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ON CONVERGENCE FOR THE GR_k^*-INTEGRAL

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ABSTRACT. The GR_k^*-integral was introduced by the authors. In this paper, we study some convergence results for the GR_k^*-integral.

1. Introduction

The authors introduced the GR_k-integral in [3]. It is a Stieltjes type integral which for $k = 1$ includes classical Henstock Stieltjes integral in particular case. Later, in [4], the authors extended the GR_k-integral and called the new integral as the GR_k^*-integral. Some new concepts of "local tagging" and "regulated δ^k-fine division" etc. were introduced to define GR_k^*-integral. In [4] some elementary results for the GR_k^*-integral and also an analogue of the Saks-Henstock lemma are studied. Furthermore, one version of the Fundamental theorem of calculus is given.

In this paper, we obtain some convergence results for the GR_k^*-integral. First we obtain the uniform convergence theorem. Then we prove monotone convergence theorem and the basic convergence theorem for the GR_k^*-integral. As an application of basic convergence theorem, we obtain the mean convergence theorem for the GR_k^*-integral.

2. Preliminaries

Let k be a fixed positive integer and δ be a positive function defined on $[a,b]$. We shall call a division D of $[a,b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$

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with associated points $\{\xi_0, \xi_1, \dots, \xi_{n-k}\}$ satisfying

$$\xi_i \in [x_i, x_{i+k}] \subset \big(\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)\big) \qquad \text{for} \quad i = 0, 1, \dots, n-k$$

a *δ^k-fine division of* $[a,b]$. For a given positive function δ, we denote a δ^k-fine division D by $\big\{[x_i, x_{i+k}], \xi_i\big\}_{i=0,1,\dots,n-k}$. Using compactness of $[a,b]$ it is easy to verify that such δ^k-fine division exists. When $k = 1$, it coincides with the usual definition of δ-fine division.

In [3], the GR_k-integral is defined as follows:

DEFINITION 2.1. Let g be a real-valued function defined on a closed interval $[a,b]^{k+1}$ in the $(k+1)$-dimensional space, and f a real-valued function defined on $[a,b]$.

We say that f is *GR_k-integrable with respect to g to I on* $[a,b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ for $\xi \in [a,b]$ such that for any δ^k-fine division $D = \big\{[x_i, x_{i+k}], \xi_i\big\}_{i=0,1,\dots,n-k}$ we have

$$\left| \sum_{i=0}^{n-k} f(\xi_i) g(x_i, \dots, x_{i+k}) - I \right| < \varepsilon\,.$$

We shall denote the above Riemann sum by $s(f,g;D)$. If f is integrable with respect to g in the above sense, we write $(f,g) \in GR_k[a,b]$ and denote the integral by $\int_a^b f \,\mathrm{d}g$.

Let $x \in [x_i, x_{i+k}]$ where $x_i < x_{i+1} < \cdots < x_{i+k}$. The *jump of g at x*, denoted by $J(g;x)$, is defined by

$$J(g;x) = \lim_{\substack{x_i \to x \\ x_{i+k} \to x}} g(x_i, \dots, x_{i+k})\,,$$

if the limit exists finitely.

In [3], it was proved that:

THEOREM 2.2. *Let $(f,g) \in GR_k[a,c]$ and $(f,g) \in GR_k[c,b]$. If $J(g;c)$ exists, then $(f,g) \in GR_k[a,b]$ and*

$$\int_a^b f \,\mathrm{d}g = \int_a^c f \,\mathrm{d}g + \int_c^b f \,\mathrm{d}g + (k-1) f(c) J(g;c)\,.$$

We introduced in [3], δ^k-fine partial division of a special kind as follows:

Let $[a_i, b_i]$, $i = 1, 2, \ldots, p$, be pairwise non-overlapping, and $\bigcup_{i=1}^{p} [a_i, b_i] \subset [a, b]$. Then $\{D_i\}_{i=1,2,\ldots,p}$ is said to be a *δ^k-fine partial division of* $[a, b]$ if each D_i is a δ^k-fine division of $[a_i, b_i]$. Its corresponding partial Riemann sum is given by $\sum_{i=1}^{p} s(f, g; D_i)$.

With this notion of partial division we have proved in [3] the following theorem.

THEOREM 2.3. (Saks-Henstock lemma analogue for GR_k-integral) *If $(f, g) \in GR_k[a, b]$ and $J(g; c)$ exists for all $c \in (a, b)$, then for every $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that for any δ^k-fine division D of $[a, b]$ and for any δ^k-fine partial division $\{D_i\}_{i=1,2,\ldots,p}$ of $[a, b]$*

$$|s(f, g; D) - F(a, b)| < \varepsilon \qquad \text{and} \qquad \left| \sum_{i=1}^{p} \{s(f, g; D_i) - F(a_i, b_i)\} \right| < (k + 1)\varepsilon$$

where D_i is a δ^k-fine division of $[a_i, b_i]$ and $F(u, v)$ denotes the GR_k-integral on $[u, v] \subseteq [a, b]$.

In [4] the following concepts are introduced:

DEFINITION 2.4. Given a function $\delta\colon [a, b] \to \mathbb{R}_+$ and a point $x \in [a, b]$, a δ^k-fine division D of $[u, v] \subseteq [a, b]$ is said to be *locally tagged* at x if $[u, v] \subset \big(x - \delta(x), x + \delta(x)\big)$ with either $u = x$ or $v = x$.

It may be noted here that for local tagging at x we need δ to be defined in a neighbourhood of x. But for simple presentation we considered δ to be defined on $[a, b]$.

DEFINITION 2.5. A family of triplets $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ is called *regulated δ^k-fine division of* $[a, b]$ if each D_i is a δ^k-fine division of $[a_i, b_i]$ locally tagged at x_i where $[a_i, b_i]$, $i = 1, 2, \ldots, p$, are non-overlapping with $\bigcup_{i=1}^{p} [a_i, b_i] = [a, b]$.

Further, $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ is called *regulated δ^k-fine partial division of* $[a, b]$ if $\bigcup_{i=1}^{p} [a_i, b_i] \subseteq [a, b]$.

DEFINITION 2.6. Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b]^{k+1} \to \mathbb{R}$ such that $J(g;c)$ exists for all $c \in (a,b)$.

We say that f is *GR_k^*-integrable with respect to g to A on* $[a,b]$ if for all $\varepsilon > 0$ there exists $\delta\colon [a,b] \to \mathbb{R}_+$ such that for any regulated δ^k-fine division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ of $[a,b]$ we have

$$\left| \sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g;b_i) - A \right| < \varepsilon$$

We can easily verify that GR_k^*-integral is well defined.

If f is GR_k^*-integrable with respect to g, we write $(f,g) \in GR_k^*[a,b]$ and denote the integral also by $\int_a^b f \, \mathrm{d}g$.

In what follows we always assume that $J(g;x)$ exists for all $x \in (a,b)$.

The following theorem follows directly from the definition of GR_k^*-integral.

THEOREM 2.7. *Let $(f_i, g) \in GR_k^*[a,b]$ and $(f, g_i) \in GR_k^*[a,b]$ for $i = 1, 2, \dots, n$. Then for real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ we have:*

(i) $\left(\sum_{i=1}^{n} \lambda_i f_i, g \right) \in GR_k^*[a,b]$ *and* $\int_a^b \sum_{i=1}^{n} (\lambda_i f_i) \, \mathrm{d}g = \sum_{i=1}^{n} \lambda_i \left(\int_a^b f_i \, \mathrm{d}g \right)$.

(ii) $\left(f, \sum_{i=1}^{n} \lambda_i g_i \right) \in GR_k^*[a,b]$ *and* $\int_a^b f \, \mathrm{d}\left(\sum_{i=1}^{n} \lambda_i g_i \right) = \sum_{i=1}^{n} \lambda_i \int_a^b f \, \mathrm{d}g_i$.

(iii) *If $f_1(x) \le f_2(x)$ for all $x \in [a,b]$ and $g\colon [a,b]^{k+1} \to [0,\infty)$, then* $\int_a^b f_1 \, \mathrm{d}g \le \int_a^b f_2 \, \mathrm{d}g$.

The following results were proved in [4].

THEOREM 2.8. *Let $a < c < b$. If $(f,g) \in GR_k^*[a,c]$ and $(f,g) \in GR_k^*[c,b]$, then $(f,g) \in GR_k^*[a.b]$ and*

$$\int_a^b f \, \mathrm{d}g = \int_a^c f \, \mathrm{d}g + \int_c^b f \, \mathrm{d}g + (k-1) f(c) J(g;c).$$

THEOREM 2.9 (CAUCHY CONDITION). *$(f,g) \in GR_k^*[a,b]$ if and only if for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that for all regulated δ^k-fine divisions $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ and $\{y_j, P_j, [c_j,d_j]\}_{j=1}^q$ of $[a,b]$ we have*

$$\left| \left\{ \sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g;b_i) \right\} - \left\{ \sum_{j=1}^{q} s(f,g;P_j) + \sum_{j=1}^{q-1} (k-1) f(d_j) J(g;d_j) \right\} \right| < \varepsilon .$$

THEOREM 2.10. *If $(f,g) \in GR_k^*[a,b]$ and $a \le c < d \le b$, then $(f,g) \in GR_k^*[c,d]$.*

THEOREM 2.11. (Saks-Henstock lemma analogue for the GR_k^*-integral) *$(f,g) \in GR_k^*[a,b]$ if and only if there exists a function F, g-nearly additive with respect to f, satisfying the condition that for all $\varepsilon > 0$ there exists $\delta\colon [a,b] \to \mathbb{R}_+$ such that for all regulated δ^k-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$ we have*

$$\left| \sum_{i=1}^{p} \{ s(f,g;D_i) - F(a_i,b_i) \} \right| < \varepsilon .$$

In [4], we used the concept of local bounded variation of kth order of g as follows.

DEFINITION 2.12. For $X \subset [a,b]$, we define

$$LV_g^k(X) = \inf_{\delta} \sup \left\{ \sum_{i=1}^{p} |s(1,g;D_i)| \right\},$$

where supremum is taken over all regulated δ^k-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$ such that $x_i \in X$.

$X \subset [a,b]$ is said to be of *Lg^k-variation zero* if $LV_g^k(X) = 0$.

A function g is said to be $LBV^k(X)$ if $LV_g^k(X)$ is finite.

Also g is said to be $LBV^kG(X)$ if $X = \bigcup_{j=1}^{\infty} X_j$ such that g is $LBV^k(X_j)$ for each j.

A property is said *to hold Lg^k a.e.* if it holds everywhere in $[a,b]$ except on a set of Lg^k-variation zero. It is easy to verify that:

THEOREM 2.13. *If either f_1 or f_2 is GR_k^*-integrable with respect to g on $[a,b]$ and $f_1 = f_2$ Lg^k a.e. in $[a,b]$, then the other is also integrable and*

$$\int_a^b f_1 \, \mathrm{d}g = \int_a^b f_2 \, \mathrm{d}g .$$

3. Some convergence results

We now give some convergence results for the GR_k^*-integral.

We first prove the uniform convergence theorem.

Theorem 3.1 (Uniform convergence theorem). *Let $g \in LBV^k[a,b]$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $[a,b]$ such that $(f_n, g) \in GR_k^*[a,b]$ for all $n = 1, 2, \dots$. If f_n is uniformly convergent to f as $n \to \infty$, then $\int_a^b f \, \mathrm{d}g$ exists and $\lim_{n\to\infty} \int_a^b f_n \, \mathrm{d}g = \int_a^b f \, \mathrm{d}g$.*

Proof. Since $g \in LBV^k[a,b]$, there exists $M > 0$ and $\delta_0 \colon [a,b] \to \mathbb{R}_+$ such that $\sum_{i=1}^{p} |s(f, g; D_i)| < M$ for all regulated δ_0^k-fine partial division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ of $[a,b]$.

Let $A_n = \int_a^b f_n \, \mathrm{d}g$.

For $\varepsilon > 0$, by the Saks-Henstock Lemma (Theorem 2.11), there exists $\delta_n(x) \colon [a,b] \to \mathbb{R}_+$, $n = 1, 2, \dots$, where $\delta_n \le \delta_0$ such that for every regulated δ_n^k-fine partial division $\{x_i^n, D_i^n, [a_i^n, b_i^n]\}_{i=1}^{p_n}$ of $[a,b]$ we have

$$\left| \sum_{i=1}^{p_n} s(f_n, g; D_i^n) - A_n \right| < \varepsilon .$$

We may assume that $\delta_{n+1} \le \delta_n$, $n = 1, 2, \dots$.

For $m, n \in \mathbb{N}$ and $n > m$ we fix a regulated δ_n^k-fine partial division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$.

Then $|A_n - A_m| \le 2\varepsilon + \sum_{i=1}^{p} |s(f_n, g; D_i) - s(f_m, g; D_i)| \le 2\varepsilon + \|f_n - f_m\| M$ where $\|f_n - f_m\| = \sup_{a \le x \le b} |f_n(x) - f_m(x)|$.

As f_n is uniformly convergent to f, we have $\|f_n - f_m\| \to 0$ as $n \to \infty$. So, there exists positive integer N_1 such that for $n, m > N_1$, $\|f_n - f_m\| < \frac{\varepsilon}{M}$.

Thus, $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and let $A = \lim_{n\to\infty} A_n$.

Now we can find a positive integer $N_2 > N_1$ such that for $n \ge N_2$ we have $|A_n - A| < \varepsilon$.

Let $\delta(x) = \delta_{N_2}(x)$ for $x \in [a,b]$.

Then for any regulated δ^k-fine partial division $\{y_j, P_j, [c_j, d_j]\}_{j=1}^q$ of $[a,b]$ we have

$$\left|\sum_{j=1}^q s(f,g;P_j) - A\right| \le \left|\sum_{j=1}^q s(f,g;P_j) - \sum_{j=1}^q s(f_{N_2},g,P_j)\right| + \left|\sum_{j=1}^q s(f_{N_2},g;P_j) - A_{N_2}\right| + |A_{N_2} - A| < 3\varepsilon\,.$$

So, by Theorem 2.11, $(f,g) \in GR_k^*[a,b]$ and $\int_a^b f \;\mathrm{d}g = \lim_{n\to\infty} \int_a^b f_n \;\mathrm{d}g$. □

THEOREM 3.2 (MONOTONE CONVERGENCE THEOREM). *If the following conditions are satisfied*

(i) *the sequence $\{f_n(x)\}_{n=1}^\infty$ is monotonic Lg^k a.e. in $[a,b]$,*

(ii) *g is a nonnegative function defined on $[a,b]^{k+1}$ such that $(f_n,g) \in GR_k^*[a,b]$ for all n and the sequence $\left\{\int_a^b f_n \;\mathrm{d}g\right\}_{n=1}^\infty$ is bounded, i.e. $\left|\int_a^b f_n \;\mathrm{d}g\right| < M$ for some M and all $n \in \mathbb{N}$,*

(iii) *$\lim_{n\to\infty} f_n = f$ is finite Lg^k a.e.,*

then $(f,g) \in GR_k^*[a,b]$ and $\int_a^b f \;\mathrm{d}g = \lim_{n\to\infty} \int_a^b f_n \;\mathrm{d}g$.

Proof. Since the change of a function on a set of Lg^k variation zero influences neither the existence nor the value of the integral, we can assume that the functions f_n and f are defined and finite everywhere in $[a,b]$. By considering $-f_n$ or $f_n - f_1$ instead of f_n, if necessary, we can achieve that the sequence $\{f_n\}_{n=1}^\infty$ is increasing and $f_n \ge 0$. Since $g \ge 0$, $\left\{\int_a^b f_n \;\mathrm{d}g\right\}_{n=1}^\infty$ is also monotonic and bounded. So, $\lim_{n\to\infty} \int_a^b f_n \;\mathrm{d}g$ exists. Let us denote it by L. Given $\varepsilon > 0$, we can find N such that $\int_a^b f_N \;\mathrm{d}g > L - \frac{\varepsilon}{3}$.

Next we find $n(x) \ge N$ such that, for $n \ge n(x)$,

$$\frac{3L + 3\varepsilon}{3L + \varepsilon} f_n(x) \ge f(x)\,.$$

If $f(x) > 0$, this is possible because the left-hand side has a limit strictly larger than the right-hand side; if $f(x) = 0$, we can take $n(x) = N$. By Theorem 2.11

(Saks-Henstock lemma), there is $\delta_n \colon [a,b] \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{p} \Big| s(f_n, g; D_i) - \int_{a_i}^{b_i} f_n \, \mathrm{d}g \Big| < \frac{\varepsilon}{3 \cdot 2^n}$$

for all regulated δ_n^k-fine partial division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ of $[a,b]$.

We define $\delta(x) = \delta_{n(x)}(x)$.

Let $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ be a regulated δ^k-fine division of $[a,b]$.

The proof will be complete if we show that

$$\Big| \sum_{i=1}^{p} s(f, g; D_i) + (k-1) \sum_{i=1}^{p-1} f(b_i) J(g; b_i) - L \Big| < \varepsilon .$$

Now,

$$\begin{aligned} &\sum_{i=1}^{p} \int_{a_i}^{b_i} f_{n(x_i)} \, \mathrm{d}g + (k-1) \sum_{i=1}^{p-1} f_{n(x_i)}(b_i) J(g; b_i) \\ &\qquad \ge \sum_{i=1}^{p} \int_{a_i}^{b_i} f_N \, \mathrm{d}g + (k-1) \sum_{i=1}^{p-1} f_N(b_i) J(g; b_i) \\ &\qquad = \int_a^b f_N \, \mathrm{d}g > L - \frac{\varepsilon}{3} . \end{aligned}$$

Denoting by $\widehat{N}$ the largest $n(x_i)$ we also have

$$\begin{aligned} &\sum_{i=1}^{p} \int_{a_i}^{b_i} f_{n(x_i)} \, \mathrm{d}g + (k-1) \sum_{i=1}^{p-1} f_{n(x_i)}(b_i) J(g; b_i) \\ &\qquad \le \sum_{i=1}^{p} \int_{a_i}^{b_i} f_{\widehat{N}} \, \mathrm{d}g + (k-1) \sum_{i=1}^{p-1} f_{\widehat{N}}(b_i) J(g; b_i) \\ &\qquad = \int_a^b f_{\widehat{N}} \, \mathrm{d}g \le L . \end{aligned}$$

The $n(x_i)$ are not necessarily distinct; let $i_1, i_2, \dots, i_l$ be the distinct i such that $n(x_i) = l$ and we have by Theorem 2.11

$$\Big| \sum_{j=1}^{l} \Big\{ s(f_l, g; D_{i_j}) - \int_{a_{i_j}}^{b_{i_j}} f_l \, \mathrm{d}g \Big\} \Big| < \frac{\varepsilon}{3 \cdot 2^l} .$$

Consequently

$$\left|\sum_{i=1}^{p}\left[s\big(f_{n(x_i)},g;D_i\big)-\int_{a_i}^{b_i} f_{n(x_i)}\ \mathrm{d}g\right]\right|<\sum_{l=1}^{\infty}\frac{\varepsilon}{3\cdot 2^l}=\frac{\varepsilon}{3}\,.$$

Now,

$$\begin{aligned}
&\sum_{i=1}^{p}s(f,g;D_i)+(k-1)\sum_{i=1}^{p-1}f(b_i)J(g;b_i)\\
&\quad\ge\sum_{i=1}^{p}s\big(f_{n(x_i)},g;D_i\big)+(k-1)\sum_{i=1}^{p-1}f_{n(x_i)}(b_i)J(g;b_i)\\
&\quad>\sum_{i=1}^{p}\int_{a_i}^{b_i} f_{n(x_i)}\ \mathrm{d}g+(k-1)\sum_{i=1}^{p-1}f_{n(x_i)}(b_i)J(g;b_i)-\frac{\varepsilon}{3}>L-\frac{2\varepsilon}{3}\,.
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&\frac{(3L+\varepsilon)}{3(L+\varepsilon)}\left[\sum_{i=1}^{p}s(f,g;D_i)+(k-1)\sum_{i=1}^{p-1}f(b_i)J(g;b_i)\right]\\
&\quad\le\sum_{i=1}^{p}s\big(f_{n(x_i)},g;D_i\big)+(k-1)\sum_{i=1}^{p-1}f_{n(x_i)}(b_i)J(g;b_i)\\
&\quad<\sum_{i=1}^{p}\int_{a_i}^{b_i} f_{n(x_i)}\ \mathrm{d}g+\frac{\varepsilon}{3}+(k-1)\sum_{i=1}^{p-1}f_{n(x_i)}(b_i)J(g;b_i)\le L+\frac{\varepsilon}{3}\,.
\end{aligned}$$

So,

$$\left|\sum_{i=1}^{p}s(f,g;D_i)+(k-1)\sum_{i=1}^{p-1}f(b_i)J(g;b_i)-L\right|<\varepsilon\,.$$

This completes the proof. □

THEOREM 3.3 (BASIC CONVERGENCE THEOREM). *Let the following conditions hold*

(i) $(f_n,g)\in GR_k^*[a,b]$ *where* $g\in LBV^k[a,b]$ *and* $J(g;c)$ *exists for all* $c\in(a,b)$.

(ii) $f_n(x)\to f(x)$ *as* $n\to\infty$, Lg^k *a.e. in* $[a,b]$.

Then $(f,g) \in GR_k^*[a,b]$ if and only if for all $\varepsilon > 0$ there is a function $M(x)$ defined on $[a,b]$ taking integer values such that for infinitely many $m(x) \geq M(x)$, there is $\delta\colon [a,b] \to \mathbb{R}_+$ such that for any regulated δ^k-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$ we have

$$\left| \sum_{i=1}^{p} \{F_{m(x)}(a_i,b_i) - F(a_i,b_i)\} \right| < \varepsilon ,$$

where $F_n(u,v)$, $F(u,v)$ denote the integral of f_n, f over $[u,v] \subseteq [a,b]$ with respect to g respectively.

P r o o f. We can assume that $f_n(x) \to f(x)$ as $n \to \infty$ everywhere in $[a,b]$.

Let $(f,g) \in GR_k^*[a,b]$. Since $f_n(x) \to f(x)$ as $n \to \infty$, we have for $x \in [a,b]$ there exists integer $M(x)$ such that whenever $m(x) \geq M(x)$,

$$|f_{m(x)}(x) - f(x)| < \frac{\varepsilon}{LV_g^k[a,b]} .$$

Since each $(f_n, g) \in GR_k^*[a,b]$, by Theorem 2.11, there exists $\delta_n(\xi) > 0$ for $\xi \in [a,b]$ such that for any regulated δ_n^k-fine partial division $\{x_i^n, D_i^n, [a_i^n, b_i^n]\}_{i=1}^{p_n}$, we have

$$\left| \sum_{i=1}^{p_n} \{s(f_n, g; D_i^n) - F_n(a_i^n, b_i^n)\} \right| < \frac{\varepsilon}{2^n} .$$

Since $(f,g) \in GR_k^*[a,b]$, there exists $\delta_0(\xi) > 0$ for $\xi \in [a,b]$ such that for all regulated δ_0^k-fine partial division $\{y_j, P_j, [c_j, d_j]\}_{j=1}^q$ of $[a,b]$, we have

$$\left| \sum_{j=1}^{q} \{s(f,g;P_j) - F(c_j,d_j)\} \right| < \varepsilon .$$

Also, since $g \in LBV_k[a,b]$, there exists $\eta(x) > 0$ such that for all regulated η^k-fine partial division $\{z_l, Q_l, [u_l, v_l]\}_{l=1}^r$ we have

$$\sum_{l=1}^{r} |s(1,g;Q_l)| \leq LV_g^k[a,b] .$$

For $x \in [a,b]$, we choose any integer $m(x) \geq M(x)$ and we take $\delta(x) = \min\{\delta_{m(x)}(x), \delta_0(x), \eta(x)\}$. Let $\{x_i, D_i, [a_i,b_i]\}_{i=1}^{p}$ be any regulated δ^k-fine partial division of $[a,b]$. Then

$$\begin{aligned}
&\left|\sum_{i=1}^{p}\{F_{m(x)}(a_i,b_i) - F(a_i,b_i)\}\right| \\
&\leq \left|\sum_{i=1}^{p}\{F_{m(x)}(a_i,b_i) - s(f_{m(x)},g;D_i)\}\right| + \left|\sum_{i=1}^{p}\{s(f_{m(x)},g;D_i) - s(f,g;D_i)\}\right| \\
&\qquad + \left|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}\right| \\
&< \varepsilon + \frac{\varepsilon}{LV_g^k[a,b]}\sum_{i=1}^{p}|s(1,g;D_i)| + \varepsilon \leq 3\varepsilon\,.
\end{aligned}$$

Hence the condition is proved to be true for every $m(x) \geq M(x)$.

Conversely, let the condition hold. So there exists an integer valued function $M(x)$ defined on $[a,b]$ such that for infinitely many $m(x) \geq M(x)$, there is a $\delta_0\colon [a,b] \to \mathbb{R}_+$ such that for any regulated δ_0^k-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^{p}$

$$\left|\sum_{i=1}^{p}\{F_{m(x)}(a_i,b_i) - F(a_i,b_i)\}\right| < \varepsilon\,.$$

Also since $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in [a,b]$, we can find $m(x) \geq M(x)$ such that

$$|f_{m(x)}(x) - f(x)| < \frac{\varepsilon}{LV_g^k[a,b]}\,.$$

Using the same notations as in the first part, we choose

$$\delta(x) = \min\{\delta_{m(x)}(x), \delta_0(x), \eta(x)\}\,, \qquad x \in [a,b]\,.$$

Let $\{x_i, D_i, [a_i,b_i]\}_{i=1}^{p}$ be a regulated δ^k-fine partial division of $[a,b]$. Then

$$\begin{aligned}
&\left|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}\right| \\
&\leq \left|\sum_{i=1}^{p}\left\{s(f,g;D_i) - s\big(f_{m(x)},g;D_i\big)\right\}\right| + \left|\sum_{i=1}^{p}\left\{s\big(f_{m(x)},g;D_i\big) - F_{m(x)}(a_i,b_i)\right\}\right| \\
&\qquad + \left|\sum_{i=1}^{p}\{F_{m(x)}(a_i,b_i) - F(a_i,b_i)\}\right| < 3\varepsilon\,.
\end{aligned}$$

□

THEOREM 3.4 (MEAN CONVERGENCE THEOREM). *If the following conditions are satisfied*

(i) $f_n(x) \to f(x)$ Lg^k *a.e. in* $[a,b]$ *as* $n \to \infty$ *where each* $(f_n, g) \in GR^*_k[a,b]$,

(ii) $g \in LBV^k[a,b]$ *and* $J(g,c)$ *exists for all* $c \in (a,b)$,

(iii) $[a,b] = \bigcup_{i=1}^{\infty} X_i$ *such that for every* i *and for every* $\varepsilon > 0$ *there exists an integer* N *and* $\delta\colon [a,b] \to \mathbb{R}_+$ *such that for all regulated* δ^k*-fine partial division* $\{x_l, D_l, [a_l,b_l]\}_{l=1}^{p}$ *tagged in* X_i *we have*

$$\left| \sum_{l=1}^{p} \{F_n(a_l,b_l) - F(a_l,b_l)\} \right| < \varepsilon \qquad \text{for all} \quad n \geq N$$

for some function F *where* $F_n(u,v) = \int_u^v f_n \, \mathrm{d}g$ *for* $[u,v] \subseteq [a,b]$,

(iv) $F_n(u,v)$ *converges to* $F(u,v)$ *as* $n \to \infty$ *for all* $[u,v] \subseteq [a,b]$,

then $(f,g) \in GR^*_k[a,b]$ *with primitive* F *and* $\int_a^b f_n \, \mathrm{d}g \to \int_a^b f \, \mathrm{d}g$ *as* $n \to \infty$.

Proof. Let $\varepsilon > 0$. In view of (iii) above, for every i and every j there exists integer N_{ij} and $\delta_{ij}\colon [a,b] \to \mathbb{R}_+$ such that for any regulated δ^k_{ij}-fine partial division $\{x_l, D_l, [a_l,b_l]\}_{l=1}^{p}$ of $[a,b]$ with $x_l \in X_i$

$$\left| \sum_{l=1}^{p} \{F_n(a_l,b_l) - F(a_l,b_l)\} \right| < \frac{\varepsilon}{2^{i+j}} \qquad \text{for all} \quad n \geq N_{ij}\,.$$

Take $n = n(i,j)$ so that the above inequality holds. We may assume that for each i, $\{F_{n(i,j)}\}_{j=1}^{\infty}$ is a subsequence of $\{F_{n(i-1,j)}\}_{j=1}^{\infty}$. Now consider $F_{n(j)} = F_{n(j,j)}$ in place of F_n and write $Y_i = X_i - (X_1 \cup \dots \cup X_{i-1})$ for $i = 1,2,\dots$ with X_0 empty.

Put $M(x) = n(i)$ when $x \in Y_i$. We note that there are infinitely many $m(x) \geq M(x)$, namely all $n(j) \geq n(i)$.

If $m(x)$ takes values in $\{n(j) : \ j \geq i\}$ when $m(x) \geq M(x) = n(i)$, we put $\delta(x) = \delta_{m(x)}(x)$.

Let $\{x_l, D_l, [a_l,b_l]\}_{l=1}^{p}$ be any regulated δ^k-fine partial division of $[a,b]$.

$$\left| \sum_{l=1}^{p} \{F_{m(x)}(a_l,b_l) - F(a_l,b_l)\} \right| \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+j}} = \varepsilon\,.$$

Hence conditions of the Basic convergence theorem is satisfied. Hence $(f,g) \in GR^*_k[a,b]$ with $\int_a^b f_n \, \mathrm{d}g \to \int_a^b f \, \mathrm{d}g$. □

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