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UNCONDITIONALLY CONVERGENT OPERATORS ON $C_0(X_0)$

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ABSTRACT. Let X_0 be a locally compact Hausdorff space, $C_0(X_0)$ the space of all scalar-valued bounded continuous functions on X_0 vanishing at infinity, and X a one-point compactification of X_0. We prove that weak compactness property of unconditionally convergent operators on $C_0(X_0)$ can be easily deduced by considering the space $C(X)$ and its dual $M(X)$. The result is proved for the vector case $C_0(X_0, F)$, F being a reflexive Banach space. It is also proved that, for a quasi-complete locally convex space E, if $c_0 \not\subseteq E$, then every linear continuous operator $u\colon C_0(X_0, F) \to E$ is weakly compact.

1. Introduction and notations

We start with some notations. In this paper X_0 is a locally compact Hausdorff space, K the field of real or complex numbers (called scalars), and X a one-point compactification of X_0; this point is called the point at infinity and we will denote it by q. For a Banach space F, $C_0(X_0, F)$ denotes the space of all F-valued bounded continuous functions on X_0 vanishing at infinity, and $C(X, F)$ denotes the space of all F-valued continuous functions on X. We have $C_0(X_0, F) = \{f \in C(X, F) : f(q) = 0\}$. The duals of $C_0(X_0, F)$ and $C(X, F)$ are denoted by $M_0(X_0, F')$ and $M(X, F')$. Elements of $M(X, F')$ are F'-valued regular Borel measures of finite variations on X ([5]) (the variation of a $\mu \in M(X, F')$ is denoted by $|\mu|$). Also $M_0(X_0, F') = \{\mu \in M(X, F') : |\mu|(\{q\}) = 0\}$ and $M_0(X_0, F')$ is a closed subspace, with induced norm, of the Banach space $M(X, F')$. For an $f \in C(X, F)$, $\|f\|$ will be considered an element of $C(X)$, $\|f\|(x) = \|f(x)\|$ ([5]).

For locally convex spaces, the notations and results of [9] will be used. For topological measure theory, notations and results of [2], [8] and [5] will be used.

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All locally convex spaces are assumed to be Hausdorff and over K. For a locally convex space G, G' and G'' will denote its dual and bidual.

E will always stand for a quasi-complete locally convex space whose topology is generated by an increasing family of semi-norms $\{p: p \in P\}$. For an $h \in E'$ and $p \in P$, $h \le p$ means $\left|h(p^{-1}([0,1]))\right| \le 1$. It is obvious that, for a $p \in P$, $\{h \in E' : h \le p\}$ is equicontinuous. For two locally convex spaces G_1, G_2, a linear continuous mapping $u\colon G_1 \to G_2$ is called unconditionally convergent if for any weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in G_1, $\sum_{n=1}^{\infty} u(x_n)$ is unconditionally convergent in G_2.

2. Main results

We start with a simple lemma which is an immediate consequence of the fact that a continuous mapping between two Banach spaces is weakly compact if and only if its adjoint is weakly compact ([3; p. 485, Theorem 8]).

LEMMA 1. *Suppose F is a Banach space and $u\colon F \to E$, a linear continuous mapping. Assume that the adjoint mapping $u'\colon E' \to F'$ maps equicontinuous subsets of E' into relatively weakly compact subsets of the Banach space F'. Then the mapping u is weakly compact.*

P r o o f . For every $p \in P$, let E_p be the completion of the normed space arising from the quasi-norm p, and $\phi_p\colon E \to E_p$ be the canonical mapping. Then $E \subset \prod_{p\in P} E_p$. For every $p \in P$, the mapping $\phi_p \circ u\colon F \to E_p$ is weakly compact since, by the given hypothesis, its adjoint is weakly compact. Since E is quasi-complete, the result follows. □

Now we state and prove the main theorem.

THEOREM 2. *Let F be a reflexive Banach space and $u\colon C_0(X_0,F) \to E$ be a linear unconditionally convergent operator. Then it is weakly compact.*

P r o o f . Fix a $p \in P$. We first prove that $\{h \circ u : h \in E', h \le p\}$ is relatively weakly compact in $M(X,F')$. For that we use [1; p. 151, Theorem 3.1]. To prove relatively weak compactness, we have to prove that $\{|h \circ u| : h \in E', h \le p\}$ is relatively weakly compact in $M(X)$. Suppose there is a disjoint sequence $\{V_n\}_{n=1}^{\infty}$ of open sets in X, a $c > 0$, and a sequence $\{h_n\}_{n=1}^{\infty} \subset E'$, $h_n \le p$ for all n, such that $|h_n \circ u|(V_n) > c$ for all n. Taking $U_n = V_n \setminus \{q\}$, we have $|h_n \circ u|(V_n) = |h_n \circ u|(U_n)$ for all n. Take $\{f_n\}_{n=1}^{\infty} \subset C(X,F)$ such that $0 \le \|f_n\| \le \chi_{U_n}$ and $|(h_n \circ u)(f_n)| > c$ for all n ([5; p .198, Theorem 2.1]). Since $\sum_{n=1}^{\infty} f_n$ is unconditionally weakly Cauchy in $C_0(X_0,F)$, we get $\sum_{n=1}^{\infty} u(f_n)$

is unconditionally convergent in E. This means $p(u(f_n)) \to 0$, which is a contradiction. Thus $\{h \circ u : h \in E', h \leq p\}$ is relatively weakly compact in $M(X, F')$. Since $M_0(X_0, F')$ is closed in $M(X, F')$, $\{h \circ u : h \in E', h \leq p\}$ is relatively weakly compact in $M_0(X_0, F')$. The result follows from Lemma 1. □

Remark 3. This result contains main part of the [7; Theorem 1]; see also [6; p. 4864, Theorem 12].

When $c_0 \not\subseteq E$, we get the following corollary:

COROLLARY 4. *Let F be a reflexive Banach space and $c_0 \not\subseteq E$. Then every linear continuous $u: C_0(X_0, F) \to E$ is weakly compact.*

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset C_0(X_0, F)$ be such that $\sum_{n=1}^{\infty} |\mu(f_n)| < \infty$ for every $\mu \in M_0(X_0, F')$. Since for every $h \in E'$, $h \circ u \in M_0(X_0, F')$, we get $\sum_{n=1}^{\infty} |h(u(f_n))| < \infty$, for every $h \in E'$. Since $c_0 \not\subseteq E$, by [10; Theorem 4], u is unconditionally convergent. By Theorem 2, u is weakly compact. □

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