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DYNAMICAL STABILITY OF THE TYPICAL CONTINUOUS FUNCTION

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ABSTRACT. We consider the typical behavior of two maps. The first is the set valued function Λ taking f in $C(I,I)$ to its collection of ω-limit points $\Lambda(f) = \bigcup_{x\in I} \omega(x,f)$, and the second is the map Ω taking f in $C(I,I)$ to its collection of ω-limit sets $\Omega(f) = \{\omega(x,f) : x \in I\}$. After reviewing results which characterize those functions f in $C(I,I)$ at which each of our maps Λ and Ω is continuous, we show that both Λ and Ω are continuous on a residual subset of $C(I,I)$. We go on to investigate the relationship between the continuity of Λ and Ω at some function f in $C(I,I)$ with the chaotic nature of that function.

1. Introduction

At the Twentieth Summer Symposium in Real Analysis, A. M. Bruckner posed several questions regarding the iterative stability of continuous functions as they undergo small perturbations, as well as why these questions are of general interest ([B]). In particular, how are the set of ω-limit points and the collection of ω-limit sets of a function affected by slight changes in that function? As Bruckner discusses in [B], we may also want to ask these questions when restricting our attention to particular subsets of $C(I,I)$, such as those functions that are in some way nonchaotic, or those functions that satisfy a particular smoothness condition. As one sees from various examples found in [B] and [TH], in general, both the set of ω-limit points and the collection of ω-limit sets of a typical function are affected dramatically by arbitrarily small perturbations. In [TH2] we make some progress towards understanding the continuity structure of the maps $f \mapsto \bigcup_{x\in I} \omega(x,f)$ and $f \mapsto \{\omega(x,f) : x \in I\}$. We take $(\boldsymbol{K},\boldsymbol{H})$ to be the class of nonempty closed sets $\boldsymbol{K}$ in I endowed with the Hausdorff metric $\boldsymbol{H}$, and let $(\boldsymbol{K}^*,\boldsymbol{H}^*)$ consist of the nonempty closed subsets of $\boldsymbol{K}$. We are then

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able to characterize those functions at which $\Lambda\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}, \boldsymbol{H})$ given by $f \mapsto \Lambda(f) = \bigcup_{x \in I} \omega(x, f)$ is continuous, as well as characterize the points of continuity of the map $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ given by $f \mapsto \Omega(f) = \{\omega(x,f) : x \in I\}$ when we restrict the domain of Ω to those continuous functions possessing zero topological entropy. These results are presented in the following two theorems.

Theorem 1.1. *The map $\Lambda\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}, \boldsymbol{H})$ is continuous at f if and only if the stable periodic points of f are dense in the set of chain recurrent points of f.*

Theorem 1.2. *Let $\boldsymbol{E} = \{f \in C(I,I) : f \text{ has zero topological entropy}\}$. Then $\Omega\colon \big(\boldsymbol{E}, \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is continuous at f if and only if either of the following equivalent conditions hold:*

1. *The stable periodic points of f are dense in the set of chain recurrent points of f.*
2. *Every periodic point of f is stable, and every simple system of f has nonempty interior.*

Our next result characterizes those functions at which $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is upper semicontinuous; that is, when we place no restrictions on the domain of Ω ([TH3]).

Theorem 1.3. *The map $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is upper semicontinuous at the function f if and only if $L \in \Omega(f)$ whenever $L \in \boldsymbol{K}$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \emptyset$ for every nonempty proper closed subset F of L.*

It is interesting to note that if a continuous function possesses zero topological entropy, then $\Omega\colon \big(\boldsymbol{E}, \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is continuous there if and only if $\Lambda\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}, \boldsymbol{H})$ is continuous there. Also, from Theorem 1.3, one sees that the basic properties of strong invariance and transport for ω-limit sets must in fact characterize the ω-limit sets of a function f for the map Ω to be upper semicontinuous there. In [TH3] we also develop the following sufficient condition for f to be a point of lower semicontinuity of our map $\Omega : \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$.

Theorem 1.4. *Let $f \in C(I,I)$. If the stable periodic orbits of f are dense in $\Omega(f)$, then $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is lower semicontinuous at f.*

The following result from [STH] builds upon the results from [TH3] to characterize those continuous self-maps of a unit interval at which $\Omega : \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is continuous.

THEOREM 1.5. *Let $f \in C(I,I)$. The map $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is continuous at f if and only if*

1. $\overline{S(f)} = CR(f)$,
2. *all the periodic points of f are stable,*

and

3. *$L \in \Omega(f)$ whenever $L \in \boldsymbol{K}$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \emptyset$ for every nonempty proper closed subset F of L.*

Our purpose in this paper is to consider the likelihood that the maps Λ and Ω are continuous at an arbitrary function f in $C(I,I)$. We show in section three that the map $\Lambda\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}, \boldsymbol{H})$ is, in fact, continuous on a residual subset of $C(I,I)$. Section four is dedicated to the analogous result for $\Omega : \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$, where we establish Ω's continuity on a residual subset of $C(I,I)$. In section five we study the relationship between the continuity of Λ and Ω and the chaotic nature of the function f in which we are interested. We find, in general, that the chaotic nature of f has little to do with the continuity of the maps Λ and Ω at f. It is only in working with functions chaotic in the sense of L i – Y o r k e, but possessing zero topological entropy that we are able to make a connection between the chaotic behavior of f and the continuity of Λ and Ω.

2. Preliminaries

We shall be concerned with the class $C(I,I)$ of continuous functions mapping the unit interval $I = [0,1]$ into itself, and the iterative properties this class of functions possesses. For f in $C(I,I)$ and any integer $n \geq 1$, f^n denotes the nth iterate of f. Let $P(f)$ represent those points $x \in I$ that are periodic under f, and if x_0 is a periodic point of period n for which $f^n(x) - x$ is not unisigned in any deleted neighborhood of x_0, then x_0 is called a *stable periodic point*; we let $S(f)$ represent the stable periodic points of f. For each x in I, we call the set of all subsequential limits of the sequence $\big\{f^n(x)\big\}_{n=0}^{\infty}$ the *ω-limit set of f generated by x*, and write $\omega(x,f)$. The following theorem summarizes three elementary properties of ω-limit sets.

THEOREM 2.1. *Suppose $f\colon I \to I$ is continuous and ω is an ω-limit set of f. Then*

1. *ω is closed;*
2. *$f(\omega) = \omega$, that is, ω is strongly invariant under f;*

and

3. *for any nonempty proper closed $F \subset \omega$ one has $F \cap \overline{f(\omega - F)} \neq \emptyset$.*

Let $\Lambda(f) = \bigcup_{x \in I} \omega(x,f)$ represent the ω-limit points of f, while $\Omega(f) = \{\omega(x,f) : x \in I\}$ denotes the set composed of the ω-limit sets of f. As alluded to in Theorems 1.3 and 1.4, the following collections of objects will be of particular importance in our analysis. We let $\boldsymbol{S}(f) = \{\omega : \omega$ is a stable periodic orbit of f in $C(I,I)\}$ be the collection of stable periodic orbits of f, and set $\widetilde{\Omega}(f) = \{L : L \subset [0,1]$ is closed, $f(L) = L$ and for any nonempty proper closed $F \subset L$ one has $F \cap \overline{f(L-F)} \neq \emptyset\}$. Now, let $\varepsilon > 0$ be given, and take x and y to be any points in $[0,1]$. An *ε-chain from x to y* with respect to a function f is a finite set of points $\{x_0, x_1, \dots, x_n\}$ in $[0,1]$ with $x = x_0$, $y = x_n$ and $|f(x_{k-1}) - x_k| < \varepsilon$ for $k = 0, 1, \dots, n-1$. We call x a *chain recurrent point of f* if there is an ε-chain from x to itself for any $\varepsilon > 0$, and write $x \in CR(f)$. We note that for every f in $C(I,I)$, $\Lambda(f) \subseteq CR(f)$.

In addition to the usual, Euclidean metric d on $I = [0,1]$, we will be working in three metric spaces. Within $C(I,I)$ we will use the supremum metric given by $\|f - g\| = \sup\{|f(x) - g(x)| : x \in I\}$. Our second metric space $(\boldsymbol{K}, \boldsymbol{H})$ is composed of the class of nonempty closed sets $\boldsymbol{K}$ in I endowed with the Hausdorff metric $\boldsymbol{H}$ given by $\boldsymbol{H}(E,F) = \inf\{\delta > 0 : E \subset B_\delta(F),\ F \subset B_\delta(E)\}$. where $B_\delta(F) = \{x \in I : d(x,y) < \delta,\ y \in F\}$. This space is compact ([BBT]). Our final metric space $(\boldsymbol{K}^*, \boldsymbol{H}^*)$ consists of the nonempty closed subsets of $\boldsymbol{K}$. Thus, $K \in \boldsymbol{K}^*$ if K is a nonempty family of nonempty closed sets in I such that K is closed in $\boldsymbol{K}$ with respect to $\boldsymbol{H}$. We endow $\boldsymbol{K}^*$ with the metric $\boldsymbol{H}^*$ so that K_1 and K_2 are close with respect to $\boldsymbol{H}^*$ if each member of K_1 is close to some member of K_2 with respect to $\boldsymbol{H}$, and vice versa. This metric space is also compact ([B]). Our interest in, and the utility of, the spaces $(\boldsymbol{K}, \boldsymbol{H})$ and $(\boldsymbol{K}^*, \boldsymbol{H}^*)$ stem from the following two theorems from [BCl] and [BBHS], respectively.

THEOREM 2.2. *For any f in $C(I,I)$, the set $\Lambda(f)$ is closed in I.*

THEOREM 2.3. *For any f in $C(I,I)$, the set $\Omega(f)$ is closed in $(\boldsymbol{K}, \boldsymbol{H})$.*

The questions posed by Bruckner require us to investigate the iterative stability of $f \in C(I,I)$ under small perturbations by studying the continuity structure of the maps $\Lambda\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}, \boldsymbol{H})$ given by $f \mapsto \Lambda(f)$, and $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ given by $f \mapsto \Omega(f)$.

Throughout much sequel we will consider various notions of chaos. The first notion comprises a closed subset $\boldsymbol{E}$ of $C(I,I)$ made up of those functions f having zero topological entropy, denoted by $\mathbf{h}(f) = 0$. The reader is referred to [FSS; Theorem A] for an extensive list of equivalent formulations of topological entropy zero. For our purposes, it suffices to note that every periodic orbit of a continuous function with zero topological entropy has cardinality of a power of two. The following theorem, due to Smítal [S], sheds considerable light on the structure of infinite ω-limit sets for functions with zero topological entropy.

THEOREM 2.4. *If ω is an infinite ω-limit set of $f \in C(I,I)$ possessing zero topological entropy, then there exists a sequence of closed intervals $\{J_k\}_{k=1}^{\infty}$ in $[0,1]$ such that*

1. *for each k, $\{f^i(J_k)\}_{i=1}^{2^k}$ are pairwise disjoint, and $J_k = f^{2^k}(J_k)$;*
2. *for each k, $J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$;*
3. *for each k, $\omega \subset \bigcup_{i=1}^{2^k} f^i(J_k)$,*
4. *for each k and i, $\omega \cap f^i(J_k) \neq \emptyset$.*

Given the very specific behavior that functions of zero topological entropy must demonstrate on their infinite ω-limit sets, it may not be too surprising that Bruckner and Smítal have been able to characterize these sets ([BS]).

THEOREM 2.5. *An infinite compact set $W \subset (0,1)$ is an ω-limit set of a map $f \in C(I,I)$ with zero topological entropy if and only if $W = Q \cup P$ where Q is a Cantor set and P is empty or countably infinite, disjoint with Q, and satisfies the following conditions:*

1. *every interval contiguous to Q contains at most two points of P;*
2. *each of the intervals $[0, \min Q)$, $(\max Q, 1]$ contains at most one point of P;*
3. $\overline{P} = Q \cup P$.

We now define chaos in the sense of Li and Yorke [LY].

Take $\delta \geq 0$, with f in $C(I,I)$. Suppose $S \subseteq I$ such that for any $x, y \in S$ with $x \neq y$ we have $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta$ and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$. We call S a *scrambled set of* f, and if f possesses an uncountable scrambled set, then f is said to be *chaotic in the sense of Li and Yorke.* While not immediately apparent, a function f is chaotic in the sense of Li and Yorke if and only if there is a point $x \in I$ which is not approximately periodic with respect to f. Moreover, if f is chaotic in the sense of Li and Yorke, there exists a simple system of f with nonempty interior.

Our third notion of chaos comes from Bruckner and Ceder [BC].

To each function $f \in C(I,I)$ associate the map $\omega_f \colon I \to \boldsymbol{K}$ given by $x \mapsto \omega(x,f)$. Bruckner and Ceder show that the Baire class of the map $\omega_f \colon I \to (\boldsymbol{K}, \boldsymbol{H})$ well reflects the chaotic nature of the function f. In fact, those functions f for which ω_f is in the first Baire class exhibit a form of nonchaos that allows scrambled sets but not positive topological entropy. That is, f not chaotic in the sense of Li and Yorke $\Longrightarrow$ $\omega_f\colon I \to (\boldsymbol{K}, \boldsymbol{H})$ is in the first Baire class $\Longrightarrow$ f possesses zero topological entropy, but none of the reverse implications is true. Bruckner and Ceder show that for a function f in $C(I,I)$, we have $\omega_f \notin B_1$ if and only if f has an ω-limit set of the form $W = Q \cup P$ as described in Theorem 2.5, with $P \neq \emptyset$.

In Section 4 we make use of the notion of semicontinuity for a set valued function. Suppose we have the set valued function $F\colon \big(C(I,I), \|\cdot\|\big) \to (X, \rho)$ with $f \in C(I,I)$. We say that F is *upper semicontinuous* at f if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $F(g) \subset B_\varepsilon\big(F(f)\big)$ whenever $\|f - g\| < \delta$. Similarly, F is *lower semicontinuous* at f if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $F(f) \subset B_\varepsilon\big(F(g)\big)$ whenever $\|f - g\| < \delta$.

We now turn our attention to the Baire category theorem. Let (X, ρ) be a metric space. A set is of the *first category* in (X, ρ) if it can be written as a countable union of nowhere dense sets; otherwise, the set is of the *second category.* A set is *residual* if it is the complement of a first category set; an element of a residual subset of (X, ρ) is called a *typical element* of X. With these definitions in mind, we recall Baire's theorem on category.

THEOREM 2.6. *Let (X, ρ) be a complete metric space, with S a first category subset of X. Then $X - S$ is dense in X.*

3. Typical continuous functions and the map $\Lambda\colon C(I,I) \to \boldsymbol{K}$

In this section we show that the map $\Lambda\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}, \boldsymbol{H})$ is continuous on a residual subset of $C(I,I)$. This will follow immediately from Propositions 3.1 and 3.2. We note that Proposition 3.1 is just a restatement of part of Theorem 1.1; we restate it here for the sake of convenience.

PROPOSITION 3.1. *Let $g \in C(I,I)$ for which $\overline{S(g)} = CR(g)$. Then the map taking f in $C(I,I)$ to $\Lambda(f)$ in $(\boldsymbol{K}, \boldsymbol{H})$ is continuous at g.*

We must show that $\overline{S(g)} = CR(g)$ for the typical f in $C(I,I)$ in order for our conclusion to follow.

PROPOSITION 3.2. *The set $S = \big\{f \in C(I,I) : \overline{S(f)} = CR(f)\big\}$ is residual in $\big(C(I,I), \|\cdot\|\big)$.*

Proof. Since $S(f) \subseteq CR(f)$ and $CR(f)$ is closed in I, it follows that $\overline{S(f)} \subseteq CR(f)$. To show that $\boldsymbol{H}\big(\overline{S(f)}, CR(f)\big) < \varepsilon$, it suffices to show that for any $x \in CR(f)$ there exists $y \in S(f)$ so that $|x - y| < \varepsilon$. Set

$S_n = \{f \in C(I,I) : \boldsymbol{H}(\overline{S(f)}, CR(f)) < \frac{1}{n}\}$. Since $S = \bigcap_{n=1}^{\infty} S_n$, we need to show that S_n is both dense and open in $C(I,I)$.

We first verify that S_n is a dense subset of $C(I,I)$. Let $f \in C(I,I) - S_n$ with $\varepsilon > 0$. Since $CR\colon C(I,I) \to \boldsymbol{K}$ is upper semicontinuous, there exists $\delta > 0$ so that $\|f - g\| < \delta$ implies $CR(g) \subset B_\varepsilon(CR(f))$. Take $\delta > 0$ so that $CR(g) \subset B_{\frac{1}{2n}}(CR(f))$ whenever $\|f - g\| < \delta$, and let $\{x_1, x_2, \dots, x_m\} \subseteq CR(f)$ be a $\frac{1}{2n}$-net of $CR(f)$. Now, choose $g \in C(I,I)$ so that $x_i \in S(g)$ for $1 \le i \le m$ and $\|f - g\| < \min\{\delta, \varepsilon\}$. Then $CR(f) \subset B_{\frac{1}{2n}}(S(g))$ since $\{x_1, x_2, \dots, x_m\} \subseteq S(g)$ and $CR(g) \subset B_{\frac{1}{2n}}(CR(f))$, so that $CR(g) \subset B_{\frac{1}{n}}(S(g))$. We conclude that $\boldsymbol{H}(\overline{S(g)}, CR(g)) < \frac{1}{n}$.

We now show that S_n is an open subset of $C(I,I)$. Let $f \in S_n$ with $n \ge 4$. Say $\boldsymbol{H}(\overline{S(f)}, CR(f)) = \alpha < \frac{1}{n}$, and set $\gamma = \frac{1}{n} - \alpha$. Let $\delta_1 > 0$ so that $\|f - g\| < \delta_1$ implies $CR(g) \subset B_{\frac{\gamma}{n}}(CR(f))$. Take $\{x_1, x_2, \dots, x_m\} \subseteq S(f)$ to be an $(\alpha + \frac{\gamma}{n})$-net of $CR(f)$. Now, there exists $\delta_2 > 0$ so that $\|f - g\| < \delta_2$ implies $S(g) \cap B_{\frac{\gamma}{n}}(x_i) \neq \emptyset$ for $i = 1, 2, \dots, m$. If $g \in C(I,I)$ for which $\|f - g\| < \min\{\delta_1, \delta_2\}$, then $\bigcup_{i=1}^{m} x_i \subset B_{\frac{\gamma}{n}}(S(g))$, $CR(f) \subset B_{\alpha + \frac{\gamma}{n}}\left(\bigcup_{i=1}^{m} x_i\right)$ and $CR(g) \subset B_{\frac{\gamma}{n}}(CR(f))$. It follows that $CR(g) \subset B_{\frac{1}{n}}(S(g))$, so that $\boldsymbol{H}(\overline{S(g)}, CR(g)) < \frac{1}{n}$, and $g \in S_n$. □

THEOREM 3.3. *The map $\Lambda\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}, \boldsymbol{H})$ given by $f \mapsto \Lambda(f)$ is continuous at a residual set of functions f in $C(I,I)$.*

4. Typical continuous functions and the map $\Omega\colon C(I,I) \to \boldsymbol{K}^*$

Our goal in this section is to show that the map $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is also continuous on a residual subset of $C(I,I)$. We begin with a study of the map $\widetilde{\Omega}\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ given by $f \mapsto \widetilde{\Omega}(f)$. We find that the set $\widetilde{\Omega}(f)$ plays a role similar to that of the chain recurrent set in section three. In particular, $\widetilde{\Omega}(f)$ is always a closed subset of $(\boldsymbol{K}, \boldsymbol{H})$ and the map $\widetilde{\Omega}\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is upper semicontinuous.

PROPOSITION 4.1. *If $f \in C(I,I)$, then $\widetilde{\Omega}(f)$ is closed in $(\boldsymbol{K}, \boldsymbol{H})$.*

P r o o f . Let $\{L_k\}_{k=1}^{\infty} \subset \boldsymbol{K}$ with $f \in C(I,I)$ so that $L_k = f(L_k)$ for any k, and $L_k \to L$ in $\boldsymbol{K}$. Since f is continuous, it follows that $f(L_k) \to f(L)$ in $\boldsymbol{K}$, too. We conclude that $L = f(L)$. Now, suppose that for each k, the following holds for L_k: If $F \neq \emptyset$ is closed such that $F \subsetneq L_k$, then $F \cap \overline{f(L_k - F)} \neq \emptyset$. We show that for any $F \neq \emptyset$ closed, $F \subsetneq L$, it follows that $F \cap \overline{f(L-F)} \neq \emptyset$. Suppose, to the contrary, that there exists such an F so that $F \cap \overline{f(L-F)} = \emptyset$; say $\boldsymbol{H}\big(F, \overline{f(L-F)}\big) = \sigma$. Let $\delta > 0$ so that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\sigma}{4}$ and choose n sufficiently large so that $\boldsymbol{H}(L_n, L) < \gamma$, $L_n \cap B_\gamma(F) \neq \emptyset$, and $L_n \cap B_\gamma(L-F) \neq \emptyset$, where $\gamma < \min\big(\delta, \frac{\sigma}{8}\big)$. Set $\widetilde{F} = \overline{L_n \cap B_\gamma(F)}$. Then $\widetilde{F} \cap \overline{f(L_n - \widetilde{F})} = \emptyset$ since $\widetilde{F} \subset B_{\frac{\sigma}{4}}(F)$ and $\overline{f(L_n - \widetilde{F})} \subset B_{\frac{\sigma}{4}}\big(\overline{f(L-F)}\big)$. □

Proposition 4.2. *The map* $\widetilde{\Omega}\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ *given by* $f \mapsto \widetilde{\Omega}(f)$ *is upper semicontinuous.*

Proof. Let $f_n \to f$ in $\big(C(I,I), \|\cdot\|\big)$ with $L_n \in \widetilde{\Omega}(f_n)$ for each n, and $L_n \to L$ in $(\boldsymbol{K}, \boldsymbol{H})$. We show that $L \in \widetilde{\Omega}(f)$.

We first show that $L = f(L)$. Since $f \in C(I,I)$, $f_n \to f$ uniformly and $L_n \to L$ in $(\boldsymbol{K}, \boldsymbol{H})$, we have $\boldsymbol{H}\big(L, f(L)\big) = \boldsymbol{H}(L, L_n) + \boldsymbol{H}\big(L_n, f_n(L_n)\big) + \boldsymbol{H}\big(f_n(L_n), f(L_n)\big) + \boldsymbol{H}\big(f(L_n), f(L)\big)$ where each of the terms on the right hand side goes to zero as $n \to \infty$. It follows that $L = f(L)$.

Now, let us suppose to the contrary that there exists an appropriate F for which our transport property does not hold for F, $L - F$ and f. In particular, $F \neq \emptyset$ is closed, $F \subsetneq L$ and $F \cap \overline{f(L-F)} = \emptyset$. Say $\boldsymbol{H}\big(F, \overline{f(L-F)}\big) = \sigma$. Since $f_n \to f$ uniformly, there is N_1 a natural number so that $n > N_1$ implies $|f(x) - f_n(x)| < \frac{\sigma}{8}$ for all $x \in I$. Since f is uniformly continuous on I, there is a $\delta > 0$ so that $|f(x) - f(y)| < \frac{\sigma}{8}$ whenever $|x - y| < \delta$. Since $L_n \to L$ in $(\boldsymbol{K}, \boldsymbol{H})$, there is N_2 a natural number so that $k > N_2$ implies $\boldsymbol{H}(L_k, L) < \gamma$, $L_k \cap B_\gamma(F) \neq \emptyset$ and $L_k \cap B_\gamma(L-F) \neq \emptyset$ where $\gamma < \min\{\delta, \frac{\sigma}{8}\}$. Now, set $\widetilde{F} = \overline{L_k \cap B_\gamma(F)}$, so that $\widetilde{F} \subset B_{\frac{\sigma}{4}}(F)$. Then $\overline{f_k(L_k - \widetilde{F})} \subset B_{\frac{\sigma}{8}}\Big(\overline{f(L_k - \widetilde{F})}\Big)$ and $\overline{f(L_k - \widetilde{F})} \subset B_{\frac{\sigma}{8}}\big(\overline{f(L-F)}\big)$ so that $\overline{f_k(L_k - \widetilde{F})} \subset B_{\frac{\sigma}{4}}\big(\overline{f(L-F)}\big)$ whenever $k > \max\{N_1, N_2\}$. This implies $\boldsymbol{H}\Big(\widetilde{F}, \overline{f_k(L_k - \widetilde{F})}\Big) > \frac{\sigma}{2}$, a contradiction. □

In our next result we tie the behavior of $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ to the upper semicontinuity of the map $\widetilde{\Omega}\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$.

Proposition 4.3. *If* $f \in C(I,I)$ *for which* $\overline{\boldsymbol{S}(f)} = \widetilde{\Omega}(f)$ *in* $(\boldsymbol{K}^*, \boldsymbol{H}^*)$, *then* $\Omega\colon \big(C(I,I), \|\cdot\|\big) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ *is continuous at* f.

Proof. Recall that for any $f \in C(I,I)$, $\overline{S(f)} \subset \Omega(f) \subset \widetilde{\Omega}(f)$. Let us fix f and $\varepsilon > 0$. Since $\widetilde{\Omega}\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is upper semicontinuous at f, there exists $\delta_1 > 0$ so that $\widetilde{\Omega}(g) \subset B_{\frac{\varepsilon}{4}}(\widetilde{\Omega}(f))$ whenever $\|f-g\| < \delta_1$. Since $\overline{S(f)}$ is dense in $\widetilde{\Omega}(f)$, there exists $\delta_2 > 0$ so that $\widetilde{\Omega}(f) \subset B_{\frac{\varepsilon}{4}}(\overline{S(g)})$ whenever $\|f-g\| < \delta_2$. If $\|f-g\| < \min\{\delta_1, \delta_2\}$, then $\Omega(g) \subset \widetilde{\Omega}(g) \subset B_{\frac{\varepsilon}{4}}(\widetilde{\Omega}(f)) \subset B_{\frac{\varepsilon}{2}}(\overline{S(g)}) \subset B_{\frac{\varepsilon}{2}}(\Omega(g))$, so that $\Omega(g) \subset B_{\frac{\varepsilon}{2}}(\Omega(f))$ and $\Omega(f) \subset \widetilde{\Omega}(f) \subset B_{\frac{\varepsilon}{4}}(\Omega(g))$. It follows that $\boldsymbol{H}(\Omega(g), \Omega(f)) < \frac{\varepsilon}{2}$, and $\Omega : (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is continuous at f. □

It remains for us to show that $\overline{S(f)} = \widetilde{\Omega}(f)$ for the typical f in $C(I,I)$.

PROPOSITION 4.4. *The set* $\boldsymbol{G} = \{f \in C(I,I) : \overline{S(f)} = \widetilde{\Omega}(f)\}$ *is residual in* $(C(I,I), \|\cdot\|)$.

Proof. Let $B_n = \{f \in C(I,I) : \boldsymbol{H}^*(\overline{S(f)}, \widetilde{\Omega}(f)) > \frac{1}{n}\}$. It suffices to show that B_n is nowhere dense for any n.

We first show that $C(I,I) - B_n$ is dense. Let $f \in B_n$. Since $\widetilde{\Omega}\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is upper semicontinuous at f, there exists $\delta > 0$ so that $\widetilde{\Omega}(g) \subset B_{\frac{1}{4n}}(\widetilde{\Omega}(f))$ whenever $\|f-g\| < \delta$. Since $\widetilde{\Omega}(f)$ is closed in $\boldsymbol{K}$, there exist $\{L_i\}_{i=1}^m \subset \widetilde{\Omega}(f)$ so that $\{L_i\}_{i=1}^m$ is a $\frac{1}{4n}$-net of $\widetilde{\Omega}(f)$. Choose $g \in C(I,I)$ so that $\|f-g\| < \delta$ and there is a stable periodic orbit $K_i \in \boldsymbol{S}(g)$ so that $\boldsymbol{H}(K_i, L_i) < \frac{1}{4n}$ for $i = 1, 2, 3, \dots, m$. It follows that $\widetilde{\Omega}(g) \subset B_{\frac{1}{4n}}(\widetilde{\Omega}(f)) \subset B_{\frac{1}{2n}}(\{L_i\}_{i=1}^m) \subset B_{\frac{1}{2n}}(\overline{S(g)})$, so that $\boldsymbol{H}^*(\overline{S(g)}, \widetilde{\Omega}(g)) < \frac{1}{2n}$.

We now show that $C(I,I) - B_n$ is open. Let $f \in C(I,I)$ such that $\boldsymbol{H}^*(\overline{S(f)}, \widetilde{\Omega}(f)) = \sigma < \frac{1}{n}$; say $\frac{1}{n} - \sigma = \varepsilon$. Choose $\delta_1 > 0$ so that $\widetilde{\Omega}(g) \subset B_{\frac{\varepsilon}{4}}(\widetilde{\Omega}(f))$ whenever $\|f-g\| < \delta_1$, and take $\{L_i\}_{i=1}^m \subset \boldsymbol{S}(f)$ with the property that $\boldsymbol{H}^*(\{L_i\}_{i=1}^m, \widetilde{\Omega}(f)) < \sigma + \frac{\varepsilon}{4}$. Since $\{L_i\}_{i=1}^m \subset \boldsymbol{S}(f)$, there exists $\delta_2 > 0$ so that $\|f-g\| < \delta_2$ implies the existence, for any $i = 1, 2, \dots, m$, of $K_i \in \boldsymbol{S}(g)$ so that $\boldsymbol{H}(K_i, L_i) < \frac{\varepsilon}{4}$. Let $g \in C(I,I)$ with $\|f-g\| < \min\{\delta_1, \delta_2\}$. Then $\widetilde{\Omega}(g) \subset B_{\frac{\varepsilon}{4}}(\widetilde{\Omega}(f)) \subset B_{\sigma+\frac{\varepsilon}{2}}(\{L_i\}_{i=1}^m) \subset B_{\sigma+\frac{3\varepsilon}{4}}(\{K_i\}_{i=1}^m) \subset B_{\sigma+\frac{3\varepsilon}{4}}(\overline{S(g)})$, so that $\boldsymbol{H}^*(\overline{S(g)}, \widetilde{\Omega}(g)) < \sigma + \frac{3\varepsilon}{4} < \frac{1}{n}$. □

From Propositions 4.3 and 4.4 it now follows immediately that $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is continuous on a residual subset of $C(I,I)$.

THEOREM 4.5. *The map* $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ *given by* $f \mapsto \Omega(f)$ *is continuous at a residual set of functions* f *in* $C(I,I)$.

5. The relationship between stability and chaos

Our goal in this section is to determine the relationship between the chaotic nature of a function f in $C(I,I)$ and the behavior of our maps $\Lambda\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K},\boldsymbol{H})$ and $\Omega\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K}^*,\boldsymbol{H}^*)$ at that function. We begin by considering functions f that are not chaotic in the sense of Li-Yorke, and then consider the evolving behavior of $\Lambda\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K},\boldsymbol{H})$ and $\Omega\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K}^*,\boldsymbol{H}^*)$ as we make our function f progressively more chaotic.

Lemma 5.1. *Suppose $f \in C(I,I)$ is not chaotic in the sense of Li-Yorke. Then one of the following possibilities must hold:*

1. *the maps $\Lambda\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K},\boldsymbol{H})$ and $\Omega\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K}^*,\boldsymbol{H}^*)$ are both continuous at f;*
2. *the maps $\Lambda\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K},\boldsymbol{H})$ and $\Omega\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K}^*,\boldsymbol{H}^*)$ are both discontinuous at f.*

Proof. If f is not chaotic in the sense of Li-Yorke, then f has zero topological entropy, so that Λ and Ω are either both continuous or discontinuous together at f. This follows from Theorems 1.1 and 1.2. $\square$

As our next pair of examples shows, each of the situations described in Lemma 5.1 is possible. Suppose $f(x) = 0$ for all $x \in I$. Then f is not Li-Yorke chaotic and both Λ and Ω are continuous there. This follows from the observation that $S(f) = CR(f) = \{0\}$. Now, let $f(x) = x$ for all $x \in I$. Then f is not Li-Yorke chaotic and both Λ and Ω are discontinuous there. We note that $S(f) = \emptyset$ whereas $CR(f) = [0,1]$.

We now wish to consider the behavior of $\Lambda\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K},\boldsymbol{H})$ and $\Omega\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K}^*,\boldsymbol{H}^*)$ at functions f that are chaotic in the sense of Li-Yorke but still have zero topological entropy.

Proposition 5.2. *Let $\boldsymbol{E} = \big\{f \in C(I,I) : f \text{ has zero topological entropy}\big\}$. If f is an element of $\boldsymbol{E}$ chaotic in the sense of Li-Yorke, then the maps $\Lambda\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K},\boldsymbol{H})$ and $\Omega\colon \big(C(I,I),\|\cdot\|\big) \to (\boldsymbol{K}^*,\boldsymbol{H}^*)$ are both discontinuous at f.*

Proof. Let $f \in \boldsymbol{E}$ be chaotic in the sense of Li-Yorke. Since $f \in \boldsymbol{E}$, Λ and Ω will either be continuous or discontinuous together at f ([TH2]). From [BC] we know that f must possess a simple system L with nonempty interior. Since $\operatorname{int}(L) \cap S(f) = \emptyset$ and $\operatorname{int}(L) \subset CR(f)$, we see that $\overline{S(f)} \subsetneq CR(f)$, and our conclusion follows from Theorem 1.1. $\square$

We now apply Proposition 5.2 to functions f for which the map $\omega_f\colon I \to \boldsymbol{K}$ is not in the first class of Baire but do still possess zero topological entropy.

COROLLARY 5.3. *Suppose f is an element of $\boldsymbol{E}$ and the map $\omega_f \colon I \to \boldsymbol{K}$ is not in the first class of Baire. Then $\Lambda\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}, \boldsymbol{H})$ and $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ are both discontinuous at f.*

With Proposition 5.4 we consider the behavior of Λ and Ω at a function f possessing positive topological entropy.

PROPOSITION 5.4. *Let $\boldsymbol{T} = \{f \in C(I,I) : f \text{ has positive topological entropy}\}$, with $f \in \boldsymbol{T}$. Then one of the following possibilities must hold:*

1. *Λ and Ω are both continuous at f;*
2. *Λ is continuous at f, but Ω is discontinuous there;*
3. *Λ and Ω are both discontinuous at f.*

Proof. This proposition follows readily from Theorems 1.1 and 1.5. □

We provide examples illustrating each of the three possibilities found in Proposition 5.4. We begin by considering our first possibility. From Theorem 4.5 we know that Ω is continuous on a residual subset of $C(I,I)$. Since $\boldsymbol{T}$ is also residual in $C(I,I)$, it follows that the set $\{f \in \boldsymbol{T} : \Omega \text{ is continuous at } f\}$ is residual in $C(I,I)$, too. Thus, our first possibility holds on a residual subset of $C(I,I)$.

As for our second possibility, consider the hat map $h(x)$ given by $x \mapsto 2x$ for $x \in [0, \frac{1}{2}]$ and $x \mapsto 2(1-x)$ for $x \in (\frac{1}{2}, 1]$. Then $\overline{S(h)} = CR(h) = [0,1]$, so that $\Lambda\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}, \boldsymbol{H})$ is continuous at h. Since $\{0\} \in P(h) - S(h)$, by Theorem 1.5 we see that $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ is discontinuous at f.

We turn our attention to the third possibility. Consider a function $f \in \boldsymbol{T}$ that has a basic set L with nonempty interior. Since $S(f) \cap \operatorname{int}(L) = \emptyset$ and $L \subset CR(f)$, we see that $\overline{S(f)} \subsetneq CR(f)$, so that $\Lambda\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}, \boldsymbol{H})$, and hence $\Omega\colon (C(I,I), \|\cdot\|) \to (\boldsymbol{K}^*, \boldsymbol{H}^*)$ must be discontinuous there.

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