Turan A. Gürkanli
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MULTIPLIERS OF SOME BANACH IDEALS AND WIENER-DITKIN SETS

A. TURAN GÜRKANLI

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ABSTRACT. Let $L^1_w(G)$ be a Beurling (convolution) algebra on a locally compact abelian group $G$. A Banach algebra $(S_w(G), \| \cdot \|_{S_w})$ continuously embedded into $L^1_w(G)$ (we may assume for its norm that $\| \cdot \|_{1,w} \leq \| \cdot \|_{S_w}$) is called a Segal algebra for $L^1_w(G)$ if it is dense subalgebra of $L^1_w(G)$, translation invariant, satisfying $\|L^*_a f\|_{S_w} \leq w(a) \| f\|_{S_w}$ for all $f \in S_w(G)$, $a \in G$, and that $y \mapsto L^*_y f$ is continuous from $G$ into $S_w(G)$. The aim of this paper is to study the properties of $S_w(G)$. In the second section we characterize the multipliers from $L^1_w(G)$ to $S_w(G)$. We also discuss the tensor product factorization $S_w(G) \otimes V = V$, where $V$ is a Banach $L^1_w(G)$-module. In the third section some applications are given. In section four we discuss the Wiener-Ditkin sets of $S_w(G)$ and show that they are the same as those of $L^1_w(G)$.

1. Introduction

Throughout the paper $G$ denotes a locally compact Abelian group (non-compact and non-discrete) with dual group $\hat{G}$ and Haar measures $dx$ and $d\hat{x}$ respectively. $C_0(G)$ denotes the space of all continuous, complex-valued functions on $G$ with compact support and by $(L^p(G), \| \cdot \|_p)$, $1 \leq p \leq \infty$, the usual Lebesgue space. Also $C_0(G)$ denotes the algebra of continuous complex-valued functions on $G$ that vanish at infinity and $M(G)$ the space of bounded regular Borel measures on $G$. A strictly positive, continuous function $w$ satisfying $w(x) \geq 1$ and $w(x+y) \leq w(x) \cdot w(y)$ for all $x, y \in G$ will be called a Beurling’s weight function on $G$. For $1 \leq p < \infty$ we set

$$L^p_w(G) := \{ f : f \in L^p(G), \ f \cdot w \in L^p(G) \}. \quad (1.1)$$

Under the norm $\| f \|_{p,w} = \| f w \|_p$ this is a Banach space. We say that $w_1 \leq w_2$ if and only if there exists a constant $c > 0$ such that $w_1(x) \leq cw_2(x)$ for all

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It is known that $L^p_{w_2}(G) \subset L^p_{w_1}(G)$ if and only if $w_1 \leq w_2$. Lastly we recall that a weight $w$ satisfies the Beurling-Domar condition ([4]) if
\[(BD) \sum_{n \geq 1} n^{-2} \cdot \log(w(nx)) < \infty \text{ for all } x \in G.
\]

An algebra $A(X)$ of complex-valued continuous functions over a locally compact Hausdorff space $X$ is called a \textit{standard algebra} if it has the following properties:

(i) If $f \in A(X)$ and $f(a) \neq 0$ at a point $a \in X$, then there is a $g \in A(X)$ such that $g(x) = \frac{1}{f(x)}$ for all $x$ in some neighbourhood of $a$.

(ii) For any closed set $E \subset X$ and any point $a \in X - E$ there is $f \in A(X)$ vanishing on $E$ and such that $f(a) \neq 0$.

For any ideal $I$ in $A(X)$ the set of points of $X$ where all functions in $I$ vanish is called the \textit{cospectrum} of $I$, denoted by $\text{cosp} I$. We shall use the known fact that any ideal with $\text{cosp} I = \emptyset$ contains $A(X) \cap C_c(X)$, i.e. all functions in $A(X)$ with compact support ([18; 1.4.(ii)]).

If, in addition to (i) and (ii), $A(X) \cap C_c(X)$ is dense in $A(X)$, then $A(X)$ will be called a \textit{Wiener algebra}.

If $V$ and $W$ are Banach $L^1_w(G)$-modules, then $\text{Hom}_A(V, W)$ will denote the Banach space of all continuous $A$-module homomorphisms from $V$ to $W$ with the operator norm. The elements of $\text{Hom}_A(V, W)$ are called \textit{multipliers} from $V$ to $W$.

Furthermore we denote the projective tensor product of $V$ and $W$ as Banach space by $V \otimes \gamma W$. Let $K$ be the closed linear subspace of $V \otimes \gamma W$ spanned by elements of the form $(f \cdot g) \otimes h - g \otimes (f \cdot h)$, $f \in L^1_w(G)$, $g \in V$ and $h \in W$. By definition, the $L^1_w(G)$-module tensor product $V \otimes_{L^1_w} W$ is the quotient Banach space $V \otimes \gamma W/K$ ([17]). It is known that any $t \in V \otimes_{L^1_w} W$ can be written in the form
\[ t = \sum_{k=1}^{\infty} g_k \otimes h_k, \quad g_k \in V, \quad h_k \in W, \quad \text{where } \sum_{k=1}^{\infty} \|g_k\| \|h_k\| < \infty. \quad (1.2) \]

Whenever we talk about $L^1_w(G)$-modules in this paper we mean Banach $L^1_w(G)$-modules with respect to convolution.

2. Multipliers from $L^1_w(G)$ to $S_w(G)$

It is known that $L^1_w(G)$ is a closed ideal in $M(w)$ and the space of multipliers of $L^1_w(G)$ is homeomorphic to the space $M(w)$, where
\[ M(w) = \{ \mu : \mu \in M(G), \int w \, d|\mu| < \infty \}. \quad (2.1) \]
Cigelj gives a generalization of Segal algebra in [2] as follows:

Let $S_w = S_w(G)$ be a subalgebra in $L_w^1(G)$ satisfying the following conditions:

1. $S_w$ is dense in $L_w^1(G)$.
2. $S_w$ is a Banach algebra under some norm $\| \cdot \|_{S_w}$ and invariant under translations.
3. $\| L_a f \|_{S_w} \leq w(a) \| f \|_{S_w}$ for all $a \in G$ and for each $f \in S_w$.
4. Given any $f \in S_w$ and $\varepsilon > 0$, there is a neighbourhood $U$ of the unit element $e$ of $G$ such that $\| L_y f - f \|_{S_w} < \varepsilon$ for all $y \in U$.
5. $\| f \|_{1,w} \leq \| f \|_{S_w}$ for all $f \in S_w$.

**Proposition 2.1.** If $\mu \in M(w)$ and $f \in S_w(G)$, then $\mu * f$ in $S_w(G)$ and $\| \mu * f \|_{S_w} \leq \| \mu \|_w \cdot \| f \|_{S_w}$ where $\| \mu \|_w = \int w \, d|\mu|$.

**Proof.** Since $y \mapsto L_y f$ is a continuous function from $G$ into $S_w(G)$ for $f \in S_w(G)$ and $\mu$ is a bounded Borel measure, then $L_y f \in L_w^1(G, \mu)$ is in $L_w^1(G)$, the space of integrable functions with values in $S_w(G)$. Hence the vector integral $\int L_y f \, d\mu(y)$ exists as in $S_w(G)$ and

$$\left\| \int L_y f \, d\mu(y) \right\|_{S_w} \leq \int \|L_y f\|_{S_w} \, d|\mu|(y) \leq \int \| f \|_{S_w} w(y) \, d|\mu|(y) = \| f \|_{S_w} \cdot \| \mu \|_w.$$  \hspace{1cm} (2.2)

By the technique of proof used in [19; p. 20, Proposition 2], we show that

$$\int L_y f \, d\mu(y) = \mu * f.$$ \hspace{1cm} (2.3)

It follows from (2.2) and (2.3) that

$$\| \mu * f \|_{S_w} \leq \| \mu \|_w \cdot \| f \|_{S_w}.$$ \hspace{1cm} (2.4)

**Proposition 2.2.** $S_w(G)$ is an essential Banach ideal in $L_w^1(G)$.

**Proof.** We know that $S_w(G)$ is a dense Banach ideal in $L_w^1(G)$ by assumption (S1) and Proposition 2.1. Now let $f \in S_w(G)$ and $\varepsilon > 0$ be given. By the definition of $S_w(G)$ there is a neighbourhood $U$ of the unit element $e$ of $G$ such that

$$\| L_y f - f \|_{S_w} < \varepsilon$$ \hspace{1cm} (2.5)

for all $y \in U$. Let $(e_\alpha)_{\alpha \in I}$ be a non-negative bounded approximate identity in $L_w^1(G)$ satisfying $\|e_\alpha\|_1 = 1$ and $\text{supp} e_\alpha \subseteq U$ for all $\alpha \in I$, [22].
Then there exists $\alpha_0 \in I$ such that
\[
\|e_\alpha * f - f\|_{S_w} = \left\| \int e_\alpha(y) \{L_y f - f\} \, dy \right\|_{S_w} \leq \int e_\alpha(y) \|L_y f - f\|_{S_w} \, dy \leq \varepsilon \quad (2.6)
\]
for all $\alpha > \alpha_0$. Hence $S_w(G)$ is an essential Banach ideal in $L^1_w(G)$ by [5; 15.3. Corollary].

**Proposition 2.3.** Suppose that $w$ satisfies (BD). Then $L^1_w(G)$ has a bounded approximate identity $(e_\alpha)_{\alpha \in I}$ whose Fourier transforms have compact support and $e_\alpha \in S_w(G)$ for all $\alpha \in I$.

**Proof.** It is known that the Fourier transform of the functions in $L^1_w(G)$ form an algebra of continuous complex-valued functions with the ordinary multiplication (pointwise) algebraic operations. We denote it by $F(L^1_w(G)) = F_w(G)$ and carry the $L^1_w$-norm over to $F_w(\hat{G})$ by putting
\[
\|\hat{f}\|_{F_w} = \|f\|_{1,w}, \quad \hat{f} \in F_w(\hat{G}). \quad (2.7)
\]
We denote by $F_{0,w}$ the set of all $f \in L^1_w(G)$ such that $\hat{f} \in F_w^1(\hat{G})$ has compact support. Since $w$ satisfies (BD), then $F_w^1(\hat{G})$ is a Wiener algebra ([18]). We denote by $F(S_w(G))$ the image of $S_w(G)$ under the Fourier transform. Since $S_w(G)$ is dense in $L^1_w(G)$, then it is easily proved that $\text{cosp}(F(S_w(G))) = \emptyset$. Hence, by the properties [18; p. 20, 1.4.ii], we have the inclusion $F_{0,w} \subset S_w(G)$.

Moreover, since $w$ satisfies (BD), then $L^1_w(G)$ admits a bounded approximate identity $(e_\alpha)_{\alpha \in I}$ such that $e_\alpha \in F_{0,w}$ for all $\alpha \in I$ ([6; Lemma 4.1]).

One can also prove Proposition 2.3 with another way using [8; Proposition 1.1].

Next we denote by $M_{S_w}$ the space of $\mu \in M(w)$ such that $\|\mu\|_M \leq C(\mu)$, where
\[
\|\mu\|_M = \sup \left\{ \frac{\|\mu * f\|_{S_w}}{\|f\|_{1,w}} : f \in L^1_w(G), \ f \neq 0, \ \hat{f} \in C_c(\hat{G}) \right\}. \quad (2.8)
\]
By the Proposition 2.1 we have $M_{S_w} \neq \{0\}$.

**Proposition 2.4.** If $w$ satisfies (BD), then for a linear operator $T : L^1_w(G) \to S_w(G)$ the following are equivalent:

1. $T \in M(L^1_w(G), S_w(G))$ (the space of multipliers from $L^1_w(G)$ to $S_w(G)$).

2. There exists a unique $\mu \in M_{S_w}$ such that $T f = \mu * f$ for every $f \in L^1_w(G)$.

Moreover the correspondence between $T$ and $\mu$ defines an isomorphism between $M(L^1_w(G), S_w(G))$ and $M_{S_w}$.

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Proof. Suppose \( \mu \in M_{S_w} \) such that \( T_1 f = \mu * f \) for every \( f \in L^1_w(G) \). \( \mu \in M_{S_w} \) implies that \( T_1 \) is a bounded linear operator from \( F_{0,w} \) (which is a dense subspace of \( L^1_w(G) \) endowed with the norm \( \| \cdot \|_{1,w} \)) into \( (S_w, \| \cdot \|_{S_w} ) \). Using a standard approximation argument \( T_1 \) extends to a unique bounded linear operator \( T \) on all of \( L^1_w(G) \). Clearly \( T \) still commutes with convolutions and maps \( L^1_w(G) \) into \( S_w(G) \).

Conversely suppose \( T \in M(L^1_w(G), S_w(G)) \). Then according to [12] there exists a unique \( \mu \in M(w) \) such that \( Tf = \mu * f \) for all \( f \in L^1_w(G) \). Since

\[
\|Tf\|_{S_w} = \|\mu * f\|_{S_w} \leq C\|f\|_{1,w}, \tag{2.9}
\]

it is obvious that \( \mu \in M_{S_w} \). The proof also shows that the norm \( \| \cdot \|_M \) and the operator norm \( \| \cdot \| \) are equivalent. \( \Box \)

DEFINITION 2.5. Let \( V \) be a \( L^1_w(G) \)-Banach convolution module. We write \( S_w(G) \otimes V \) for the space of all \( t \in V \) for which there are sequences \( \{g_k\}_{k=1}^\infty \subset S_w(G), \{h_k\}_{k=1}^\infty \subset V \) with \( t = \sum_{k=1}^\infty g_k * h_k \) and \( \sum_{k=1}^\infty \|g_k\|_{S_w} \cdot \|h_k\|_V < \infty \).

It follows immediately from [20; Theorem 6] that \( S_w(G) \otimes V \) is a Banach space with the norm

\[
\|t\| = \inf \left\{ \sum_{k=1}^\infty \|g_k\|_{S_w} \cdot \|h_k\|_V : \{g_k\}_{k=1}^\infty \subset S_w(G), \{h_k\}_{k=1}^\infty \subset V, \right. \tag{2.10}
\]

\[
\left. t = \sum_{k=1}^\infty g_k * h_k \right\} .
\]

Also it is easy to see that \( \|t\|_V \leq \|t\| \).

PROPOSITION 2.6. Let \( V \) be a \( L^1_w(G) \)-convolution Banach module. Then

\[
S_w(G) \otimes_{L^1_w} V \cong S_w(G) \otimes V .
\]

The isomorphism being an isometric one.

Proof. Consider the mapping \( B \) from the projective tensor product \( S_w(G) \otimes_{\gamma} V \) to \( S_w(G) \otimes V \) determined by \( B(f \otimes g) = f * g, f \in S_w(G) \) and \( g \in V \). It is easy to see that \( B \) is surjective. Also \( B \) is an isomorphism being isometric by the arguments used in the proof of [17; Theorem 3.3]. \( \Box \)

THEOREM 2.7. For an \( L^1_w \)-convolution Banach module \( V \) the following are equivalent:

1) \( S_w(G) \otimes V = V . \)

2) \( \text{Hom}_{L^1_w}(S_w(G), V^*) \cong V^* \) in the sense of a topological isomorphism.
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Proof. If $S_w(G) \otimes V = V$, then it is easy to see that $S_w(G) \otimes V \cong V$, using the closed graph theorem. Hence

$$(S_w(G) \otimes V)^* \cong V^*.$$  \hspace{1cm} (2.11)

Then we have

$$\text{Hom}_{L^1_w}(S_w(G), V^*) \cong (S_w(G) \otimes V)^* = V^*$$ \hspace{1cm} (2.12)

by [17; Theorem 1.4].

Conversely suppose $\text{Hom}_{L^1_w}(S_w(G), V^*) = V^*$. Then for $\alpha: V^* \to \text{Hom}_{L^1_w}(S_w(G), V^*),$

$$\langle v, \alpha(v^*)(g) \rangle = \langle g \ast v, v^* \rangle, \quad v \in V, \; v^* \in V^*, \; g \in S_w(G),$$  \hspace{1cm} (2.13)

is a surjective topological isomorphism. Define the function $\beta \circ \alpha: V^* \to (S_w(G) \otimes V)^*$, where $\beta: \text{Hom}_{L^1_w}(S_w(G), V^*) \to (S_w(G) \otimes V)^*$ is defined as in [17]. Since $\alpha$ and $\beta$ are surjective also $\beta \circ \alpha$ will be surjective. The proof that $i^* = \beta \circ \alpha$ proceeds then as for [15; Theorem 2], where $i: S_w(G) \otimes V \to V$ is the identity map and $i^*$ is the usual adjoint of $i$. Hence $i$ is also surjective and in this case we have $S_w(G) \otimes V = V$. \hfill \Box

3. Applications

1) Let $w, \omega$ be weight functions on $G$ and $\hat{G}$ respectively. For $1 \leq p < \infty$ we set

$$A^{p}_{w,\omega}(G) = \{ f : f \in L^1_w(G), \; \hat{f} \in L^p_{\omega}(\hat{G}) \}$$

and

$$\|f\|^{p}_{w,\omega} = \|f\|_{1,w} + \|\hat{f}\|_{p,\omega}. \hspace{1cm} (3.1)$$

These spaces were introduced by Feichtinger–Gürkanlı in [6]. Another generalization has been given by Fischer–Gürkanlı–Liu in [10], [11], where it is proved that $(A^{p}_{w,\omega}(G), \| \cdot \|^{p}_{w,\omega})$ is a Banach algebra with respect to convolution. It is also proved that if the first weight $w$ satisfies (BD), then $A^{p}_{w,\omega}(G)$ is a dense Banach ideal in $L^1_w(G)$ having an approximate identity bounded in the norm of $L^1_w(G)$ with compactly supported Fourier transforms. Furthermore for given any $f \in A^{p}_{w,\omega}(G)$, the function $a \mapsto L_a f$ is continuous. Finally $\|f\|_{1,w} \leq \|f\|^{p}_{w,\omega}$ and

$$\|L_a f\|^{p}_{w,\omega} = \|L_a f\|_{1,w} + \|\hat{L_a f}\|_{p,\omega} \leq w(a)\|f\|_{1,w} + \|\hat{f}\|_{p,\omega} \leq w(a)\|f\|^{p}_{w,\omega}. \hspace{1cm} (3.2)$$

Therefore if $w$ satisfies (BD), then $A^{p}_{w,\omega}(G)$ is a $S_w(G)$ space. Applying the Proposition 2.4 to the space $A^{p}_{w,\omega}(G)$ one obtains that the space of multipliers
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from $L^1_w(G)$ to $A^p_{w,\omega}(G)$ (briefly $M(L^1_w(G), A^p_{w,\omega}(G))$) is homeomorphic to $M_A$, where $A = A^p_{w,\omega}(G)$.

We denote by $B_w(G)$ the space of all the measures $\mu$ in $M(w)$ such that the Fourier-Stieltjes transform $\hat{\mu}$ of $\mu$ belongs to $L^p(\hat{G})$. It is easily seen that $B_w(G)$ is a Banach space with the norm

$$
\|\mu\|_{B_w} = \|\mu\|_w + \|\hat{\mu}\|_{p,\omega}.
$$

Indeed, it is proved in [1] that $B_w(G) = M_A(G)$ is a Banach space for $w = 1$ and $\omega = 1$, with the norm

$$
\|\mu\|_{B_1} = \|\mu\|_w + \|\hat{\mu}\|_{p,\omega},
$$

where $\|\mu\|$ denotes the usual total variation norm of $\mu \in M(G)$. Also $\|\mu\| \leq \|\mu\|_w$ and $\|\hat{\mu}\|_{p,\omega} \leq \|\hat{\mu}\|_{p,w}$ for all $\mu \in B_w(G)$. Now let $\{\mu_n\}_{n=1}^\infty$ be a Cauchy sequence in $B_w(G)$. Then $\{\mu_n\}_{n=1}^\infty$ and $\{\hat{\mu}_n\}_{n=1}^\infty$ are Cauchy sequences in $M(w)$ and in $L^p(\hat{G})$ respectively. Since $M(w)$ and $L^p(\hat{G})$ are Banach spaces, then $\{\mu_n\}_{n=1}^\infty$ converges to a measure $\mu \in M(w)$ and $\{\hat{\mu}_n\}_{n=1}^\infty$ converges to a function $h \in L^p(G)$. Hence $\{\mu_n\}_{n=1}^\infty$ converges to $\mu$ in $M(G)$ and $\{\hat{\mu}_n\}_{n=1}^\infty$ converges to $h$ in $L^p(G)$. Then using (3.4) we write $\hat{\mu} = h$ as in [1]. This completes the proof.

**Theorem 3.1.** If $w$ satisfies (BD), then $B_w(G) = M_A(G)$ and the corresponding natural norms are equivalent.

**Proof.** Suppose $\mu \in B_w(G)$ and $f \in L^1_w(G)$, $f \neq 0$. Then we write

$$
\|\mu * f\|_{p,w,\omega} = \|\mu * f\|_{1,w} + \|\hat{\mu} \hat{f}\|_{p,\omega} = \|\mu * f\|_{1,w} + \|\hat{\mu} \hat{f}\|_{p,\omega}.
$$

By the technique of proof used in Proposition 2.1, we see that

$$
\|\mu * f\|_{1,w} \leq \|\mu\|_w \cdot \|f\|_{1,\omega}.
$$

It follows from (3.5) and (3.6) that

$$
\|\mu * f\|_{p,w,\omega} \leq \|\mu * f\|_{1,w} + \|\hat{\mu} \hat{f}\|_{p,\omega}
\leq \|\mu\|_w \cdot \|f\|_{1,w} + \|\hat{\mu}\|_{p,\omega} \cdot \|\hat{f}\|_{\infty}
\leq \|\mu\|_w \cdot \|f\|_{1,w} + \|\hat{\mu}\|_{p,\omega} \cdot \|f\|_{1,w}
$$

(3.7)

$$
= \|f\|_{1,w} (\|\mu\|_w + \|\hat{\mu}\|_{p,\omega}) = \|f\|_{1,w} \cdot \|\mu\|_{B_w} \leq \infty.
$$

Hence we have

$$
\frac{\|\mu * f\|_{p,w,\omega}}{\|f\|_{1,w}} \leq \|\mu\|_{B_w}.
$$

(3.8)

This implies that $\mu \in M_A$ and $\|\mu\|_M \leq \|\mu\|_{B_w,\omega}$.

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Conversely suppose \( \mu \in M_A \). Since \( w \) satisfies (BD), the space \( A_{w,\omega}^p(G) \) admits an approximate identity \( (e_\alpha) \subset F_{0,w} \), bounded in \( L^1_w(G) \) ([6]). Also since \( \|\mu\|_{M_A} < \infty \), then \( \mu \in M(w) \) and
\[
\|\mu * e_\alpha\|_{p,w,\omega}^p \leq \|\mu\|_M \cdot \sup_{\alpha \in I} \|e_\alpha\|_{1,w} = K(\mu)
\] (3.9)
for every \( \alpha \in I \). This implies
\[
\|\hat{\mu} \cdot \hat{e}_\alpha\|_{p,\omega} \leq \|\mu * e_\alpha\|_{p,w,\omega} \leq K(\mu)
\] (3.10)
for all \( \alpha \in I \). The reflexivity of \( L^p_\omega(\hat{G}) \) and the Banach-Alaoglu theorem imply that there exists a subnet \( (\hat{\mu} \cdot \hat{e}_\beta) \) of \( (\hat{\mu} \cdot \hat{e}_\alpha) \) and \( g \in L^p_\omega(\hat{G}) \) such that \( (\hat{\mu} \cdot \hat{e}_\beta) \) converges weakly to \( g \). Since \( (\hat{\mu} \cdot \hat{e}_\beta) \) converges uniformly to \( \hat{\mu} \) on compact subsets of \( \hat{G} \) it is easy to see that \( \hat{\mu} = g \) almost everywhere. Thus \( \hat{\mu} \in L^p_\omega(\hat{G}) \). Consequently we obtain that \( \mu \in B_w(G) \) and
\[
\|\mu\|_{B_w} \leq \|\mu\|_M \cdot \sup_{\alpha \in I} \|e_\alpha\|_{1,w}.
\] (3.11)
Hence we have \( B_w(G) = M_A(G) \).

2) Let \( S(G) \) be a solid ordinary Segal algebra, i.e. assume that \( |f(x)| \leq |g(x)| \) a.e. for \( g \in S(G) \), \( f \in L^1(G) \) (or just measurable) implies \( f \in S(G) \) and \( \|f\|_S \leq \|g\|_S \). Define a set
\[
S^w(G) = \{ f \in L^1_w(G) : f \cdot w \in S(G) \}.
\] (3.12)
It is easy to see that \( (S^w(G), \|\cdot\|_{S^w}) \) is a normed space with the natural norm
\[
\|f\|_{S^w} = \|f \cdot w\|_S.
\] (3.13)

**Proposition 3.2.**

a) \( (S^w(G), \|\cdot\|_{S^w}) \) is a Banach convolution algebra.

b) \( S^w(G) \) is dense in \( L^1_w(G) \).

**Proof.**

a) It is easy to prove that \( S^w(G) \) is a Banach space. Now let \( f, g \in S^w(G) \) be given. We write
\[
\|(f * g)w\|_S \leq \int \|f(t) \cdot g(x - t)w(x)\|_S dt
\]
\[
\leq \int \|g(u) \cdot w(u)\|_S \cdot |w(t)f(t)| \ dt = \|gw\|_S \cdot \|f\|_{1,w},
\] (3.14)
where \( x - t = u \). Hence we have
\[
\|f * g\|_{S^w} \leq \|f\|_{1,w} \cdot \|g\|_{1,w} + \|gw\|_S \cdot \|f\|_{1,w} = \|f\|_{1,w} \cdot \|g\|_{S^w} \leq \|f\|_{S^w} \cdot \|g\|_{S^w}.
\]
This completes the proof of a).

b) Let \( \varepsilon > 0 \) and any \( f \in L^1(G) \) be given. Since \( fw \in L^1(G) \) and \( S(G) \) is dense in \( L^1(G) \), there exists \( g \in S(G) \) such that

\[
\|fw - g\|_1 < \varepsilon.
\]

Hence

\[
\left\| f - \frac{g}{w} \right\|_{1,w} = \left\| \left( f - \frac{g}{w} \right) w \right\|_1 = \|fw - g\|_1 < \varepsilon. \tag{3.15}
\]

Also we have

\[
\left\| \frac{g}{w} \right\|_{S^w} = \|g\|_S < \infty.
\]

That means \( g \in S^w(G) \). This completes the proof. \( \square \)

**Proposition 3.3.** If \( w \) satisfies (BD), then to every compact subset \( \hat{K} \subset \hat{G} \) there is a constant \( C_K > 0 \) such that, for every \( f \in S^w(G) \) whose Fourier transformation vanishes outside of \( \hat{K} \), it holds that \( \|f\|_{S^w} \leq C_K \cdot \|f\|_{1,w} \).

**Proof.** Due to (BD), for any such \( \hat{K} \subset \hat{G} \) there is \( g \in S^w(G) \) with \( \hat{g}(x) = 1 \) for all \( x \in \hat{K} \). Hence \( f * g \in S^w(G) \) and

\[
\|f * g\|_{S^w} \leq \|f\|_{1,w} \cdot \|g\|_{S^w} \tag{3.16}
\]

for all \( f \in S^w(G) \) satisfying \( \text{supp} \hat{f} \subset \hat{K} \) by the proof of Proposition 3.2. If we set \( C_K = \|g\|_{S^w} \), we obtain the desired estimate

\[
\|f\|_{S^w} = \|f * g\|_{S^w} \leq C_K \|f\|_{1,w}. \tag{3.17}
\]

\( \square \)

**Proposition 3.4.** If \( w \) satisfies (BD), then for any \( f \in S^w(G) \) the map \( y \mapsto L_y f \) is continuous from \( G \) into \( S^w(G) \).

**Proof.** For \( g \in F_{0,w} \), \( \text{supp}(L_y g - g)^\wedge \) is compact and \( y \mapsto L_y g \) is continuous from \( G \) into \( S^w(G) \) by Proposition 3.3. Since \( w \) satisfies (BD), \( F_{0,w} \) is dense in \( L^1_w(G) \) and statement is true for any \( f \in S^w(G) \).

In summary we have shown: If \( w \) satisfies (BD), then \( S^w(G) \) is a \( S^w(G) \) space. \( \square \)

**Examples.** Let \( G \) be a non-discrete and non-compact locally compact abelian group and \( w \) be Beurling’s function weight on \( G \).

1) Choose \( L^1(G) \cap L^p(G), 1 \leq p < \infty \), as a solid Segal algebra with norm

\[
\|f\| = \|f\|_1 + \|f\|_p, \quad f \in L^1(G) \cap L^p(G). \tag{3.18}
\]

One can define \( S^w(G) \) using this Segal algebra.

2) Take the Wiener amalgam space \( W(L^p(G), L^1(G)) \). It is known that \( W(L^p(G), L^1(G)) \) is a solid Segal algebra by [9; Corollary 1]. So one can define the space \( S^w(G) \) using this Segal algebra.
4. Wiener-Ditkin sets for $S_w(G)$-spaces

In this section we will discuss the Wiener-Ditkin sets for $S_w(G)$-spaces. In the spirit of [18] we call a closed subset $E \subset \hat{G}$ a Wiener-Ditkin set for $S_w(G)$ if each $f \in S_w(G)$ such that $\hat{f}$ vanishes on $E$ can be approximated in $S_w(G)$ with functions $f * F$ such that $\hat{F}$ vanishes in some neighbourhood on $E$.

**Theorem 4.1.** A set $E \subset \hat{G}$ is a Wiener-Ditkin set for $S_w(G)$ if and only if $E$ is a Wiener-Ditkin set for $L^1_w(G)$.

**Proof.**

1) Assume that $E$ is a Wiener-Ditkin set for $S_w(G)$. Let $f \in L^1_w(G)$ be such that $\hat{f}$ vanishes on $E$ and $(e_\alpha)_{\alpha \in I}$ be a bounded approximate identity in $L^1_w(G)$. Also let $\varepsilon > 0$ be given. Then there exists $\alpha_1 \in I$ such that

$$\|f - f * e_{\alpha_1}\|_{1,w} < \frac{\varepsilon}{2}. \tag{4.1}$$

Since $S_w(G)$ is an ideal in $L^1_w(G)$, then $f * e_{\alpha_1} \in S_w(G)$. Clearly $(f * e_{\alpha_1})^* = 0$ on $E$. Hence we can find $F_1 \in S_w(G)$ such that $\hat{F}_1$ vanishes on a neighbourhood of $E$ and

$$\|f * e_{\alpha_1} - F_1 * (f * e_{\alpha_1})\|_{S_w} < \frac{\varepsilon}{2}. \tag{4.2}$$

We set $F = F_1 * e_{\alpha_1}$. Then $\hat{F} = \hat{F}_1 * \hat{e}_{\alpha_1}$ and thus $\hat{F}$ vanishes on a neighbourhood of $E$. Since $\|\cdot\|_{1,w} \leq \|\cdot\|_{S_w}$, from (4.1) and (4.2) we have

$$\|f - f * F\|_{1,w} = \|f - f * e_{\alpha_1} + f * e_{\alpha_1} - F * f\|_{1,w} \tag{4.3}$$

$$\leq \|f - f * e_{\alpha_1}\|_{1,w} + \|f * e_{\alpha_1} - F_1 * (f * e_{\alpha_1})\|_{1,w}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof of first part.

2) Assume that $E \subset \hat{G}$ is a Wiener-Ditkin set for $L^1_w(G)$. Let $f \in S_w(G)$ with $\hat{f}$ vanishing on $E \subset \hat{G}$. Since $S_w(G)$ is an essential Banach module over $L^1_w(G)$, then for any given $\varepsilon > 0$ there exists $\alpha_0 \in I$ such that

$$\|f - f * e_{\alpha_0}\|_{S_w} < \frac{\varepsilon}{2}. \tag{4.4}$$

There also exists $F_1 \in L^1_w(G)$ such that $\hat{F}_1$ vanishes on a neighbourhood of $E$ and

$$\|f - f * F_1\|_{S_w} < \frac{\varepsilon}{2 \cdot \|e_{\alpha_0}\|_{S_w}}. \tag{4.5}$$
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Take \( F = e_{\alpha_0} * F_1 \). Then \( \hat{F} = \hat{e}_{\alpha_0} \cdot \hat{F}_1 \) and therefore \( \hat{F} \) vanishes on a neighbourhood of \( E \). Finally

\[
\|f - f * F\|_{S_w} < \|f - f * e_{\alpha_0}\|_{S_w} + \|f * e_{\alpha_0} - f * F\|_{S_w} = \|f - f * e_{\alpha_0}\|_{S_w} + \|f * e_{\alpha_0} - f * F_1 * e_{\alpha_0}\|_{S_w} \\
\leq \frac{\varepsilon}{2} + \|e_{\alpha_0}\| \cdot \frac{\varepsilon}{2 \cdot \|e_{\alpha_0}\|_{S_w}} = \varepsilon .
\] (4.6)

This completes the proof. \( \square \)

**Remark 4.2.** Let \( \alpha \) be a positive number (or zero) and consider the Beurling’s weight function

\[
w(x) = (1 + |x|)^{\alpha} , \quad x, y \in \mathbb{R}^n . \] (4.7)

We denote the corresponding weighted space by \( L^1_w(\mathbb{R}^n) = L^1_{\alpha}(\mathbb{R}^n) \) and the norm by \( \| \cdot \|_{1,w} = \| \cdot \|_{1,\alpha} . \) It is known that the closed subgroup of \( \mathbb{R}^n \) are Wiener-Ditkin sets for \( L^1_{\alpha}(\mathbb{R}^n) \) (0 \( \leq \alpha < 1 \), [21]. Take the space \( A^p_{w,w}(G) \) from Section 3. Since \( w \) satisfies (BD), then any closed subgroup of \( \mathbb{R}^n \) is a Wiener-Ditkin sets for this space.

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Ondokuz Mayıs University
Faculty of Arts and Sciences
Department of Mathematics
55139, Kurupelit, Samsun
TURKEY
E-mail: gurkanli@SAMSUN.omu.edu.tr