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MULTIPLIERS OF SOME BANACH IDEALS AND WIENER-DITKIN SETS

A. Turan Gürkanli

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ABSTRACT. Let $L^1_w(G)$ be a Beurling (convolution) algebra on a locally compact abelian group G. A Banach algebra $(S_w(G), \|\cdot\|_{S_w})$ continuously embedded into $L^1_w(G)$ (we may assume for its norm that $\|\cdot\|_{1,w} \le \|\cdot\|_{S_w}$) is called a Segal algebra for $L^1_w(G)$ if it is dense subalgebra of $L^1_w(G)$, translation invariant, satisfying $\|L_a f\|_{S_w} \le w(a)\|f\|_{S_w}$ for all $f \in S_w(G)$, $a \in G$, and that $y \mapsto L_y f$ is continuous from G into $S_w(G)$. The aim of this paper is to study the properties of $S_w(G)$. In the second section we characterize the multipliers from $L^1_w(G)$ to $S_w(G)$. We also discuss the tensor product factorization $S_w(G) \underset{\sim}{\otimes} V = V$, where V is a Banach $L^1_w(G)$-module. In the third section some applications are given. In section four we discuss the Wiener-Ditkin sets of $S_w(G)$ and show that they are the same as those of $L^1_w(G)$.

1. Introduction

Throughout the paper G denotes a locally compact Abelian group (noncompact and non-discrete) with dual group $\hat{G}$ and Haar measures dx and $d\hat{x}$ respectively. $C_c(G)$ denotes the space of all continuous, complex-valued functions on G with compact support and by $(L^p(G), \|\cdot\|_p)$, $1 \le p \le \infty$, the usual Lebesgue space. Also $C_0(G)$ denotes the algebra of continuous complex-valued functions on G that vanish at infinity and $M(G)$ the space of bounded regular Borel measures on G. A strictly positive, continuous function w satisfying $w(x) \ge 1$ and $w(x+y) \le w(x)\cdot w(y)$ for all $x, y \in G$ will be called a Beurling's weight function on G. For $1 \le p < \infty$ we set

$$L^p_w(G) := \{f :\ f \in L^p(G),\ f\cdot w \in L^p(G)\}. \tag{1.1}$$

Under the norm $\|f\|_{p,w} = \|fw\|_p$ this is a Banach space. We say that $w_1 \le w_2$ if and only if there exists a constant $c > 0$ such that $w_1(x) \le c w_2(x)$ for all

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$x \in G$. It is known that $L^p_{w_2}(G) \subset L^p_{w_1}(G)$ if and only if $w_1 \le w_2$. Lastly we recall that a weight w satisfies the *Beurling-Domar condition* ([4]) if

$$\text{(BD)} \quad \sum_{n \ge 1} n^{-2} \cdot \log\bigl(w(nx)\bigr) < \infty \text{ for all } x \in G.$$

An algebra $A(X)$ of complex-valued continuous functions over a locally compact Hausdorff space X is called a *standard algebra* if it has the following properties:

(i) If $f \in A(X)$ and $f(a) \neq 0$ at a point $a \in X$, then there is a $g \in A(X)$ such that $g(x) = \frac{1}{f(x)}$ for all x in some neighbourhood of a.

(ii) For any closed set $E \subset X$ and any point $a \in X - E$ there is $f \in A(X)$ vanishing on E and such that $f(a) \neq 0$.

For any ideal I in $A(X)$ the set of points of X where all functions in I vanish is called the *cospectrum* of I, denoted by $\operatorname{cosp} I$. We shall use the known fact that any ideal with $\operatorname{cosp} I = \emptyset$ contains $A(X) \cap C_c(X)$, i.e. all functions in $A(X)$ with compact support ([18; 1.4.(ii)]).

If, in addition to (i) and (ii), $A(X) \cap C_c(X)$ is dense in $A(X)$, then $A(X)$ will be called a *Wiener algebra.*

If V and W are Banach $L^1_w(G)$-modules, then $\operatorname{Hom}_A(V, W)$ will denote the Banach space of all continuous A-module homomorphisms from V to W with the operator norm. The elements of $\operatorname{Hom}_A(V, W)$ are called *multipliers* from V to W.

Furthermore we denote the projective tensor product of V and W as Banach space by $V \otimes_\gamma W$. Let K be the closed linear subspace of $V \otimes_\gamma W$ spanned by elements of the form $(f \cdot g) \otimes h - g \otimes (f \cdot h)$, $f \in L^1_w(G)$, $g \in V$ and $h \in W$. By definition, the $L^1_w(G)$-module tensor product $V \otimes_{L^1_w} W$ is the quotient Banach space $V \otimes_\gamma W/K$ ([17]). It is known that any $t \in V \otimes_{L^1_w} W$ can be written in the form

$$t = \sum_{k=1}^{\infty} g_k \otimes h_k\,, \quad g_k \in V\,,\ h_k \in W\,, \qquad \text{where} \quad \sum_{k=1}^{\infty} \|g_k\| \|h_k\| < \infty\,. \tag{1.2}$$

Whenever we talk about $L^1_w(G)$-modules in this paper we mean Banach $L^1_w(G)$-modules with respect to convolution.

2. Multipliers from $L^1_w(G)$ to $S_w(G)$

It is known that $L^1_w(G)$ is a closed ideal in $M(w)$ and the space of multipliers of $L^1_w(G)$ is homeomorphic to the space $M(w)$, where

$$M(w) = \bigl\{\mu :\ \mu \in M(G)\,,\ \textstyle\int w \,\mathrm{d}|\mu| < \infty\bigr\}\,. \tag{2.1}$$

Cigler gives a generalization of Segal algebra in [2] as follows:

Let $S_w = S_w(G)$ be a subalgebra in $L^1_w(G)$ satisfying the following conditions:

S1) S_w is dense in $L^1_w(G)$.

S2) S_w is a Banach algebra under some norm $\|\cdot\|_{S_w}$ and invariant under translations.

S3) $\|L_a f\|_{S_w} \le w(a)\|f\|_{S_w}$ for all $a \in G$ and for each $f \in S_w$.

S4) Given any $f \in S_w$ and $\varepsilon > 0$, there is a neighbourhood U of the unit element e of G such that $\|L_y f - f\|_{S_w} < \varepsilon$ for all $y \in U$.

S5) $\|f\|_{1,w} \le \|f\|_{S_w}$ for all $f \in S_w$.

PROPOSITION 2.1. *If $\mu \in M(w)$ and $f \in S_w(G)$, then $\mu * f$ in $S_w(G)$ and $\|\mu * f\|_{S_w} \le \|\mu\|_w \cdot \|f\|_{S_w}$ where $\|\mu\|_w = \int w \, \mathrm{d}|\mu|$.*

Proof. Since $y \mapsto L_y f$ is a continuous function from G into $S_w(G)$ for $f \in S_w(G)$ and μ is a bounded Borel measure, then $L_y f \in L^1_{S_w}(G,\mu)$ is in $L^1_{S_w}(G)$, the space of integrable functions with values in $S_w(G)$. Hence the vector integral $\int L_y f \, \mathrm{d}\mu(y)$ exists as in $S_w(G)$ and

$$\begin{aligned}\left\| \int L_y f \, \mathrm{d}\mu(y) \right\|_{S_w} &\le \int \|L_y f\|_{S_w} \, \mathrm{d}|\mu|(y) \\ &\le \int \|f\|_{S_w} w(y) \, \mathrm{d}|\mu|(y) = \|f\|_{S_w} \cdot \|\mu\|_w .\end{aligned} \tag{2.2}$$

By the technique of proof used in [19; p. 20, Proposition 2], we show that

$$\int L_y f \, \mathrm{d}\mu(y) = \mu * f . \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\|\mu * f\|_{S_w} \le \|\mu\|_w \cdot \|f\|_{S_w} . \tag{2.4}$$

□

PROPOSITION 2.2. *$S_w(G)$ is an essential Banach ideal in $L^1_w(G)$.*

Proof. We know that $S_w(G)$ is a dense Banach ideal in $L^1_w(G)$ by assumption (S1) and Proposition 2.1. Now let $f \in S_w(G)$ and $\varepsilon > 0$ be given. By the definition of $S_w(G)$ there is a neighbourhood U of the unit element e of G such that

$$\|L_y f - f\|_{S_w} < \varepsilon \tag{2.5}$$

for all $y \in U$. Let $(e_\alpha)_{\alpha \in I}$ be a non-negative bounded approximate identity in $L^1_w(G)$ satisfying $\|e_\alpha\|_1 = 1$ and $\operatorname{supp} e_\alpha \subset U$ for all $\alpha \in I$, [22].

Then there exists $\alpha_0 \in I$ such that

$$\|e_\alpha * f - f\|_{S_w} = \left\| \int e_\alpha(y)\{L_y f - f\}\,\mathrm{d}y \right\|_{S_w} \le \int e_\alpha(y)\|L_y f - f\|_{S_w} \le \varepsilon \quad (2.6)$$

for all $\alpha > \alpha_0$. Hence $S_w(G)$ is an essential Banach ideal in $L^1_w(G)$ by [5; 15.3. Corollary]. □

PROPOSITION 2.3. *Suppose that w satisfies* (BD)*. Then $L^1_w(G)$ has a bounded approximate identity $(e_\alpha)_{\alpha\in I}$ whose Fourier transforms have compact support and $e_\alpha \in S_w(G)$ for all $\alpha \in I$.*

Proof. It is known that the Fourier transform of the functions in $L^1_w(G)$ form an algebra of continuous complex-valued functions with the ordinary multiplication (pointwise) algebraic operations. We denote it by $F\big(L^1_w(G)\big) = F^1_w(\hat{G})$ and carry the L^1_w-norm over to $F^1_w(\hat{G})$ by putting

$$\|\hat{f}\|_{F^1_w} = \|f\|_{1,w}\,, \qquad \hat{f} \in F^1_w(\hat{G})\,. \quad (2.7)$$

We denote by $F_{0,w}$ the set of all $f \in L^1_w(G)$ such that $\hat{f} \in F^1_w(\hat{G})$ has compact support. Since w satisfies (BD), then $F^1_w(\hat{G})$ is a Wiener algebra ([18]). We denote by $F\big(S_w(G)\big)$ the image of $S_w(G)$ under the Fourier transform. Since $S_w(G)$ is dense in $L^1_w(G)$, then it is easily proved that $\operatorname{cosp}\big(F\big(S_w(G)\big)\big) = \emptyset$. Hence, by the properties [18; p. 20, 1.4.ii], we have the inclusion $F_{0,w} \subset S_w(G)$. Also since w satisfies (BD), then $L^1_w(G)$ admits a bounded approximate identity $(e_\alpha)_{\alpha\in I}$ such that $e_\alpha \in F_{0,w}$ for all $\alpha \in I$ ([6; Lemma 4.1]). □

One can also prove Proposition 2.3 with another way using [8; Proposition 1.1].

Next we denote by M_{S_w} the space of $\mu \in M(w)$ such that $\|\mu\|_M \le C(\mu)$, where

$$\|\mu\|_M = \sup\left\{ \frac{\|\mu * f\|_{S_w}}{\|f\|_{1,w}} :\ f \in L^1_w(G)\,,\ f \ne 0\,,\ \hat{f} \in C_c(\hat{G}) \right\}. \quad (2.8)$$

By the Proposition 2.1 we have $M_{S_w} \ne \{0\}$.

PROPOSITION 2.4. *If w satisfies* (BD)*, then for a linear operator $T\colon L^1_w(G) \to S_w(G)$ the following are equivalent:*

(1) $T \in M\big(L^1_w(G), S_w(G)\big)$ *(the space of multipliers from $L^1_w(G)$ to $S_w(G)$).*

(2) *There exists a unique $\mu \in M_{S_w}$ such that*
$Tf = \mu * f$ for every $f \in L^1_w(G)$.

Moreover the correspondence between T and μ defines an isomorphism between $M\big(L^1_w(G), S_w(G)\big)$ and M_{S_w}.

Proof. Suppose $\mu \in M_{S_w}$ such that $T_1 f = \mu * f$ for every $f \in L^1_w(G)$. $\mu \in M_{S_w}$ implies that T_1 is a bounded linear operator from $F_{0,w}$ (which is a dense subspace of $L^1_w(G)$ endowed with the norm $\|\cdot\|_{1,w}$) into $(S_w, \|\cdot\|_{S_w})$. Using a standard approximation argument T_1 extends to a unique bounded linear operator T on all of $L^1_w(G)$. Clearly T still commutes with convolutions and maps $L^1_w(G)$ into $S_w(G)$.

Conversely suppose $T \in M\big(L^1_w(G), S_w G)\big)$. Then according to [12] there exists a unique $\mu \in M(w)$ such that $Tf = \mu * f$ for all $f \in L^1_w(G)$. Since

$$\|Tf\|_{S_w} = \|\mu * f\|_{S_w} \le C\|f\|_{1,w}, \tag{2.9}$$

it is obvious that $\mu \in M_{S_w}$. The proof also shows that the norm $\|\cdot\|_M$ and the operator norm $\|\cdot\|$ are equivalent. □

Definition 2.5. Let V be a $L^1_w(G)$-Banach convolution module. We write $S_w(G) \underline{\otimes} V$ for the space of all $t \in V$ for which there are sequences $\{g_k\}_{k=1}^{\infty} \subset S_w(G)$, $\{h_k\}_{k=1}^{\infty} \subset V$ with $t = \sum_{k=1}^{\infty} g_k * h_k$ and $\sum_{k=1}^{\infty} \|g_k\|_{S_w} \cdot \|h_k\|_V < \infty$.

It follows immediately from [20; Theorem 6] that $S_w(G) \underline{\otimes} V$ is a Banach space with the norm

$$|||t||| = \inf\left\{ \sum_{k=1}^{\infty} \|g_k\|_{S_w} \cdot \|h_k\|_V : \ \{g_k\}_{k=1}^{\infty} \subset S_w(G), \ \{h_k\}_{k=1}^{\infty} \subset V, \ t = \sum_{k=1}^{\infty} g_k * h_k \right\}. \tag{2.10}$$

Also it is easy to see that $\|t\|_V \le |||t|||$.

Proposition 2.6. *Let V be a $L^1_w(G)$-convolution Banach module. Then*

$$S_w(G) \otimes_{L^1_w} V \cong S_w(G) \underline{\otimes} V.$$

The isomorphism being an isometric one.

Proof. Consider the mapping B from the projective tensor product $S_w(G) \otimes_\gamma V$ to $S_w(G) \underline{\otimes} V$ determined by $B(f \otimes g) = f * g$, $f \in S_w(G)$ and $g \in V$. It is easy to see that B is surjective. Also B is an isomorphism being isometric by the arguments used in the proof of [17; Theorem 3.3]. □

Theorem 2.7. *For an L^1_w-convolution Banach module V the following are equivalent:*

1) $S_w(G) \underline{\otimes} V = V$.
2) $\mathrm{Hom}_{L^1_w}\big(S_w(G), V^*\big) \cong V^*$ *in the sense of a topological isomorphism.*

P r o o f . If $S_w(G) \underline{\otimes} V = V$, then it is easy to see that $S_w(G) \underline{\otimes} V \cong V$, using the closed graph theorem. Hence

$$\left(S_w(G) \underline{\otimes} V\right)^* = V^* . \tag{2.11}$$

Then we have

$$\operatorname{Hom}_{L^1_w}\left(S_w(G), V^*\right) \cong \left(S_w(G) \underline{\otimes} V\right)^* = V^* \tag{2.12}$$

by [17; Theorem 1.4].

Conversely suppose $\operatorname{Hom}_{L^1_w}\left(S_w(G), V^*\right) = V^*$.
Then for $\alpha\colon V^* \to \operatorname{Hom}_{L^1_w}\left(S_w(G), V^*\right)$,

$$\left\langle v, \alpha(v^*)(g)\right\rangle = \langle g * v, v^*\rangle , \qquad v \in V , \ \ v^* \in V^* , \ \ g \in S_w(G) , \tag{2.13}$$

is a surjective topological isomorphism. Define the function $\beta \circ \alpha\colon V^* \to \left(S_w(G) \underline{\otimes} V\right)^*$, where $\beta\colon \operatorname{Hom}_{L^1_w}\left(S_w(G), V^*\right) \to \left(S_w(G) \underline{\otimes} V\right)^*$ is defined as in [17]. Since α and β are surjective also $\beta \circ \alpha$ will be surjective. The proof that $i^* = \beta \circ \alpha$ proceeds then as for [15; Theorem 2], where $i\colon S_w(G) \underline{\otimes} V \to V$ is the identity map and i^* is the usual adjoint of i. Hence i is also surjective and in this case we have $S_w(G) \underline{\otimes} V = V$. □

3. Applications

1) Let w, ω be weight functions on G and $\hat{G}$ respectively. For $1 \le p < \infty$ we set

$$A^p_{w,\omega}(G) = \left\{f : \ f \in L^1_w(G) , \ \ \hat{f} \in L^p_\omega(\hat{G})\right\}$$

and

$$\|f\|^p_{w,\omega} = \|f\|_{1,w} + \|\hat{f}\|_{p,w} . \tag{3.1}$$

These spaces were introduced by F e i c h t i n g e r – G ü r k a n l ı in [6]. Another generalization has been given by F i s c h e r – G ü r k a n l ı – L i u in [10], [11], where it is proved that $\left(A^p_{w,\omega}(G), \|\cdot\|^p_{w,\omega}\right)$ is a Banach algebra with respect to convolution. It is also proved that if the first weight w satisfies (BD), then $A^p_{w,\omega}(G)$ is a dense Banach ideal in $L^1_w(G)$ having an approximate identity bounded in the norm of $L^1_w(G)$ with compactly supported Fourier transforms. Furthermore for given any $f \in A^p_{w,\omega}(G)$, the function $a \mapsto L_a f$ is continuous. Finally $\|f\|_{1,w} \le \|f\|^p_{w,\omega}$ and

$$\|L_a f\|^p_{w,\omega} = \|L_a f\|_{1,w} + \|\widehat{L_a f}\|_{p,\omega} \le w(a)\|f\|_{1,w} + \|\hat{f}\|_{p,\omega} \le w(a)\|f\|^p_{w,\omega} . \tag{3.2}$$

Therefore if w satisfies (BD), then $A^p_{w,\omega}(G)$ is a $S_w(G)$ space. Applying the Proposition 2.4 to the space $A^p_{w,\omega}(G)$ one obtains that the space of multipliers from $L_w^1(G)$ to $A_{w,\omega}^p(G)$ (briefly $M\big(L_w^1(G), A_{w,\omega}^p(G)\big)$) is homeomorphic to M_A, where $A = A_{w,\omega}^p(G)$.

We denote by $B_w(G)$ the space of all the measures μ in $M(w)$ such that the Fourier-Stieltjes transform $\hat{\mu}$ of μ belongs to $L_\omega^p(\hat{G})$. It is easily seen that $B_w(G)$ is a Banach space with the norm

$$\|\mu\|_{B_w} = \|\mu\|_w + \|\hat{\mu}\|_{p,\omega} . \tag{3.3}$$

Indeed, it is proved in [1] that $B_w(G) = M_A(G)$ is a Banach space for $w = 1$ and $\omega = 1$, with the norm

$$\|\mu\|_{B_1} = \|\mu\|_w + \|\hat{\mu}\|_{p,\omega} , \tag{3.4}$$

where $\|\mu\|$ denotes the usual total variation norm of $\mu \in M(G)$. Also $\|\mu\| \le \|\mu\|_w$ and $\|\hat{\mu}\|_p \le \|\hat{\mu}\|_{p,w}$ for all $\mu \in B_w(G)$. Now let $\{\mu_n\}_{n=1}^\infty$ be a Cauchy sequence in $B_w(G)$. Then $\{\mu_n\}_{n=1}^\infty$ and $\{\hat{\mu}_n\}_{n=1}^\infty$ are Cauchy sequences in $M(w)$ and in $L_\omega^p(\hat{G})$ respectively. Since $M(w)$ and $L_\omega^p(\hat{G})$ are Banach spaces, then $\{\mu_n\}_{n=1}^\infty$ converges to a measure $\mu \in M(w)$ and $\{\hat{\mu}_n\}_{n=1}^\infty$ converges to a function $h \in L_\omega^p(G)$. Hence $\{\mu_n\}_{n=1}^\infty$ converges to μ in $M(G)$ and $\{\hat{\mu}_n\}_{n=1}^\infty$ converges to h in $L^p(G)$. Then using (3.4) we write $\hat{\mu} = h$ as in [1]. This completes the proof.

THEOREM 3.1. *If w satisfies* (BD), *then $B_w(G) = M_A(G)$ and the corresponding natural norms are equivalent.*

P r o o f. Suppose $\mu \in B_w(G)$ and $f \in L_w^1(G)$, $f \ne 0$. Then we write

$$\|\mu * f\|_{w,\omega}^p = \|\mu * f\|_{1,w} + \|\widehat{\mu * f}\|_{p,\omega} = \|\mu * f\|_{1,w} + \|\hat{\mu} \cdot \hat{f}\|_{p,\omega} . \tag{3.5}$$

By the technique of proof used in Proposition 2.1, we see that

$$\|\mu * f\|_{1,w} \le \|\mu\|_w \cdot \|f\|_{1,\omega} . \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned} \|\mu * f\|_{w,\omega}^p &= \|\mu * f\|_{1,w} + \|\widehat{\mu * f}\|_{p,\omega} \\ &\le \|\mu\|_w \cdot \|f\|_{1,w} + \|\hat{\mu}\|_{p,\omega} \cdot \|\hat{f}\|_\infty \\ &\le \|\mu\|_w \cdot \|f\|_{1,w} + \|\hat{\mu}\|_{p,\omega} \cdot \|f\|_{1,w} \\ &= \|f\|_{1,w}\big(\|\mu\|_w + \|\hat{\mu}\|_{p,\omega}\big) = \|f\|_{1,w} \cdot \|\mu\|_{B_w} \le \infty . \end{aligned} \tag{3.7}$$

Hence we have

$$\frac{\|\mu * f\|_{w,\omega}^p}{\|f\|_{1,w}} \le \|\mu\|_{B_w} . \tag{3.8}$$

This implies that $\mu \in M_A$ and $\|\mu\|_M \le \|\mu\|_{B_{w,\omega}}$.

Conversely suppose $\mu \in M_A$. Since w satisfies (BD), the space $A^p_{w,\omega}(G)$ admits an approximate identity $(e_\alpha) \subset F_{0,w}$, bounded in $L^1_w(G)$ ([6]). Also since $\|\mu\|_{M_A} < \infty$, then $\mu \in M(w)$ and

$$\|\mu * e_\alpha\|^p_{w,\omega} \le \|\mu\|_M \cdot \sup_{\alpha\in I} \|e_\alpha\|_{1,w} = K(\mu) \tag{3.9}$$

for every $\alpha \in I$. This implies

$$\|\hat{\mu} \cdot \hat{e}_\alpha\|_{p,\omega} \le \|\mu * e_\alpha\|^p_{w,\omega} \le K(\mu) \tag{3.10}$$

for all $\alpha \in I$. The reflexivity of $L^p_\omega(\hat{G})$ and the Banach-Alaoglu theorem imply that there exists a subnet $(\hat{\mu} \cdot \hat{e}_\beta)$ of $(\hat{\mu} \cdot \hat{e}_\alpha)$ and $g \in L^p_\omega(\hat{G})$ such that $(\hat{\mu} \cdot \hat{e}_\beta)$ converges weakly to g. Since $(\hat{\mu} \cdot \hat{e}_\beta)$ converges uniformly to $\hat{\mu}$ on compact subsets of $\hat{G}$ it is easy to see that $\hat{\mu} = g$ almost everywhere. Thus $\hat{\mu} \in L^p_\omega(\hat{G})$. Consequently we obtain that $\mu \in B_w(G)$ and

$$\|\mu\|_{B_w} \le \|\mu\|_M \cdot \sup_{\alpha\in I} \|e_\alpha\|_{1,w} \, . \tag{3.11}$$

Hence we have $B_w(G) = M_A(G)$. □

2) Let $S(G)$ be a solid ordinary Segal algebra, i.e. assume that $|f(x)| \le |g(x)|$ a.e. for $g \in S(G)$, $f \in L^1(G)$ (or just measurable) implies $f \in S(G)$ and $\|f\|_S \le \|g\|_S$. Define a set

$$S^w(G) = \left\{ f \in L^1_w(G) : \; f \cdot w \in S(G) \right\}. \tag{3.12}$$

It is easy to see that $\left(S^w(G), \|\cdot\|_{S^w}\right)$ is a normed space with the natural norm

$$\|f\|_{S^w} = \|f \cdot w\|_S \, . \tag{3.13}$$

PROPOSITION 3.2.

a) $\left(S^w(G), \|\cdot\|_{S^w}\right)$ *is a Banach convolution algebra.*

b) $S^w(G)$ *is dense in* $L^1_w(G)$.

Proof.

a) It is easy to prove that $S^w(G)$ is a Banach space. Now let $f, g \in S^w(G)$ be given. We write

$$\begin{aligned} \|(f*g)w\|_S &\le \int \|f(t) \cdot g(x-t) w(x)\|_S \, \mathrm{d}t \\ &\le \int \|g(u) \cdot w(u)\|_S \cdot |w(t) f(t)| \, \mathrm{d}t = \|gw\|_S \cdot \|f\|_{1,w} \, , \end{aligned} \tag{3.14}$$

where $x - t = u$. Hence we have

$$\|f*g\|_{S^w} \le \|f\|_{1,w} \cdot \|g\|_{1,w} + \|gw\|_S \cdot \|f\|_{1,w} = \|f\|_{1,w} \cdot \|g\|_{S^w} \le \|f\|_{S^w} \cdot \|g\|_{S^w} \, .$$

This completes the proof of a).

b) Let $\varepsilon > 0$ and any $f \in L^1_w(G)$ be given. Since $fw \in L^1(G)$ and $S(G)$ is dense in $L^1(G)$, there exists $g \in S(G)$ such that

$$\|fw - g\|_1 < \varepsilon .$$

Hence

$$\left\|f - \frac{g}{w}\right\|_{1,w} = \left\|\left(f - \frac{g}{w}\right)w\right\|_1 = \|fw - g\|_1 < \varepsilon . \tag{3.15}$$

Also we have

$$\left\|\frac{g}{w}\right\|_{S^w} = \|g\|_S < \infty .$$

That means $g \in S^w(G)$. This completes the proof. □

Proposition 3.3. *If w satisfies* (BD), *then to every compact subset $\hat{K} \subset \hat{G}$ there is a constant $C_{\hat{K}} > 0$ such that, for every $f \in S^w(G)$ whose Fourier transformation vanishes outside of $\hat{K}$, it holds that $\|f\|_{S^w} \le C_{\hat{K}} \cdot \|f\|_{1,w}$.*

Proof. Due to (BD), for any such $\hat{K} \subset \hat{G}$ there is $g \in S^w(G)$ with $\hat{g}(x) = 1$ for all $x \in \hat{K}$. Hence $f * g \in S^w(G)$ and

$$\|f * g\|_{S^w} \le \|f\|_{1,w} \cdot \|g\|_{S^w} \tag{3.16}$$

for all $f \in S^w(G)$ satisfying $\operatorname{supp}\hat{f} \subset \hat{K}$ by the proof of Proposition 3.2. If we set $C_{\hat{K}} = \|g\|_{S^w}$, we obtain the desired estimate

$$\|f\|_{S^w} = \|f * g\|_{S^w} \le C_{\hat{K}}\|f\|_{1,w} . \tag{3.17}$$

□

Proposition 3.4. *If w satisfies* (BD), *then for any $f \in S^w(G)$ the map $y \mapsto L_y f$ is continuous from G into $S^w(G)$.*

Proof. For $g \in F_{0,w}$, $\operatorname{supp}(L_y g - g)^\wedge$ is compact and $y \mapsto L_y g$ is continuous from G into $S^w(G)$ by Proposition 3.3. Since w satisfies (BD), $F_{0,w}$ is dense in $L^1_w(G)$ and statement is true for any $f \in S^w(G)$.

In summary we have shown: If w satisfies (BD), then $S^w(G)$ is a $S_w(G)$ space. □

Examples. Let G be a non-discrete and non-compact locally compact abelian group and w be Beurling's function weight on G.

1) Choose $L^1(G) \cap L^p(G)$, $1 \le p < \infty$, as a solid Segal algebra with norm

$$\|f\| = \|f\|_1 + \|f\|_p , \qquad f \in L^1(G) \cap L^p(G) . \tag{3.18}$$

One can define $S^w(G)$ using this Segal algebra.

2) Take the Wiener amalgam space $W\big(L^p(G), L^1(G)\big)$. It is known that $W\big(L^p(G), L^1(G)\big)$ is a solid Segal algebra by [9; Corollary 1]. So one can define the space $S^w(G)$ using this Segal algebra.

4. Wiener-Ditkin sets for $S_w(G)$-spaces

In this section we will discuss the Wiener-Ditkin sets for $S_w(G)$-spaces. In the spirit of [18] we call a closed subset $E \subset \hat{G}$ a *Wiener-Ditkin set* for $S_w(G)$ if each $f \in S_w(G)$ such that $\hat{f}$ vanishes on E can be approximated in $S_w(G)$ with functions $f * F$ such that $\hat{F}$ vanishes in some neighbourhood on E.

THEOREM 4.1. *A set $E \subset \hat{G}$ is a Wiener-Ditkin set for $S_w(G)$ if and only if E is a Wiener-Ditkin set for $L^1_w(G)$.*

Proof.

1) Assume that E is a Wiener-Ditkin set for $S_w(G)$. Let $f \in L^1_w(G)$ be such that $\hat{f}$ vanishes on E and $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity in $L^1_w(G)$. Also let $\varepsilon > 0$ be given. Then there exists $\alpha_1 \in I$ such that

$$\|f - f * e_{\alpha_1}\|_{1,w} < \frac{\varepsilon}{2}\,. \tag{4.1}$$

Since $S_w(G)$ is an ideal in $L^1_w(G)$, then $f * e_{\alpha_1} \in S_w(G)$. Clearly $\left(f * e_{\alpha_1}\right)^\wedge = 0$ on E. Hence we can find $F_1 \in S_w(G)$ such that $\hat{F}_1$ vanishes on a neighbourhood of E and

$$\|f * e_{\alpha_1} - F_1 * (f * e_{\alpha_1})\|_{S_w} < \frac{\varepsilon}{2}\,. \tag{4.2}$$

We set $F = F_1 * e_{\alpha_1}$. Then $\hat{F} = \hat{F}_1 \cdot \hat{e}_{\alpha_1}$ and thus $\hat{F}$ vanishes on a neighbourhood of E. Since $\|\cdot\|_{1,w} \le \|\cdot\|_{S_w}$, from (4.1) and (4.2) we have

$$\begin{aligned}\|f - f * F\|_{1,w} &= \|f - f * e_{\alpha_1} + f * e_{\alpha_1} - F * f\|_{1,w}\\ &\le \|f - f * e_{\alpha_1}\|_{1,w} + \|f * e_{\alpha_1} - F_1 * (f * e_{\alpha_1})\|_{1,w}\\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned} \tag{4.3}$$

This completes the proof of first part.

2) Assume that $E \subset \hat{G}$ is a Wiener- Ditkin set for $L^1_w(G)$. Let $f \in S_w(G)$ with $\hat{f}$ vanishing on $E \subset \hat{G}$. Since $S_w(G)$ is an essential Banach module over $L^1_w(G)$, then for any given $\varepsilon > 0$ there exists $\alpha_0 \in I$ such that

$$\|f - f * e_{\alpha_0}\|_{S_w} < \frac{\varepsilon}{2}\,. \tag{4.4}$$

There also exists $F_1 \in L^1_w(G)$ such that $\hat{F}_1$ vanishes on a neighbourhood of E and

$$\|f - f * F_1\|_{S_w} < \frac{\varepsilon}{2 \cdot \|e_{\alpha_0}\|_{S_w}}\,. \tag{4.5}$$

Take $F = e_{\alpha_0} * F_1$. Then $\hat{F} = \hat{e}_{\alpha_0} \cdot \hat{F}_1$ and therefore $\hat{F}$ vanishes on a neighbourhood of E. Finally

$$\begin{aligned} \|f - f * F\|_{S_w} &< \|f - f * e_{\alpha_0}\|_{S_w} + \|f * e_{\alpha_0} - f * F\|_{S_w} \\ &= \|f - f * e_{\alpha_0}\|_{S_w} + \|f * e_{\alpha_0} - f * F_1 * e_{\alpha_0}\|_{S_w} \\ &\le \frac{\varepsilon}{2} + \|e_{\alpha_0}\| \cdot \frac{\varepsilon}{2 \cdot \|e_{\alpha_0}\|_{S_w}} = \varepsilon . \end{aligned} \tag{4.6}$$

This completes the proof. □

Remark 4.2. Let α be a positive number (or zero) and consider the Beurling's weight function

$$w(x) = \bigl(1 + |x|\bigr)^{\alpha} , \qquad x, y \in \mathbb{R}^n . \tag{4.7}$$

We denote the corresponding weighted space by $L^1_w(\mathbb{R}^n) = L^1_\alpha(\mathbb{R}^n)$ and the norm by $\|\cdot\|_{1,w} = \|\cdot\|_{1,\alpha}$. It is known that the closed subgroup of $\mathbb{R}^n$ are Wiener-Ditkin sets for $L^1_\alpha(\mathbb{R}^n)$ $(0 \le \alpha < 1)$, [21]. Take the space $A^p_{w,w}(G)$ from Section 3. Since w satisfies (BD), then any closed subgroup of $\mathbb{R}^n$ is a Wiener-Ditkin sets for this space.

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