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Sodnomkhorloo Tumurbat; Richard Wiegandt

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Dedicated to Professor Tibor Katriňák

ON A-RADICALS

S. TUMURBAT* — R. WIEGANDT**

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ABSTRACT. We consider biideal versions of conditions imposed on left (and/or right) ideals which latter characterize normal radicals. It is proved that the bi-strong and principally bi-hereditary radicals are the A-radicals (i.e. radicals depending only on the additive groups), a special case of normal radicals. A-radicals are characterized also in terms of quasi-ideals.

1. Preliminaries

A Kurosh-Amitsur radical γ of rings is said to be an *A-radical* if the radicality depends only on the additive group of rings, that is, for any two rings A and B with isomorphic additive groups, $A \in \gamma \implies B \in \gamma$.

This notion was introduced by Gardner [2] and studied, for instance, in the papers of Jaegermann [5], Jaegermann and Sands [6].

For a ring A, we shall denote the zero-ring on the additive group A^+, by A^0. Gardner [2] (cf. [4; Lemma 3.12.7]) proved that

$\gamma^0(A) = \sum(S \subseteq A \mid S^0 \in \gamma)$ *is an ideal of* A.

We shall make use of Gardner's Lemma ([3]) (cf. [4; Lemma 3.19.17]):

A nilpotent ring A belongs to a radical γ if and only if $A^0 \in \gamma$.

A-radicals are special cases of normal radicals which are defined by Morita contexts. A radical γ is called a *normal radical* if for every Morita context (R, V, W, S) the inclusion $V_\gamma(S)W \subseteq \gamma(R)$ holds. For details we refer to [4].

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A subring B of a ring A is called a *biideal* if $BAB \subseteq B$, this fact will be denoted by $B \triangleleft_b A$. A *quasi-ideal* Q of a ring A (denoted by $Q \triangleleft_q A$) is a subring satisfying $QA \cap AQ \subseteq Q$. Biideals and quasi-ideals are useful tools in structural investigations of rings (cf. for instance, [1], [4] and [9]); in [4; p. 164] it was observed that supernilpotent normal radicals can be defined in terms of biideals as upper radicals. In accordance with the notations $\triangleleft_b$ and $\triangleleft_q$, ideals, left ideals and right ideals will be denoted by $\triangleleft$, $\triangleleft_\ell$ and $\triangleleft_r$, respectively. One readily sees that

i) *every quasi-ideal is a biideal,*
ii) *if $B \triangleleft_b A$, then $B \triangleleft_\ell B + AB \triangleleft_r A$,*
iii) *if $B \triangleleft_\ell R \triangleleft_r A$, then $B \triangleleft_b A$.*

Let γ be a radical and $\mathcal{S}\gamma$ its semisimple class. The radical γ is *bi-stable* (*q-stable*) if $S \triangleleft_b A$ ($S \triangleleft_q A$), respectively, implies $\gamma(S) \subseteq \gamma(A)$. γ is bi-stable (q-stable) if and only if the semisimple class $\mathcal{S}\gamma$ is *bi-hereditary* (*q-hereditary*), that is $S \triangleleft_b A \in \mathcal{S}\gamma$ ($S \triangleleft_q A \in \mathcal{S}\gamma$), respectively, implies $S \in \mathcal{S}\gamma$. We say that γ is *bi-strong* (*q-strong*) if $\gamma(S) = S \triangleleft_b A$ ($\gamma(S) = S \triangleleft_q A$) implies $S \subseteq \gamma(A)$. Left (right) stability, strongness and hereditariness are defined correspondingly.

Obviously stability implies strongness, but a left and right strong radical need not be left or right stable. *For biideals*, however, *bi-strongness is equivalent to bi-stability* (cf. [10; Proposition 8]).

Sands [7] characterized *normal radicals* as *left strong and principally left hereditary* (i.e. $A \in \gamma \implies Aa \in \gamma$ for all $a \in A$) *radicals* (cf. [4; Theorem 3.18.5]). Nevertheless, *the left stable and principally left hereditary radicals are just the A-radicals*, as proved by Jaegermann and Sands [6], see also [4; Theorem 3.19.13]).

The main objective of this note is to replace here "left" and/or "right" by "bi-", and characterize radicals with these properties. Since bi-strongness is equivalent to bi-stability, characterizations of A-radicals are anticipated. For that purpose we need to define principally *bi-hereditariness*: $A \in \gamma \implies aAa \in \gamma$ for all $a \in A$.

A-radicals will be characterized also in terms of quasi-ideals, therefore beside q-stability and q-strongness, we have to define a suitable notion for hereditariness. A radical γ is said to be *principally left q-hereditary* if $A \in \gamma$ implies $A(Aa \cap aA) \in \gamma$ for all $a \in A$.

In the proofs we shall work with matrix rings, more precisely with the ring $M_2(A)$ of 2×2 matrices over a ring A. Doing so, we use the notations $(A)_{ij}$ for the set of matrices which have elements from A at the i, j position and 0 everywhere else, and $\begin{pmatrix} X & Y \\ U & V \end{pmatrix}$ for the set of matrices which have elements from X, Y, U and V at the corresponding positions. We recall from Snider [8] (cf. [4; Proposition 4.9.1]): *if* γ *is a radical, then* $\gamma(M_n(A)) = M_n(I)$ *for some ideal* I *of* A *for every ring* A.

2. Results

Lemma 1. *Let* γ *be a bi-strong or a* q*-strong or* q*-stable radical. If* $A \in \gamma$ *or* $A^0 \in \gamma$, *then* $M_2(A) \in \gamma$.

Proof. Since every bi-strong or q-stable radical is q-strong and every quasi-ideal is a biideal, it suffices to prove the statement only for q-strong radicals.

Suppose that $A \in \gamma$. Then $(A)_{11} \triangleleft_q M_2(A)$ and $(A)_{11} \cong A \in \gamma$. Since γ is q-strong, in view of Snider [8] we have

$$(A)_{11} \subseteq \gamma(M_2(A)) = \begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

with an appropriate ideal I of A. Hence $A \subseteq I$, and so $\gamma(M_2(A)) = M_2(A)$.

Writing $(A)_{12}$ in place of $(A)_{11}$, we get the proof for the case $A^0 \in \gamma$. □

Lemma 2. *Let* γ *be a* q*-strong and principally left* q*-hereditary radical. If* $A^0 \in \gamma$ *for a ring* A, *then* $A \in \gamma$.

Proof. By Lemma 1, we have $M_2(A) \in \gamma$. Since for every element $a \in A$ we have

$$\begin{aligned}\begin{pmatrix} A(Aa \cap aA) & 0 \\ A(Aa \cap aA) & 0 \end{pmatrix} &= M_2(A)(Aa \cap aA)_{11} \\ &= M_2(A)\left(\begin{pmatrix} Aa & 0 \\ Aa & 0 \end{pmatrix} \cap \begin{pmatrix} aA & aA \\ 0 & 0 \end{pmatrix}\right) \\ &= M_2(A)\big(M_2(A)(a)_{11} \cap (a)_{11}M_2(A)\big) \triangleleft_q M_2(A),\end{aligned}$$

the principally q-hereditariness of γ yields that

$$M_2(A)\big(M_2(A)(a)_{11} \cap (a)_{11}M_2(A)\big) \in \gamma.$$

Hence

$$\big(A(Aa \cap aA)\big)_{21} \triangleleft \begin{pmatrix} A(Aa \cap aA) & 0 \\ A(Aa \cap aA) & 0 \end{pmatrix} \in \gamma,$$

and so also $A(Aa \cap aA) \in \gamma$. But $A(Aa \cap aA) \triangleleft_q A$ and γ is q-strong, therefore

$$AaAa \subseteq A(Aa \cap aA) \subseteq \gamma(A).$$

Hence $(Aa + \gamma(A))/\gamma(A)$ is a homomorphic image of $A^0 \in \gamma$, and so $(Aa + \gamma(A))/\gamma(A) \in \gamma$. Since γ is q-strong and $(Aa + \gamma(A))/\gamma(A) \triangleleft_q A/\gamma(A)$, it follows that $Aa \subseteq \gamma(A)$ for all $a \in A$, that is, $A^2 \subseteq \gamma(A)$. Thus $A/\gamma(A) \cong A^0/(\gamma(A))^0 \in \gamma$ holds implying $A = \gamma(A)$. □

Lemma 3. *Let γ be a q-strong and principally left q-hereditary radical. If $A \in \gamma$, then $A^0 \in \gamma$.*

Proof. Assume that $A \in \gamma$ and $A^0 \notin \gamma$. As already mentioned, we know that $\gamma^0(A) \triangleleft A$, so without loss of generality we may confine ourselves to the case $A \in \gamma$ and $0 \neq A^0 \in \mathcal{S}\gamma$.

Clearly $(A)_{11}$, $(A)_{22} \triangleleft_q \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$ and $(A)_{ii} \cong A \in \gamma$. Since γ is q-strong,

$$(A)_{11} + (A)_{22} \subseteq \gamma \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$$

and

$$\begin{pmatrix} A^2 & A^2 \\ 0 & A^2 \end{pmatrix} = ((A)_{11} + (A)_{22}) \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \subseteq \gamma \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}.$$

These relations imply

$$\begin{pmatrix} A & A^2 \\ 0 & A \end{pmatrix} \subseteq \gamma \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}.$$

Hence the factor ring $\left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right) / \gamma \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$ is a homomorphic image of $A/A^2 \in \gamma$, and so we conclude that $\left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right) \in \gamma$. Since for each element $a \in A$ we have

$$\big(A(Aa \cap aA)\big)_{12} = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \left(\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} (a)_{12} \cap (a)_{12} \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \right) \triangleleft_q \begin{pmatrix} A & A \\ 0 & A \end{pmatrix},$$

and γ is principally left q-hereditary, it follows that

$$\big(A(Aa \cap aA)\big)^0 \cong \big(A(Aa \cap aA)\big)_{12} \in \gamma.$$

Hence $\big(A(Aa \cap aA)\big)^0 \subseteq \gamma^0(A) = 0$. This proves that $AaAa \subseteq A(Aa \cap aA) = 0$ and $(Aa)^2 = 0$ for every element $a \in A$. Thus for every $a, b \in A$ we have

$$M_2(A) \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} M_2(A) \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = 0,$$

and therefore $AaAb = 0$, that is $A^4 = 0$. Hence $A \in \gamma$ is a nilpotent ring, and so $A^0 \in \gamma$ by Gardner's Lemma. Hence $A^0 \in \gamma \cap \mathcal{S}\gamma = 0$, which is a contradiction. Thus $A^0 \in \gamma$ as claimed. □

Lemma 4. *Let γ be a q-stable and principally bi-hereditary radical. If $A^0 \in \gamma$ for a ring A, then $A \in \gamma$.*

Proof. We know from Lemma 1 that $M_2(A) \in \gamma$. Since γ is principally bi-hereditary, for any $a \in A$ we have

$$aAa \cong (aAa)_{11} = (a)_{11} M_2(A) (a)_{11} \in \gamma.$$

Taking into account that γ is q-stable, we get that

$$aAa \subseteq \gamma(Aa \cap aA) \subseteq \gamma(A)\,.$$

Hence $(Aa)^2 \subseteq \gamma(A)$ for all $a \in A$, and proceeding as the proof of Lemma 2, we get that $A^2 \subseteq \gamma(A)$ and $A \in \gamma(A)$. □

LEMMA 5. *Let γ be a q-stable and principally bi-hereditary radical. If $A \in \gamma$, then $A^0 \in \gamma$.*

P r o o f . As in the proof of Lemma 4, we get $(aAa)^0 \cong (aAa)_{12} \in \gamma$. Suppose that $A^0 \notin \gamma$. As in the proof of Lemma 3, it suffices to deal with the case $0 \neq A^0 \in \mathcal{S}\gamma$. But then $(aAa)^0 \in \mathcal{S}\gamma$ implying $aAa = 0$ for all $a \in A$. Hence from $(x+y)A(x+y) = 0$ we conclude that $xAy = 0$ for all $x, y \in A$, that is, $A^3 = 0$. Now Gardner's Lemma is applicable yielding $0 \neq A^0 \in \gamma$, which is a contradiction. □

LEMMA 6. *If γ is an A-radical, then γ is bi-stable, principally bi-hereditary and principally left q-hereditary.*

P r o o f . As mentioned, the A-radical γ is left and right stable and principally left and right hereditary. Thus, if $B \triangleleft_b A \in \mathcal{S}\gamma$, then $B \triangleleft_\ell B + AB \triangleleft_r A$, and so $B \in \mathcal{S}\gamma$, proving that γ is bi-stable. Further, if $A \in \gamma$, then by $aAa \triangleleft_\ell aA \triangleleft_r A$ it follows that $aAa \in \gamma$, whence γ is principally bi-hereditary.

If $A \in \gamma$, then also $A^0 \in \gamma$. Now, for every element $x \in Aa \cap aA$ the ring $(Ax)^0$ is in γ as a homomorphic image of A^0. So by $(Ax)^0 \triangleleft A^0$ we have

$$\big(A(Aa \cap aA)\big)^0 = \sum\big((Ax)^0 \mid x \in Aa \cap aA\big) \in \gamma\,.$$

Taking into consideration that γ is an A-radical, also $A(Aa \cap aA) \in \gamma$ proving that γ is principally left q-hereditary. □

Summarizing the so far proved results, we get several characterizations for A-radicals.

THEOREM. *For a radical γ of rings the following conditions are equivalent:*

(i) *γ is an A-radical,*
(ii) *γ is bi-strong and principally bi-hereditary,*
(iii) *γ is q-stable and principally bi-hereditary,*
(iv) *γ is bi-strong and principally left q-hereditary,*
(v) *γ is q-strong and principally left q-hereditary.*

P r o o f . By Lemma 6, any A-radical γ is bi-strong, principally bi-hereditary and principally left q-hereditary.

As proved in [10], bi-strongness is equivalent to bi-stability. Further, the implications

$$\text{bi-stable} \Longrightarrow q\text{-stable} \Longrightarrow q\text{-strong}$$

are obvious.

Lemmas 2 and 3 state that for a q-strong and principally left q-hereditary radical γ, $A \in \gamma \iff A^0 \in \gamma$. Lemmas 4 and 5 assert that for a q-stable and principally bi-hereditary radical γ, $A \in \gamma \iff A^0 \in \gamma$. This property characterizes the A-radicals (cf. [4; Proposition 3.19.2]). □

COROLLARY 1. *A radical γ of rings is an A-radical if and only if γ is normal and bi-strong.*

Proof. By [4; Corollary 3.19.14], γ is an A-radical if and only if γ is normal and left (and right) stable. As one readily verifies, left and right stability is equivalent to bi-stability, that is, to bi-strongness, by [10]. □

Remark. A normal radical is always left and right strong, but not necessarily an A-radical. So Corollary 2 shows that left and right strongness (even together with normality) does not imply bi-strongness. □

COROLLARY 2. *For a radical γ the following conditions are equivalent:*

(i) *γ is a hereditary A-radical,*
(ii) *γ is bi-strong and bi-hereditary,*
(iii) *$\gamma(B) = B \cap \gamma(A)$ for every $B \triangleleft_b A$.*

Proof.

(i) $\iff$ (ii): Left and right hereditariness is obviously equivalent to bi-hereditariness. Furthermore, a bi-hereditary radical is also hereditary.

(i) $\iff$ (iii): By [4; Corollary 3.19.5], γ is a hereditary A-radical if and only if $\gamma(L) = L \cap \gamma(A)$ for every $L \triangleleft_\ell A$ (and also for every $L \triangleleft_r A$). Hence, if $B \triangleleft_\ell A$, then $\gamma(B) = B \cap \gamma(B + AB)$ and $\gamma(B + AB) = (B + AB) \cap \gamma(A)$, whence $\gamma(B) = B \cap \gamma(A)$. □

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* *Department of Algebra*
University of Mongolia
P.O. Box 75
Ulaan Baatar 20
MONGOLIA
E-mail: stumurbat@hotmail.com

** *A. Rényi Institute of Mathematics*
Hungarian Academy of Sciences
P.O. Box 127
H-1364 Budapest
HUNGARY
E-mail: wiegandt@renyi.hu