_1, b_1, c) be a Mackey decomposition of (a, b). The following statements are equivalent.*

(i) *(a_1, b_1, c) is a strong Mackey decomposition of (a, b).*
(ii) *c is a maximal lower bound of a, b.*
(iii) *$d := a_1 \oplus b_1 \oplus c$ is a minimal upper bound of a, b.*

Proof.

(i) $\Longrightarrow$ (ii): Let (a_1, b_1, c) be a strong Mackey decomposition of (a, b). Let $\bar{c} \in E$ be such that $c \leq \bar{c} \leq a, b$. There is an $x \in E$ such that $\bar{c} = c \oplus x$. We have

$$a = \bar{c} \oplus (a \ominus \bar{c}) \tag{1}$$

$$= c \oplus x \oplus (a \ominus \bar{c}) = c \oplus a_1 , \tag{2}$$

hence $a_1 = x \oplus (a \ominus \bar{c})$,

$$b = \bar{c} \oplus (b \ominus \bar{c}) \tag{3}$$

$$= c \oplus x \oplus (b \ominus \bar{c}) = c \oplus b_1 , \tag{4}$$

hence $b_1 = x \oplus (b \ominus \bar{c})$. It follows that $x \leq a_1, b_1$, so $x = 0$. Therefore c is a maximal lower bound of a, b.

(ii) $\Longrightarrow$ (i): Let c be a maximal lower bound of a, b and let (a_1, b_1, c) be a Mackey decomposition of (a, b). Assume that $x \leq a_1, b_1$. Then $a_1 = x \oplus \bar{a}_1$,

$b_1 = x \oplus \bar{b}_1$, so that $a = x \oplus c \oplus \bar{a}_1$, $b = x \oplus c \oplus \bar{b}_1$. Hence $c \le x \oplus c \le a, b$. As c is a maximal lower bound of a, b, then $x = 0$, and (a_1, b_1, c) is a strong Mackey decomposition of (a, b).

(ii) $\iff$ (iii): Assume that (a_1, b_1, c) is a Mackey decomposition of (a, b). Put $d := a_1 \oplus b_1 \oplus c$. Then $a \ominus c = d \ominus b$. Let there be a $\bar{d}$ with $a, b \le \bar{d} \le d$. There is $y \in E$ such that $d = \bar{d} \oplus y$. So we obtain $a \ominus c = (\bar{d} \oplus y) \ominus b = (\bar{d} \ominus b) \oplus y$, and hence $(a \ominus c) \ominus y = \bar{d} \ominus b$, which implies that $a \ominus (c \oplus y) = \bar{d} \ominus b$. Hence $c \oplus y \le a$, and similarly we prove that $c \oplus y \le b$. So if d is not a minimal upper bound of a, b, then c is not a maximal lower bound of a, b. By reversing the implications we obtain the converse. It follows that c is a maximal lower bound of a, b if and only if d is a minimal upper bound of a, b. □

As $a \leftrightarrow b$ implies $a' \leftrightarrow b'$, and if c is a maximal lower bound of a, b, then c' is a minimal upper bound of a', b', we conclude that $a \overset{c}{\leftrightarrow} b$ if and only if $a' \overset{c}{\leftrightarrow} b'$. But in general it need not hold that $a \overset{c}{\leftrightarrow} b$ implies $a' \overset{c}{\leftrightarrow} b$.

We will say that E satisfies the *strong difference compatibility* property (SDC) if the following condition is satisfied:

(SDC) $a \overset{c}{\leftrightarrow} b$, $a \overset{c}{\leftrightarrow} c$ and $b \le c \implies a \overset{c}{\leftrightarrow} (c \ominus b)$.

If E is lattice ordered, then for $a, b \in E$, $a \leftrightarrow b$ if and only if $a \overset{c}{\leftrightarrow} b$ if and only if $(a \vee b) \ominus a = b \ominus (a \wedge b)$ ([6], [8; Theorem 1.10.6]), so that Lemma 3.2 implies that (SDC) is satisfied.

Observe that by Lemma 3.1(vi), pairwise compatibility in orthomodular posets coincides with strong (pairwise) compatibility. We may conclude from Lemma 3.2 that (SDC) is satisfied. In the case of orthomodular posets it turned out that to describe their blocks (maximal Boolean subalgebras), pairwise compatibility may be not sufficient and a stronger notion of so called f-compatibility is to be introduced ([20]). Recall that pairwise compatibility coincides with f-compatibility if and only if the orthomodular poset is *regular*, i.e., for every three pairwise compatible elements elements a, b, c we have $a \leftrightarrow b \vee c$ ([20]). As an example of a non-regular orthomodular poset, we may consider the family of all subsets with even cardinality of the set $\{1,2,3,4,5,6,7,8\}$. Recall that two elements a, b are compatible if and only if $a \cap b$ is of even cardinality (see [20; Exercise 1.4.11]). Put $a = \{1,2,3,4\}$, $b = \{1,2,5,6\}$, $c = \{2,3,5,7\}$. Then a, b, c are pairwise compatible, but $b \vee c$ is not compatible with a.

On the other hand, if E is an effect algebra with (RIP), then $a \overset{c}{\leftrightarrow} b$ implies that $a \wedge b$ and $a \vee b$ exist. In addition, for every three pairwise strongly compatible elements a, b, c it holds that $a \overset{c}{\leftrightarrow} b \vee c$ ([7]), however, (SDC) may fail. Indeed, consider the following example (see [7; Example 2.1]). Let G be the additive group $\mathbb{R}^2$ with the positive cone of all (x, y) such that either $x = y = 0$ or $x > 0$ and $y > 0$. The interval $E = [0, u]$, where $u = (1, 1)$, is an effect algebra with (RDP), which is an anti-lattice[2]. Take $a = (0.6, 0.8)$, $b = (0.3, 0.2)$, $c = (0.5, 0.5)$. Then $b \leq c \leq a$, hence $a \wedge c$, $b \wedge c$ exist in E, but $a \ominus b = (0.3, 0.6)$ is not comparable with c, hence $(a \ominus b) \wedge c$ does not exist in E.

A generalization of f-compatibility to effect algebras is the notion of joint compatibility described below.

Let $C := (c_1, \dots, c_n)$ be a finite sequence of elements of E. We say that C is *jointly orthogonal* (or simply *orthogonal*) if the sum $c_1 \oplus c_2 \oplus \cdots \oplus c_n$ exists in E. We then write $\bigoplus C := c_1 \oplus c_2 \oplus \cdots \oplus c_n$.

A finite family M of elements of E is said to be *jointly compatible* (or simply *compatible*) if there is a finite orthogonal sequence $C = (c_1, \dots, c_n)$ such that for every $a \in M$ there is a set $I_a \subset \{1, \dots, n\}$ such that $a = \bigoplus_{i \in I_a} c_i$. We say that C *covers* M, and call C a *cover* of M. If M is an arbitrary subset of E, we say that M is (jointly) compatible if every finite subset of M is compatible. Let F be a subset of E. We say that elements $A = (a_1, \dots, a_n)$ are compatible in F if A is compatible with a cover $C \subset F$. We say that $M \subset E$ is *internally compatible* if M is compatible in M. We say that (a, b) are Mackey compatible in F if there is a Mackey decomposition (a_1, b_1, c) of (a, b) with $a_1, b_1, c \in F$. It is straightforward to show that if $(a_1, \dots, a_n)$ are compatible in F, then elements in every subsequence of $(a_1, \dots, a_n)$ are compatible in F. In particular, for every $i, j \in \{1, 2, \dots, n\}$, (a_i, a_j) are Mackey compatible in F.

In [12], the following weakening of the (RDP) was introduced: We say that an effect algebra E is *homogeneous* if, whenever $a, b, c \in E$ are such that $a \leq b \oplus c$, $a \leq (b \oplus c)'$, there are $a_1 \leq b$ and $a_2 \leq c$ with $a = a_1 \oplus a_2$.

Notice that

(i) every effect algebra satisfying (RDP) is homogeneous,
(ii) every orthoalgebra is homogeneous,
(iii) every lattice ordered effect algebra is homogeneous.

(i) follows directly from the definition. (ii): if $a \leq b \oplus c$ and $a \leq (b \oplus c)'$, then $a \perp a$, hence $a = 0$. (iii): $a \leq b \oplus c$, $a \leq (b \oplus c)'$ imply that a, b, c are pairwise compatible, hence by [25], they are contained in a block which is MV-sub-effect algebra of E, and since MV-algebras satisfy (RDP), E is homogeneous.

A *block* in a homogeneous effect algebra is defined as a maximal sub-effect algebra B of E satisfying (RDP). The following modification of Lemma 3.2 was proved ([12; Corollary 3.3]).

PROPOSITION 3.5. *Let M be a finite compatible subset of a homogeneous effect algebra E. Let $a, b \in M$ be such that $a \geq b$. Then $M \cup \{a \ominus b\}$ is a compatible set.*

[2] i.e., $a \wedge b$ and $a \vee b$ exist if and only if a and b are comparable.

Observe that Proposition 3.5 implies that if M is a finite compatible set, then $M \cup \{1\}$ is a compatible set and, if $a \in M$, then $1 \geq a$ implies $M \cup \{a'\}$ is a compatible set. From the equality $a \oplus b = a' \ominus b$ whenever $a \perp b$, we derive that $M \cup \{a \oplus b\}$ is a compatible set.

In [12], the following results were obtained.

THEOREM 3.6. *Let E be a homogeneous effect algebra.*

(i) *Blocks can be characterized as maximal internally compatible subsets of E containing 1.*
(ii) *Every finite compatible subset of E can be embedded into a block.*

4. Blocks and orthocompleteness

Let m be any cardinal. An effect algebra E is called *m-orthocomplete* if every orthogonal family in E of cardinality at most m has an $\oplus$-sum in E.

We will say that a sub-effect algebra L of an m-orthocomplete effect algebra E is *sub-m-orthocomplete* if for every orthogonal subfamily $J \subset L$ with cardinality at most m, we have $\bigoplus J \in L$.

An effect algebra E is *monotone m-complete* if every ascending chain of elements of E of cardinality at most m has a least upper bound in E (dually, every descending chain of cardinality at most m has a greatest lower bound in E).

An effect algebra S has the *m-complete Riesz interpolation property* if for any collections $(a_i)_{i\in I}$, $(b_j)_{j\in J}$, where I, J are arbitrary index sets with cardinality at most m, we have $a_i \leq b_j$ for all $i \in I$, $j \in J$, there exists $c \in E$ such that $a_i \leq c \leq b_j$ for all $i \in I$, $j \in J$.

If $m = \aleph_0$, we obtain the following.

An effect algebra E is *σ-orthocomplete* if every at most countable orthogonal set has an $\oplus$-sum in E. E is *monotone σ-complete* if every countable ascending chain has a supremum in E (and dually, every countable descending chain has an infimum in E).

We say that E satisfies *countable* (RIP) if for every countable collections $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ of elements of E such that $a_i \geq b_j$ for all $i \in I$, $j \in J$ there is an element $x \in E$ such that $a_i \geq x \geq b_j$ for all $i \in I$, $j \in J$.

It was proved in [13] that an effect algebra E is σ-orthocomplete if and only if it is monotone σ-complete. More generally, for every cardinal m, an effect algebra E is m-orthocomplete if and only if E is monotone m-complete ([14]).

Recall that an effect algebra E is *Archimedean* if for every $a \neq 0$, the isotropic index $\iota(a)$ is finite.

A σ-orthocomplete effect algebra is Archimedean. Indeed, assume that na is defined for all $n \in \mathbb{N}$, then $(a_i)_{i\in\mathbb{N}}$, $a_i = a$ for all $i \in \mathbb{N}$, is an orthogonal family, so $\bigoplus\limits_{i\in\mathbb{N}} a_i$ exists in E, and $\bigoplus\limits_{i\in\mathbb{N}} a_i = a \oplus \bigoplus\limits_{i\in\mathbb{N}} a_i$ entails $a = 0$.

PROPOSITION 4.1. *Let E be a monotone σ-complete effect algebra with* (RIP). *Then E satisfies countable* (RIP).

P r o o f. Let $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ be countable collections of elements of E such that $a_i \geq b_j$ for all $i \in I$, $j \in J$. We may assume that $I = J = \mathbb{N}$, the set of positive integers. Select x_1 in E such that $a_1, a_2 \geq x_1 \geq b_1, b_2$. Now select $x_2 \in E$ such that $a_1, a_2 \geq x_2 \geq x_1, b_3$. This process may obviously be continued to obtain an ascending sequence $x_1 \leq x_2 \leq \dots$ such that $a_1, a_2 \geq x_i \geq b_1, b_2, \dots, b_{i+1}$. Since E is monotone σ-complete, $y_1 = \bigvee\limits_{i\in\mathbb{N}} x_i$ exists in E, and $a_1, a_2 \geq y_1 \geq b_j$ for all $j \in \mathbb{N}$. We may repeat this process to obtain $y_2 \in E$ such that $y_1, a_3 \geq y_2 \geq b_j$ and we may continue thereby constructing a descending chain $y_1 \geq y_2 \geq \dots$. As E is monotone σ-complete, $x = \bigwedge\limits_{i\in\mathbb{N}} y_i$ exists in E, and the fact that

$$(\forall i, j \in \mathbb{N})\,(a_i \geq x \geq b_j)$$

follows easily. □

An effect algebra E is *orthocomplete* if it is m-orthocomplete for any cardinal m. Similarly we define monotone complete effect algebra and complete (RIP).

In [22], the following observation can be found.

PROPOSITION 4.2. *For all elements a, b of an orthocomplete effect algebra, $a \leftrightarrow b$ implies $a \overset{c}{\leftrightarrow} b$.*

THEOREM 4.3. *Let E be an effect algebra. The following statements are equivalent.*

(i) *E is orthocomplete and satisfies* (RIP).
(ii) *E is monotone complete and satisfies* (RIP).
(iii) *E is lattice ordered and satisfies complete* (RIP).
(iv) *E is a complete lattice.*

P r o o f. The equivalence of (i) and (ii) follows from [14].

(iii) $\Longrightarrow$ (iv): Let M be any subset of elements of E. Let $\nabla(M)$ denote the set of all upper bounds of M. Since $1 \in \nabla(M)$, $\nabla(M) \neq \emptyset$. For all $a \in M$, $b \in \nabla(M)$ we have $a \leq b$ and from complete (RIP), there exists $c \in E$ such that $a \leq c \leq b$ for all $a \in M$ and $b \in \nabla(M)$. It follows that $c \in \nabla(M)$, and since $c \leq b$ for any other $b \in \nabla(M)$, c is the least upper bound of M, i.e. $c = \bigvee M$. The existence of $\bigwedge M$ can be proved analogously.

(iv) $\Longrightarrow$ (iii): obvious.

(ii) $\Longrightarrow$ (iv): Let $\nabla(a,b)$ denote the set of all upper bounds of elements $a,b \in E$. Since E is monotone complete, every descending chain in $\nabla(a,b)$ has a greatest lower bound in E, which clearly belongs to $\nabla(a,b)$. By Zorn's lemma, there is a minimal element, say d, in $\nabla(a,b)$. Let $c \in \nabla(a,b)$ be arbitrary. Then $a,b \le c,d$ and owing to (RIP), there is $x \in E$ with $a,b \le x \le c,d$. By minimality of d, $x = d$, hence $d \le c$. This proves that $d = a \vee b$, and E is a lattice. To prove that E is a complete lattice, let M be any subset of elements of E, and let $\nabla(M)$ be the set of all upper bounds of M. Since E is monotone complete, every descending chain in $\nabla(M)$ has a greatest lower bound in E, which belongs to $\nabla(M)$. By Zorn's lemma, there is a minimal element in $\nabla(M)$, say v. For any other $b \in \nabla(M)$, $b \wedge v \in \nabla(M)$, and minimality of v implies that $v \le b$. So v is the least upper bound of M. This proves that E is a complete lattice.

(iv) $\Longrightarrow$ (i): obvious. $\square$

We note that for the case of lattice ordered effect algebras, the equivalence of (i) and (iv) was proved in [27].

It is well known that an orthomodular lattice L is orthocomplete (i.e., it is a complete lattice) if and only if all its blocks are orthocomplete (i.e., are complete Boolean algebras), see e.g. [28; p. 23].

For the case of orthoalgebras, some relations between orthocompleteness of the orthoalgebra and its blocks were shown in [29].

In this section, we prove that for lattice ordered effect algebras, similar results to those for orthomodular lattices can be obtained.

THEOREM 4.4. *A lattice ordered effect algebra E is orthocomplete if and only if every block of E is orthocomplete.*

Proof. Let E be orthocomplete. By Theorem 4.3, E is a complete lattice. Let B be a block of E, and M an arbitrary subset of B. Let $a = \bigvee M$ (in E). By [15], for every $b \in B$, $b \leftrightarrow m$ for every $m \in M$ implies $a \leftrightarrow b$. Since B is maximal pairwise compatible set, $a \in B$, and therefore B is a complete MV-subalgebra of E (see also [26]).

Conversely, let every block in E be a complete MV algebra. Choose an orthogonal family M in E, and denote by $\mathcal{F}(M)$ the class of all finite subfamilies of M. We may assume that there is an upper bound of $\{\bigoplus F : F \in \mathcal{F}(M)\}$ different from 1. Then there is an element $a \in E$, $a \neq 0$, $a \perp \bigoplus F$ for all $F \in \mathcal{F}(M)$. Denote by $\mathcal{A}$ the class of all orthogonal families A of E consisting of nonzero elements and such that $(G \cup F)$ is an orthogonal family for all $F \in \mathcal{F}(M)$, $G \in \mathcal{F}(A)$, $A \in \mathcal{A}$, i.e., $\bigoplus F \le (\bigoplus G)'$ for all $F \in \mathcal{F}(M)$, $G \in \mathcal{F}(A)$. The $\mathcal{A}$ is nonempty and can be partially ordered by $A_1 \le A_2$ if

$A_1 \subset A_2$. If $(A_i)_i$ is a chain in $\mathcal{A}$ and $A_0 = \bigcup_i A_i$, then $A_0 \in \mathcal{A}$. By Zorn's lemma, there is a maximal element $\bar{A}$ in $\mathcal{A}$. Since $\bar{A} \cup M$ is a compatible family, there is a block B of E which contains $\bar{A}$ and M. By hypothesis, B is orthocomplete, hence there exists

$$d = \bigoplus_B \bar{A} = \bigvee_B \left\{ \bigoplus G : \ G \in \mathcal{F}(\bar{A}) \right\}$$

in B, and we have $\bigoplus F \leq d'$ for all $F \in \mathcal{F}(M)$. We will show that d' is equal to $\bigoplus M = \bigvee \{ \bigoplus F : \ F \in \mathcal{F}(M) \}$ in E.

Let there exist $b \in E$ such that $\bigoplus F \leq b$ for all $F \in \mathcal{F}(M)$ such that $d' \wedge b \neq d'$. Put $c = d' \ominus (d' \wedge b)$, then $c \neq 0$ and $\bar{A} \cup \{c\} \in \mathcal{A}$ (indeed, $\bigoplus_B (\bar{A} \cup \{c\}) = d \oplus (d' \ominus d' \wedge b) = (d' \wedge b)'$, and for every $F \in \mathcal{F}(M)$, $\bigoplus F \leq d' \wedge b$), contradicting maximality of $\bar{A}$. Hence $d' = \bigoplus M$. □

It is not known whether an orthocomplete homogeneous effect algebra has orthocomplete blocks. In the opposite direction, we have the following result, which says in short, that if all blocks are orthocomplete and the $\oplus$-sums are block-independent, then E is orthocomplete (compare with [29; Theorem 2.5]).

THEOREM 4.5. *Let E be a homogeneous effect algebra. Assume that for every orthogonal set X in E, we have that for all blocks B of E with $X \subseteq B$, the $\oplus$-sum $\bigoplus_B X$ in B exists, and if A, B are two blocks with $X \subseteq A \cap B$, then $\bigoplus_B X = \bigoplus_A X$. Then E is orthocomplete, and has sub-orthocomplete blocks.*

Proof. Let S be an orthogonal subset of E, then there is a block B such that $S \subset B$. By hypotheses, $c := \bigoplus_B S$ exists. For a finite subset $F \subset S$, define $a_F = \bigoplus F$. Let d be any upper bound of the set $\{a_F : \ F \subset S, \ F \text{ finite}\}$. We may suppose that $d \neq 1$. Then $S \cup \{d'\}$ is an orthogonal subfamily of E, therefore there is a block $B_1 \supset S \cup \{d'\}$. It follows that $a_F \leq d$ for all F, and by hypotheses, $c = \bigvee_{B_1} a_F$, hence $c \leq d$. Hence $c = \bigoplus S$. □

5. Blocks and MV-algebras

The question which effect algebras can be covered by MV-algebras was studied in [7]. It was shown there that every effect algebra E satisfying RIP and an additional condition, called the *difference-meet property* (DMP):

(DMP) If $x \leq y$, $x \wedge z \in E$ and $y \wedge z \in E$, then $(y \ominus x) \wedge z \in E$

is a set-theoretical union of MV-algebras [7; Theorem 3.3]. These MV-algebras are formed by maximal pairwise strongly compatible subsets of E.

LEMMA 5.1. *In an effect algebra E with* (RDP)*, conditions* (SDC) *and* (DMP) *are equivalent.*

Proof. Owing to (RDP), for every $a, b \in E$ we have $a \leftrightarrow b$ (it easily follows from $a \oplus a' = b \oplus b'$). If $a \wedge b \in E$, then Lemma 3.3 and Proposition 3.4 imply that $a \overset{c}{\leftrightarrow} b$. Conversely, if $a \overset{c}{\leftrightarrow} b$, then Proposition 3.4 implies that there is a maximal lower bound of a, b, and from (RIP) we obtain that $a \wedge b$ exists in E. Assume that (SDC) holds. If $x \le y$ and $x \wedge z, y \wedge z \in E$, then $x \overset{c}{\leftrightarrow} z$ and $y \overset{c}{\leftrightarrow} z$, which by (SDC) implies $z \overset{c}{\leftrightarrow} y \ominus x$, which in turn implies $z \wedge (y \ominus x) \in E$. The converse statement is obtained analogously. □

Applying [7; Theorem 3.3], we obtain the following result.

THEOREM 5.2. *Every homogeneous effect algebra E such that every block B of E satisfies* (DMP) (*or* (SDC)*, equivalently*) *can be covered by MV-algebras.*

Proof. By [12], every homogeneous effect algebra E is a set-theoretical union of blocks which are sub-effect algebras of E with (RDP). Since (RDP) implies (RIP), together with (DMP) it implies, by [7], that every block is a set-theoretical union of MV-algebras. □

The condition that every block in a homogeneous effect algebra satisfies (SDC) is equivalent to the following condition. For every (jointly) compatible elements a, b, x in E and every block B containing a, b, x, if $b \le a$ and both a and b are strongly compatible with x in B, then $a \ominus b$ is strongly compatible with x in B. In particular, if a belongs to a block B, then the elements $a \wedge_B a'$, $a \vee_B a'$ must exist in B.

Notice that joint compatibility cannot be omitted. E.g., in the orthoalgebra with orthogonal triples of atoms $(a, b, c), (c, d, e), (e, f, a)$ (Wright triangle), c is strongly compatible with a and e', and $a \le e'$, but (c, a, e') are not jointly compatible, hence do not belong to a block.

Lattice operations in the MV-algebras of Theorem 5.2 are taken in the corresponding blocks, and do not necessarily agree with the lattice operations in the whole E. E.g., in orthoalgebras which are not orthomodular posets, blocks are Boolean algebras, in which the lattice operations are only local. On the other hand, orthomodular posets and lattice ordered effect algebras are covered with MV-algebras, in which the lattice operations agree with the global lattice operations, taken in the whole E.

From Theorem 4.3, we obtain the following.

COROLLARY 5.3. *Every homogeneous effect algebra with orthocomplete blocks is a set-theoretical union of MV-algebras, which coincide with its blocks.*

We note that in the above theorem, the lattice operations are only local, taken with respect to the corresponding blocks.

An example of an effect algebra which is not homogeneous and can be covered by MV-algebras, is the Hilbert space effect algebra $\mathcal{E}(H)$ $(\dim H \geq 1)$. Here the MV-algebras consist of maximal sets of pairwise commuting effects ([22]). We note the lattice operations in these MV-algebras do not agree with the lattice operations in the whole $E(H)$ (the latter may even not exist).

An *observable* on an effect algebra E is a σ-morphism $x\colon B \to E$, where B is a Boolean σ-algebra. In applications in physics, B usually is the σ-algebra of Borel subsets of $\mathbb{R}^n$. If $B = \mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of subsets of the real line $\mathbb{R}$, then $x\colon B \to E$ is called a *real observable.*

The following theorem is a generalization of [20; Proposition 1.3.8].

THEOREM 5.4. *Let E be an effect algebra. Let $(a_i)_i$ be a countable family of elements of E, and let $b \in E$ be such that $a_i \wedge b$, $a_i \vee b$ exist in E, and $b \ominus (a_i \wedge b) = (a_i \vee b) \ominus a_i$ for all i, and the elements $\bigvee a_i$, $\bigvee b \wedge a_i$ exist in E. Then $b \overset{c}{\leftrightarrow} \bigvee a_i$, and $\bigvee b \wedge a_i$ is a maximal lower bound of b and $\bigvee a_i$.*

P r o o f. Notice that the hypotheses imply that $b \overset{c}{\leftrightarrow} a_i$ for all i. Denote $c := \bigvee b \wedge a_i$, $a := \bigvee a_i$. Clearly, $c \leq a, b$, and $b \ominus (b \wedge a_i) = (b \vee a_i) \ominus a_i$ implies that $b \ominus c \leq b \ominus (b \wedge a_i) \leq a_i'$, which in turn implies $b \ominus c \perp a$. So we may write $a = (a \ominus c) \oplus c$, $b = (b \ominus c) \oplus c$, and $(b \ominus c) \oplus (a \ominus c) \oplus c$ exists, hence a is compatible with b. Moreover, for every i, $b \ominus (b \wedge a_i) \geq b \ominus c$. Assume that $d \leq \big(b \ominus (b \wedge a_i)\big) = (b \vee a_i) \ominus a_i$. Then $d \oplus a_i \leq b \vee a_i \leq (b \ominus c) \oplus a$ for all i, hence $a_i \leq \big((b \ominus c) \oplus a\big) \ominus d$ for all i, so that $a \leq \big((b \ominus c) \oplus a\big) \ominus d$, consequently, $a \oplus d \leq (b \ominus c) \oplus a$, and finally, $d \leq b \ominus c$. This entails that

$$b \ominus c = \bigwedge \big(b \ominus (b \wedge a_i)\big) .$$

We will prove that $(b \ominus c) \wedge (a \ominus c) = 0$. Assume that $z \leq b \ominus c$, $z \leq (a \ominus c)$. This yields for all i, $b \wedge a_i \oplus c \leq z \oplus c \leq a, b$. Applying Lemma 3.3, we obtain $a \ominus (z \oplus c) \perp b \ominus (z \oplus c)$. Repeating the same process with $z \oplus c$ instead of c, we obtain $b \ominus c = b \ominus (z \oplus c)$, hence $z = 0$. Hence a, b are strongly compatible. □

COROLLARY 5.5. *Let E be a homogeneous effect algebra with σ-orthocomplete blocks satisfying* (DMP). *Then E can be covered by Archimedean MV-algebras.*

P r o o f. Since the blocks are σ-orthocomplete, hence monotone σ-complete, for every ascending sequence $(a_i)_i$ of elements of a block B, the supremum $\bigvee_i a_i$ exists in B. If $(a_i)_i$ and b belong to an MV-subalgebra A, say, of B, then $a_i \overset{c}{\leftrightarrow} b$ for all i entails $a_i \wedge b$ exist in B for all i, as well as $\bigvee_i a_i \wedge b \in B$. Theorem 5.4 implies that $\bigvee_i a_i \overset{c}{\leftrightarrow} b$, hence $\bigvee_i a_i \in A$. Hence A is monotone σ-complete, so that A is an Archimedean MV-algebra. □

According to [8; Theorem 6.1.32], every Archimedean MV-algebra can be considered as an MV-algebra of fuzzy sets $\mathcal{F}(\Omega) \subseteq [0,1]^\Omega$ for $\Omega \neq \emptyset$. By [23], every MV-algebra of fuzzy sets is the range of an observable $\lambda\colon \mathcal{B}([0,1])^\Omega \to [0,1]^\Omega$. Therefore, under the assumptions of Corollary 5.5 E can be covered by ranges of observables.

REFERENCES

[1] BERAN, L.: *Orthomodular Lattices — Algebraic Approach*, Academia, Czechoslovak Academy of Sciences/D. Reidel Publishing Company, Praha/Dordrecht, 1984.

[2] BUSCH, P.—LAHTI, P. J.—MITTELSTADT, P.: *The Quantum Theory of Measurement.* Lecture Notes in Phys. New Ser. m Monogr. 31, Springer-Verlag, Berlin-Heidelberg-New York-London-Budapest, 1991.

[3] BUSCH, P.—GRABOWSKI, M.—LAHTI, P. J.: *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995.

[4] CHANG, C. C.: *Algebraic analysis of many-valued logic*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[5] CHOVANEC, F.—KÔPKA, F.: *D-lattices*, Internat. J. Theoret. Phys. **34** (1995), 1297–1302.

[6] CHOVANEC, F.—KÔPKA, F.: *Boolean D-posets*, Tatra Mt. Math. Publ. **10** (1997), 183–197.

[7] DVUREČENSKIJ, A.: *On effect algebras that can be covered by MV-algebras*, Internat. J. Theoret. Phys. **41** (2002), 221–229.

[8] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Academic Publ./Ister Science, Dordrecht-Boston-London/Bratislava, 2000.

[9] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1331–1362.

[10] FOULIS, D. J.—GREECHIE, R.—RÜTTIMANN, G.: *Filters and supports in orthoalgebras*, Internat. J. Theoret. Phys. **35** (1995), 789–802.

[11] GIUNTINI, R.—GRUEULING, H.: *Toward a formal language for unsharp properties*, Found. Phys. **19** (1994), 769–780.

[12] JENČA, G.: *Blocks in homogeneous effect algebras*, Bull. Austral. Math. Soc. **64** (2001), 81–98.

[13] JENČA, G.: *A Cantor-Bernstein type theorem for effect algebras*, Algebra Universalis **48** (2002), 399–411.

[14] JENČA, G.—PULMANNOVÁ, S.: *Quotients of partial abelian monoids and the Riesz decomposition property*, Algebra Universalis **47** (2002), 443–477.

[15] JENČA, G.—RIEČANOVÁ, Z.: *On sharp elements in lattice ordered effect algebras*, BUSEFAL **80** (1999), 24–29.

[16] KALMBACH, G.: *Orthomodular Lattices*, Academic Press, London-New York, 1983.

[17] KÔPKA, F.: *Compatibility in D-posets*, Internat. J. Theoret. Phys. **34** (1995), 1525–1531.

[18] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[19] LOCK, P. L.—HARDEGREE, G. M.: *Connections among quantum logics: Part 2. Quantum event logic*, Internat. J. Theoret. Phys. **24** (1985), 55–61.

[20] PTÁK, P.—PULMANNOVÁ, S.: *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publ., Dordrecht-Boston-London, 1991.

[21] PULMANNOVÁ, S.: *On connections among some orthomodular structures*, Demonstratio Math. **30** (1997), 313–328.

[22] PULMANNOVÁ, S.: *Compatibility and decompositions of effects*, J. Math. Phys. **43** (2002), 2817–2830.

[23] PULMANNOVÁ, S.: *A note on observables on MV-algebras*, Soft Computing **4** (2000), 45–48.

[24] RAVINDRAN, K.: *On a Structure Theory of Effect Algebras*. PhD Theses, Kansas State Univ., Manhattan, Kansas, 1996.

[25] RIEČANOVÁ, Z.: *A generalization of blocks for lattice effect algebras*, Internat. J. Theoret. Phys. **39** (2000), 855–865.

[26] RIEČANOVÁ, Z.: *On order topological continuity of effect algebra operations*. In: Contributions to General Algebra 12, Verlag Johannes Heyn, Klagenfurt, 2000, pp. 349–354.

[27] RIEČANOVÁ, Z.: *Orthogonal sets in effect algebras*, Demonstratio Math. **34** (2001), 525–531.

[28] SARYMSAKOV, T. A. et al: *Uporyadochennye algebry*, FAN, Tashkent, 1983. (Russian)

[29] SCHRÖDER, B.: *On three notions of orthosummability in orthoalgebras*, Internat. J. Theoret. Phys. **34** (1999), 3305–3313.

Received July 1, 2003

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAKIA

E-mail: pulmann@mat.savba.sk