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C_λ-WEDGE AND WEAK C_λ-WEDGE FK-SPACES

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ABSTRACT. In this paper we study the (weak) C_λ-wedge FK-spaces for C_λ methods defined by deleting a set of rows from the Cesáro matrix C_1 and give some characterizations. We also apply these results to summability domains.

1. Introduction and notation

In Section 1 we introduce the notation and terminology while in Section 2 we study the C_λ-wedge and weak C_λ-wedge FK-spaces, some characterizations related to these spaces and compactness of the inclusion mapping are found. In Section 3 we give some applications of results given above to general summability domains. Also some important applications are obtained for some particular summability domains.

Let E be an infinite subset of $\mathbb{N}$ and consider E as the range of a strictly increasing sequence of positive integers, say $E = \{\lambda(n)\}_{n=1}^{\infty}$. The *Cesáro submethod* C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k \qquad (n = 1, 2, \dots),$$

where $\{x_k\}_{k=1}^{\infty}$ is a sequence of a real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. The basic properties of C_λ-method may be found in [1] and [11].

Let w denote the space of all real or complex-valued sequences. It can be topologized with the seminorms $p_n(x) = |x_n|$ $(n = 1, 2, \dots)$. Any vector subspace X of w is a sequence space. A sequence space X with a vector space topology τ is a *K-space* provided that the inclusion map $i\colon (X, \tau) \to w$, $i(x) = x$,

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is continuous. If, in addition, τ is complete, metrizable and locally convex, then (X, τ) is an *FK-space*. So an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals $P_n(x) = x_n$, $(n = 1, 2, \dots)$ are continuous. An FK-space whose topology is normable is called a *BK-space*. The basic properties of FK-spaces may be found in [13], [14] and [16].

By c, c_0, ℓ^∞ we denote the spaces of all convergent sequences, null sequences and bounded sequences, respectively. These are FK-spaces under $\|x\| = \sup_{j\in\mathbb{N}} |x_j|$. ℓ^p, $1 \le p < \infty$, is the space of all absolutely p-summable sequences,

$$\mathfrak{cs} = \Big\{x \in w : \sum_{j=1}^{\infty} x_j \text{ exists}\Big\}$$

is the space of all summable sequences, and $\mathfrak{bs}$ is as the following

$$\mathfrak{bs} = \Big\{x \in w : \sup_{k\in\mathbb{N}} \Big|\sum_{j=1}^{k} x_j\Big| < \infty\Big\}.$$

As usual, ℓ^1 is replaced by ℓ. The sequence spaces

$$\sigma_0(\lambda) = \Big\{x \in w : \lim_{n\to\infty} \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} x_j = 0\Big\}$$

and

$$h(\lambda) = \Big\{x \in w : \lim_{j\to\infty} x_j = 0 \text{ and } \sum_{j=1}^{\infty} \lambda(j)\,|\Delta x_j| < \infty\Big\}$$

are BK-spaces with the norms

$$\|x\|_{\sigma_0(\lambda)} = \sup_{n\in\mathbb{N}} \frac{1}{\lambda(n)} \Big|\sum_{j=1}^{\lambda(n)} x_j\Big|$$

and

$$\|x\|_{h(\lambda)} = \sum_{j=1}^{\infty} \lambda(j)\,|\Delta x_j| + \sup_{j\in\mathbb{N}} |x_j|$$

respectively, where $\Delta x_j = x_j - x_{j+1}$. Also, $\mathfrak{bv}$ and $\mathfrak{bv}_0$ can be shown as the following

$$\mathfrak{bv} = \Big\{x \in w : \sum_{j=1}^{\infty} |x_j - x_{j+1}| < \infty\Big\}, \qquad \mathfrak{bv}_0 = \mathfrak{bv} \cap c_0$$

(see [3], [4], [6] and [7]).

Throughout the paper, e denotes the sequence of ones, i.e., $e = (1, 1, \dots \dots, 1, \dots)$; δ^j $(j = 1, 2, \dots)$ the sequence $(0, 0, \dots, 0, 1, 0, \dots)$ with the one in the jth position; ϕ the linear span of the δ^j's. The topological dual of X is denoted by X'. A sequence x in a locally convex sequence space X is said to have the *property AK* if $x^{(n)} \to x$ in X, where $x^{(n)} = (x_1, x_2, \dots, x_n, 0, \dots) = \sum_{k=1}^{n} x_k \delta^k$. Let $z = \{z_j\}_{j=1}^{\infty} \in w$ be such that $z_j \neq 0$ for every $j = 1, 2, \dots$. Then

$$V_0(z) := \Big\{x \in c_0 : \sum_{j=1}^{\infty} |z_j||\Delta x_j| < \infty\Big\}$$

is an FK-AK space with norm $\|x\|_{V_0(z)} = \sum_{j=1}^{\infty} |z_j|\,|\Delta x_j|$ ([7]). We recall (see [7]) that the β-dual of a subset X of w is defined to be

$$\begin{aligned} X^\beta &= \Big\{y \in w : \sum_{j=1}^{\infty} x_j y_j \text{ converges for all } x \in X\Big\} \\ &= \big\{y \in w : x \cdot y \in \mathfrak{cs} \text{ for all } x \in X\big\}. \end{aligned}$$

For example $\sigma_0^\beta = h$ with $h := \Big\{x \in w : \sum_{j=1}^{\infty} j\,|\Delta x_j| < \infty \text{ and } x \in c_0\Big\}$ (see [4] and [6]).

Following Bennett [3] we say that a K-space (X, τ) containing ϕ is a *weak wedge space* if $\delta^j \to 0$ (weakly) in X. It is a *wedge space* if $\delta^j \to 0$ in X. İnce, in [8], continued to work on Cesáro wedge and weak Cesáro wedge FK-spaces and to give some characterizations.

2. C_λ-Wedge FK-spaces

In this section, the concept of C_λ-wedgeness for an FK-space X containing ϕ is defined, and some characterizations related to this space and compactness of the inclusion mapping are studied.

DEFINITION 2.1. Let (X, τ) be a K-space containing ϕ and

$$\mu^n := \frac{e^{(\lambda(n))}}{\lambda(n)} = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \delta^k = \Big(\underbrace{\tfrac{1}{\lambda(n)}, \tfrac{1}{\lambda(n)}, \dots, \tfrac{1}{\lambda(n)}}_{\lambda(n)}, 0, \dots\Big). \tag{1}$$

If $\mu^n \to 0$ in X, then (X, τ) is called a C_λ*-wedge space*; and if $\mu^n \to 0$ (weakly) in X, then (X, τ) is called a *weak* C_λ*-wedge space.*

We shall now present several examples of C_λ-wedge FK-spaces which are not wedge space. For example $c, c_0, \ell^\infty, \mathfrak{bv}, \mathfrak{bv}_0$, and ℓ^p $(p > 1)$ are C_λ-wedge FK-spaces, but these are not wedge spaces. Also, $\mathfrak{bv}_0$ is weak C_λ-wedge space but not wedge space.

Let $s = \{s_n\}_{n=1}^{\infty}$ denote throughout a strictly increasing sequence of non-negative integers with $s_1 = 0$. Let $c|s|(\lambda)$ designate the space defined by

$$c|s|(\lambda) = \left\{x \in c_0 : \sup_{n\in\mathbb{N}} \sum_{j=s_n+1}^{s_{n+1}} \lambda(j)\,|\Delta x_j| < \infty\right\}.$$

Then $c|s|(\lambda)$ is a FK-space under the norm

$$\|x\|_{c|s|(\lambda)} = \sup_{n\in\mathbb{N}} \sum_{j=s_n+1}^{s_{n+1}} \lambda(j)\,|\Delta x_j|.$$

Also, it is obvious that $h(\lambda) \subset c|s|(\lambda) \subset c_0 \subset \ell^{\infty}$.

LEMMA 2.2. *Let $\lim_{j\to\infty} \frac{z_j^n}{\lambda(j)} = 0$ for $n = 1,2,\dots$. Then there exists $z \in w$ with $\lim_{j\to\infty} \frac{z_j}{\lambda(j)} = 0$ such that $\lim_{j\to\infty} \frac{z_j^n}{z_j} = 0$ $(n = 1,2,\dots)$. Moreover, for any such z, we get*

$$V_0(z) \subset \bigcap_{n=1}^{\infty} V_0(z^n).$$

The proof uses the same technique as in [3] and [8], therefore it is omitted.

Now we give the sufficient conditions for an FK-space X to be a C_λ-wedge space.

LEMMA 2.3. *Let X be an FK-space and $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$. Consider the following propositions:*

(i) *$V_0(z) \subset X$ for some $z \in w$ such that $z_j = o(\lambda(j))$;*

(ii) *X contains $c|s|(\lambda)$ for some s, and the identity map*

$$I\colon \left(c|s|(\lambda), \|\cdot\|_{c|s|(\lambda)}\right) \to (X,\tau)$$

is compact;

(iii) *$h(\lambda) \subset X$, and the identity map $I\colon \left(h(\lambda), \|\cdot\|_{c|s|(\lambda)}\right) \to (X,\tau)$ is compact;*

(iv) *(X,τ) is a C_λ-wedge space.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).

Proof.

(i) $\Longrightarrow$ (ii): Let $s_1 = 0$ and $s = \{s_n : n \geq 1\}$ denote a strictly increasing sequence satisfying $\frac{|z_j|}{\lambda(j)} < \frac{1}{2^n}$, $j \geq s_n$ $(n = 1,2,\dots)$. Let $x \in c|s|(\lambda)$. Suppose $t,m \in \mathbb{N}$, $t \leq m$. Then

$$\sum_{j=s_t+1}^{s_{m+1}} |z_j||\Delta x_j| \leq \sum_{n=t}^{m} \frac{1}{2^n} \sum_{j=s_n+1}^{s_{n+1}} \lambda(j)|\Delta x_j| \leq \|x\|_{c|s|(\lambda)} \sum_{n=t}^{m} \frac{1}{2^n},$$

hence $x \in V_0(z)$. So $c|s|(\lambda) \subset X$. Let now $K \subset c|s|(\lambda)$ be such that $\|x\|_{c|s|(\lambda)} \leq M$ for all $x \in K$. For $s_n < m \leq s_{n+1}$ and $x \in K$,

$$\begin{aligned}\|x - x^{(m)}\|_{V_0(z)} &= \sum_{j=m+1}^{\infty} |z_j||\Delta x_j| \\ &\leq \sum_{i=n}^{\infty} \sum_{j=s_i+1}^{s_{i+1}} |z_j||\Delta x_j| \leq \sum_{i=n}^{\infty} \frac{1}{2^i} \sum_{j=s_i+1}^{s_{i+1}} \lambda(j)|\Delta x_j| \leq \|x\|_{c|s|(\lambda)} \sum_{i=n}^{\infty} \frac{1}{2^i} \\ &\leq M \sum_{i=n}^{\infty} \frac{1}{2^i} \to 0 \text{ (uniformly).}\end{aligned}$$

Hence, the convergence with respect to topology of the space $V_0(z)$ is uniform on K. On the other hand, since $V_0(z)$ is AK-space by [3; Lemma 2], we find that K is τ-relatively compact.

(ii) $\implies$ (iii): Since $h(\lambda) \subset c|s|(\lambda)$, by [9; Proposition 3.1] the identity map from $h(\lambda)$ into $c|s|(\lambda)$ is continuous, hence (iii) follows from (ii).

(iii) $\implies$ (iv): Since $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$, first observe that $\psi := \{\mu^n : n = 1, 2, \ldots\}$ is a bounded subset of $h(\lambda)$ and so it must be relatively compact in X. Therefore, it is easy to see that, for each i, $p_i(\mu^n) = \frac{1}{\lambda(n)}$ if $i \leq \lambda(n)$, and 0 if $i > \lambda(n)$. Hence, for each i, $p_i(\mu^n) \to 0$ as $n \to \infty$. Now [9; Theorem 2.3.11] implies that $\mu^n \to 0$ in (X, τ), giving (iv). □

Using the fact that the space $z^{-1} \cdot X = \{x \in w : z \cdot x \in X\}$ is an FK-space ([14]) one can get immediately the following:

LEMMA 2.4. *Let (X, q) be an FK-space with $\phi \subset X$ and $z \in w$, then $z^{-1} \cdot X$ is a C_λ-wedge space if and only if $\frac{z^{(\lambda(n))}}{\lambda(n)} \to 0$ in X.*

Proof.

Sufficiency: Consider [14; Theorem 4.3.6] to obtain the seminorms of $z^{-1} \cdot X$. Hence it easy to see that, for each i, $p_i(\mu^n) = \frac{1}{\lambda(n)}$ if $i \leq \lambda(n)$, and 0 if $i > \lambda(n)$. Thus we have for each i, that $p_i(\mu^n) \to 0$ as $n \to \infty$. Also,

$$h(\mu^n) = q(z \cdot \mu^n) = q\biggl(\frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} z_k \delta^k\biggr) \to 0 \qquad \text{as} \quad n \to \infty.$$

□

THEOREM 2.5. *If $z \in \sigma_0(\lambda)$, then z^β is a C_λ-wedge FK-space.*

Proof. Recall that $z^\beta = \left\{x : \sum_{k=1}^{\infty} x_k z_k \text{ converges}\right\}$ is an FK-space under the topology given by the seminorms

$$p_n(x) = |x_n| \quad (n = 1, 2, \dots) \qquad \text{and} \qquad p_0(x) = \sup_{m\in\mathbb{N}} \left|\sum_{k=1}^{m} z_k x_k\right|$$

([14]). Observe that

$$p_n(\mu^r) = \begin{cases} \frac{1}{\lambda(r)}, & n \le \lambda(r), \\ 0, & n > \lambda(r). \end{cases}$$

Hence, for each n, $p_n(\mu^r) \to 0$ as $r \to \infty$. Now a few calculation yields that $p_0(\mu^r) = \max_{1\le m\le\lambda(r)} \frac{1}{\lambda(r)} \left|\sum_{k=1}^{m} z_k\right|$. By hypothesis, since $z \in \sigma_0(\lambda)$, choose an index sequence $(\nu_N)_{n\in\mathbb{N}}$ such that $\frac{\nu_N}{\nu_{N-1}} \ge 2^N$ and for each $\lambda(\nu) \ge \nu_N$, $\frac{1}{\lambda(\nu)} \left|\sum_{k=1}^{\lambda(\nu)} z_k\right| \le 2^{-N}$.

Let $\lambda(r) \ge \nu_N$; then for an arbitrary $N > 2$,

(i) $m = \lambda(r)$, $\qquad \frac{1}{\lambda(r)} \left|\sum_{k=1}^{\lambda(r)} z_k\right| \le 2^{-N}$;

(ii) $m < \nu_{N-1}$, $\qquad \frac{m}{\lambda(r)} \frac{1}{m} \left|\sum_{k=1}^{m} z_k\right| \le 2^{-N} \sup_{m\in\mathbb{N}} \frac{1}{m} \left|\sum_{k=1}^{m} z_k\right|$;

(iii) $\nu_{N-1} \le m < \lambda(r)$, $\qquad \frac{m}{\lambda(r)} \frac{1}{m} \left|\sum_{k=1}^{m} z_k\right| \le 2^{-(N-1)}$.

Hence, since

$$p_0(\mu^r) = \max\left\{ \sup_{m<\nu_{N-1}} \frac{1}{\lambda(r)} \left|\sum_{k=1}^{m} z_k\right|, \sup_{\nu_{N-1}\le m<\lambda(r)} \frac{1}{\lambda(r)} \left|\sum_{k=1}^{m} z_k\right|, \frac{1}{\lambda(r)} \left|\sum_{k=1}^{\lambda(r)} z_k\right| \right\},$$

this proves the theorem. □

COROLLARY 2.6. *The intersection of all (weak) C_λ-wedge FK-spaces is h.*

Proof. Let the set of all (C_1-wedge) C_λ-wedge FK-spaces be $(\Gamma(C_1))$ $\Gamma(C_\lambda)$. Since every C_1-wedge FK-space is C_λ-wedge, we get $\Gamma(C_1) \subset \Gamma(C_\lambda)$. Also,

$$\bigcap\{X : X \in \Gamma(C_1)\} \subset \bigcap\{X : X \in \Gamma(C_\lambda)\}.$$

On the other hand the intersection of all (weak) C_1-wedge FK-spaces is h in [8]. Hence $h \subset \bigcap\{X : X \in \Gamma(C_\lambda)\}$. Therefore, we have

$$h \subset \bigcap\{X : X \in \Gamma(C_\lambda)\} \subset \bigcap\{z^\beta : z \in \sigma_0\} = \sigma_0^\beta = h,$$

thus the result. □

THEOREM 2.7.

(i) *An FK-space that contains a (weak) C_λ-wedge FK-space must be a (weak) C_λ-wedge FK-space.*

(ii) *A closed subspace containing ϕ of a (weak) C_λ-wedge FK-space is a (weak) C_λ-wedge FK-space.*

(iii) *A countable intersection of (weak) C_λ-wedge FK-spaces is a (weak) C_λ-wedge FK-spaces.*

The proof is easily obtained from elementary properties of FK-spaces (see, e.g, [14]).

THEOREM 2.8.

(i) *If X is a C_λ-wedge space, then $X \cap (\mathfrak{bs} \setminus \mathfrak{cs}_0)$ is non-empty.*

(ii) *If X is a C_λ-wedge space, then $X \cap (\mathfrak{cs} \setminus \ell)$ is non-empty, where*

$$\mathfrak{cs}_0 = \Big\{x : \sum_{j=1}^{\infty} x_j = 0\Big\}.$$

Proof.

(i): It is clear that $\mathfrak{cs}$ is not C_λ-wedge, and hence, by Theorem 2.7(i), nor is $\mathfrak{cs} \cap X$. Theorem 2.7(ii) implies that $\mathfrak{cs} \cap X$ is not closed in X. Thus we consider the one-to-one and onto mapping $S\colon X \to Y$, $Sx = \Big(x_1, x_1+x_2, \dots, \sum_{k=1}^{n} x_k, \dots\Big)$ ([9] and [3]). Hence $S(\mathfrak{cs} \cap X) = c \cap Y$ is not closed in Y. If $c \cap Y$ is not closed in Y, then c_0 is of codimension 1 in c so it follows from [2] that $c_0 \cap Y$ is not closed in Y. Therefore, by [12; Corollary 1], $Y \cap (\ell^\infty \setminus c_0)$ is non-empty. We have that $S^{-1}(Y \cap (\ell^\infty \setminus c_0))$ is non-empty. Moreover, since $S^{-1}(Y \cap (\ell^\infty \setminus c_0)) = X \cap (\mathfrak{bs} \setminus \mathfrak{cs}_0)$, we get $X \cap (\mathfrak{bs} \setminus \mathfrak{cs}_0)$ is non-empty.

(ii): Since ℓ is not a C_λ-wedge space, then by Theorem 2.7(i), $\ell \cap X$ is not C_λ-wedge space, too. Hence, Theorem 2.7(ii) implies that $\ell \cap X$ is not closed in X. Therefore, by [2; Theorem 2(i)], $X \cap (\mathfrak{cs} \setminus \ell)$ is non-empty. □

THEOREM 2.9. *If X is a C_λ-wedge space, then $X \cap \mathfrak{bs}$ is a non-separable subspace of $\mathfrak{bs}$.*

Proof. Since $\mathfrak{cs}$ is not a C_λ-wedge space, then by Theorem 2.7(i), $\mathfrak{cs} \cap X$ is not a C_λ-wedge space either. Theorem 2.7(ii) implies that $\mathfrak{cs} \cap X$ is not closed in X. Therefore, $S(\mathfrak{cs} \cap X) = c \cap Y$ is not closed in Y. Hence [2; Theorem 8] implies that the space $\ell^\infty \cap Y$ is a non-separable subspace of ℓ^∞. In this case we claim that the space $S^{-1}(\ell^\infty \cap Y) = \mathfrak{bs} \cap X$ is a non-separable subspace of $S^{-1}(\ell^\infty) = \mathfrak{bs}$. To see this, suppose that $\mathfrak{bs} \cap X$ is a separable subspace of $\mathfrak{bs}$. Then there exists a countable set $\kappa \subset \mathfrak{bs} \cap X$ such that $\bar{\kappa}^{\mathfrak{bs} \cap X} = \mathfrak{bs} \cap X$. Thus,

$$S(\kappa) \subset S(\mathfrak{bs} \cap X) = S(\bar{\kappa}^{\mathfrak{bs} \cap X}) = (\ell^\infty \cap Y) \cap S(\bar{\kappa}^{\mathfrak{bs}}).$$

However, since $\bar{\kappa}^{bs} \subset \bar{\kappa}^{\ell^\infty}$, then $S(\bar{\kappa}^{bs}) \subset S(\bar{\kappa}^{\ell^\infty})$ therefore, we get

$$\ell^\infty \cap Y = \overline{S(\kappa)}^{\ell^\infty \cap Y} .$$

Since $S(\kappa)$ is countable and $\ell^\infty \cap Y$ is dense in the topology of ℓ^∞, then it is a separable subspace of ℓ^∞, which is a contradiction. This completes the proof. □

THEOREM 2.10. *Let X be an FK-space and $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$. If $h(\lambda) \subset X$, and the identity map $I\colon (h(\lambda), \|\cdot\|_{h(\lambda)}) \to (X,\tau)$ is weakly compact, then X is weak C_λ-wedge space.*

Proof. Suppose that $h(\lambda) \subset X$ and $I\colon (h(\lambda), \|\cdot\|_{h(\lambda)}) \to (X,\tau)$ is weak compact. Since $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$, $\psi := \{\mu^n : n = 1,2,\dots\}$ is a bounded subset of $h(\lambda)$ and it is $\sigma(X,X')$-relatively compact. Observe that $p_i(\mu^n) = \frac{1}{\lambda(n)}$ if $i \le \lambda(n)$, and zero if $i > \lambda(n)$. Hence, for each i, $p_i(\mu^n) \to 0$ as $n \to \infty$. The same is also true in $\sigma(X,X')$ by [9; Theorem 2.3.11]. This proves the theorem. □

THEOREM 2.11. *Let X be an FK-space with $\phi \subset X$ and $z \in w$, then $z^{-1}\cdot X$ is a weak C_λ-wedge space if and only if $\frac{z^{(\lambda(n))}}{\lambda(n)} \to 0$ (weakly) in X.*

Proof.

Necessity: Let $f \in (z^{-1}\cdot X)'$. By [14; Theorem 4.4.10], $f \in (z^{-1}\cdot X)'$ if and only if $f(x) = \alpha x + g(z\cdot x)$, $\alpha \in \phi$, $g \in X'$. Also,

$$x^n := (x^n_k) = \frac{e^{(\lambda(n))}}{\lambda(n)} = \left(\frac{1}{\lambda(n)}, \frac{1}{\lambda(n)}, \dots, \frac{1}{\lambda(n)}, 0, \dots\right).$$

Hence we get that

$$\begin{aligned} f(x^n) &= \alpha x^n + g(z\cdot x^n) \\ &= \sum_{k=1}^{\infty} \alpha_k x^n_k + g((z_k x^n_k)) \\ &= \left\{ \begin{array}{ll} \frac{1}{\lambda(n)} \sum_{k=1}^{p} \alpha_k, & p \le \lambda(n) \\ \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \alpha_k, & p > \lambda(n) \end{array} \right\} + g\left(\frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} z_k \delta^k\right). \end{aligned} \tag{2}$$

Therefore, for each $f \in (z^{-1}\cdot X)'$, $f\left(\frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} \delta^k\right) \to 0$ as $n \to \infty$, which proves the theorem.

Sufficiency is trivial by (2). □

3. Summability domains and applications

In this sections we give simple conditions for a summability domains E_A to be (weak) C_λ-wedge. We shall be concerned with matrix transformations $y = Ax$, where $x, y \in w$, $A = \{a_{ij}\}_{i,j=1}^{\infty}$ is an infinite matrix with complex coefficients, and

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j \qquad (i = 1, 2, \dots).$$

The sequence $\{a_{ij}\}_{j=1}^{\infty}$ is called the ith row of A and is denoted by a^i $(i = 1, 2, \dots)$; similarly, the jth column of the matrix A, $\{a_{ij}\}_{i=1}^{\infty}$ is denoted by a^j, $(j = 1, 2, \dots)$. For an FK-space E, we consider the summability domain E_A defined by

$$E_A = \{x \in w : \ Ax \text{ exists and } Ax \in E\}.$$

Then E_A is an FK-space under the seminorms $p_n(x) = |x_n|$ $(n = 1, 2, \dots)$;

$$h_i(x) = \sup_{m \in \mathbb{N}} \left| \sum_{j=1}^{m} a_{ij} x_j \right| \quad (i = 1, 2, \dots) \qquad \text{and} \qquad (q \circ A)(x) = q(Ax)$$

([14] and [16]).

The following theorem is an application of Lemma 2.3 to summability domains.

Theorem 3.1. *Let E be an FK-space, A be a matrix and $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$. Then consider the propositions below.*

(i) *$h(\lambda) \subset E_A$, $a^i \in \sigma_0(\lambda)$ for all $i \geq 1$ and the mapping $A\colon h(\lambda) \to E$ is compact;*

(ii) *the sequence defined by $A\left(\frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} \delta^j \right) = \left\{ \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \right\}_{i=1}^{\infty}$ for each n belongs to E and converges to zero there;*

(iii) *E_A is a C_λ-wedge space.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof.

(i) $\Longrightarrow$ (ii): Observe that $\delta^j \in h(\lambda)$ for all j, and since $h(\lambda) \subset E_A$, we have $a^j = A(\delta^j) \in E$ for all $j \geq 1$. Since $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$, $\psi := \{\mu^n : \ n = 1, 2, \dots\}$ is a bounded subset of $h(\lambda)$ and $A\colon h(\lambda) \to E$ is compact, $A(\psi) = \{A(\mu^n) : \ n = 1, 2, \dots\}$ is relatively compact in E. Thus, by [9; Theorem 2.3.11], $A(\mu^n) \to 0$ in w implies that $A(\mu^n) \to 0$ in E.

The proof (ii) $\Longrightarrow$ (iii) is similar to Theorem 2.5 and hence is omitted. □

The following theorem is an application of Theorem 2.12 to summability domains.

THEOREM 3.2. *Let E be an FK-space, A be a matrix and $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$. Then consider the propositions below.*

(i) *$h(\lambda) \subset E_A$, $a^i \in \sigma_0(\lambda)$ for all $i \geq 1$ and the mapping $A\colon h(\lambda) \to E$ is weakly compact;*

(ii) *the sequence defined by $A\left(\frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} \delta^j\right) = \left\{\frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij}\right\}_{i=1}^{\infty}$ for each n belong to E and converge weakly to zero there;*

(iii) *E_A is a weak C_λ-wedge space.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof.

(i) $\Longrightarrow$ (ii): Proceed as in the proof (i) $\Longrightarrow$ (ii) of Theorem 3.1.

(ii) $\Longrightarrow$ (iii): By [14; Theorem 4.4.2], $f \in E'_A$ if and only if $f(x) = \sum_{k=1}^{\infty} \alpha_k x_k + g(Ax)$ for all $x \in E_A$, where $\alpha \in w_A^\beta = \left\{x : \sum_{n=1}^{\infty} x_n y_n \text{ converges for all } y \in w_A\right\}$, and $g \in E'$. Thus we get for each $i \in \mathbb{N}$

$$|\alpha x| \leq M \sup_{m\in\mathbb{N}} \left|\sum_{j=1}^{m} a_{ij} x_j\right| = M h_i(x), \qquad M > 0.$$

Therefore

$$|\alpha(\mu^n)| \leq M h_i(\mu^n) \qquad \text{for all} \quad i \geq 1. \tag{3}$$

Since $\frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \to 0$ (weakly) in E for each $i \in \mathbb{N}$, and E is a K-space, we get

$$p_i\left(\left\{\frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij}\right\}_{i=1}^{\infty}\right) = \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \to 0 \qquad \text{as} \quad n \to \infty$$

for each $i \in \mathbb{N}$. Hence as in the proof of (ii) $\Longrightarrow$ (iii) of Theorem 3.1, $h_i(\mu^n) \to 0$ as $n \to \infty$ for each $i \in \mathbb{N}$. Thus (3) implies that

$$\alpha(\mu^n) \to 0 \qquad \text{as} \quad n \to \infty. \tag{4}$$

Also,

$$f(\mu^n) = \alpha(\mu^n) + g\big(A(\mu^n)\big), \qquad \alpha \in w_A^\beta, \quad g \in E'. \tag{5}$$

By hypothesis, $g\big(A(\mu^n)\big) \to 0$ as $n \to \infty$. Therefore, by (3), (4) and (5), $f(\mu^n) \to 0$, for each $f \in E'_A$ as $n \to \infty$. □

Let $\lambda := \{\lambda(n)\}_{n=1}^{\infty}$ be an infinite subset of $\mathbb{N}$ and $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$. Then we have some important applications.

COROLLARY 3.3. *If*

(i) $\sup_{i\in\mathbb{N}} \frac{1}{\lambda(n)} \left| \sum_{j=1}^{\lambda(n)} a_{ij} \right| < \infty \ (n = 1, 2, \dots)$

$\lim_{i\to\infty} \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij}$ *exists for each* n, *respectively* $\lim_{i\to\infty} \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} = 0$,

(ii) $\lim_{n\to\infty} \sup_{i\in\mathbb{N}} \frac{1}{\lambda(n)} \left| \sum_{j=1}^{\lambda(n)} a_{ij} \right| = 0$, *then* $(\ell^\infty)_A$ $(c_A$, *respectively* $(c_0)_A)$ *is a* C_λ*-wedge space.*

Proof. This is just Theorem 3.1, (ii) $\implies$ (iii), with $E = \ell^\infty$ (c, respectively c_0). □

COROLLARY 3.4. *If* $\lim_{n\to\infty} \sum_{i=1}^{\infty} \left| \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \right| = 0$, *then* ℓ_A *is a* C_λ*-wedge space.*

Proof. This follows at once from Theorem 3.1, (ii) $\implies$ (iii), with $E = \ell$. □

COROLLARY 3.5. *If* $\lim_{n\to\infty} \left\{ \sum_{i=1}^{\infty} \left| \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} (a_{ij} - a_{i+1,j}) \right| + \lim_{i\to\infty} \left| \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \right| \right\}$ $= 0$, *then* $(\mathfrak{bv})_A$ *is a* C_λ*-wedge space.*

Proof. This is just Theorem 3.1, (ii) $\implies$ (iii), with $E = \mathfrak{bv}$. □

PROPOSITION 3.6. *Let* $A \in (\ell, \ell; p)$ *and* $\sup_{n\in\mathbb{N}} \frac{\lambda(\lambda(n))}{\lambda(n)} < \infty$. *Then* ℓ_A *is not* C_λ*-wedge space.*

Proof. $A \in (\ell, \ell; p)$ if and only if

(a) $\sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{nk}| < \infty$;

(b) $\sum_{n=1}^{\infty} a_{nk} = 1$ for all $k \geq 1$

(see, [10; p. 189]). Hence we get the following

$$\sum_{i=1}^{\infty} \left| \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \right| \geq \sum_{i=1}^{\infty} \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} = \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} \left(\sum_{i=1}^{\infty} a_{ij} \right) = \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} 1 .$$

Therefore, $\sum_{i=1}^{\infty} \left| \frac{1}{\lambda(n)} \sum_{j=1}^{\lambda(n)} a_{ij} \right| \nrightarrow 0$ as $n \to \infty$. Thus ℓ_A is not C_λ-wedge space by Corollary 3.4. □

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