COMPANION d-ALGEBRAS

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ABSTRACT. In this paper we develop a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Included among the latter are results on certain natural posets associated with companion d-algebras as well as constructions on Bin(X), the collection of binary operations on the set X, which permit construction of new companion d-algebras from companion d-algebras X also in natural ways.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([I], [Ia]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [HL1], [HL2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCK-algebras also have some connections with other areas: A. Dvurečenskij and M. G. Graziano [DvGr], C. S. Hoo [Hoo], J. M. Font, A. J. Rodríguez and A. Torrens [FRT], D. Mundici [Mun] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [Me] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful

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generalization of BCK-algebras, and then investigated several relations between $d$-algebras and BCK-algebras as well as several other relations between $d$-algebras and oriented digraphs ([NK3]). After that some further aspects were studied ([LK], [NJK], [JNK]). As a generalization of BCK-algebras $d$-algebras are obtained by deleting identities. Given one of these deleted identities a related identities are constructed by replacing one of the terms involving the original operation by an identical term involving a second (companion) operation, thus producing the notion of companion $d$-algebra which (precisely) generalizes the notion of BCK-algebra and is such that not every $d$-algebra is one of these. In this paper we develop a theory of companion $d$-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Included among the latter are results on certain natural posets associated with companion $d$-algebras as well as constructions on Bin($X$), the collection of binary operations on the set $X$, which permit construction of new companion $d$-algebras from companion $d$-algebras $X$ also in natural ways.

2. Companion $d$-algebras

A $d$-algebra ([NK3]) is a non-empty set $X$ with a constant 0 and a binary operation “$*$” satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y$ in $X$.

A BCK-algebra is a $d$-algebra $(X ; *, 0)$ satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$

for all $x, y, z$ in $X$.

A BCK-algebra $(X ; *, 0)$ is said to have a condition (S) ([MeJu]) if

$A(a, b) := \{ x \in X : x * a \leq b \}$

has a greatest element for any $a, b \in X$.

**Definition 2.1.** Let $(X ; *, 0)$ be a $d$-algebra. Define a binary operation $\odot$ on $X$ by

(VI) $((x \odot y) * x) * y = 0$

for any $x, y \in X$, which is called a subcompanion operation of $X$. A subcompanion operation $\odot$ is said to be a companion operation of $X$ if

(VII) if $(z * x) * y = 0$, then $z * (x \odot y) = 0$ for any $x, y, z \in X$.
COMPANION $d$-ALGEBRAS

Example 2.2. Let $X := \{0,1,2,3\}$ be a set with the following tables:

\[
\begin{array}{cccc}
* & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 2 & 2 & 2 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
\circ & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 & 3 \\
1 & 1 & 1 & 3 & 3 \\
2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

Then $(X; *, 0)$ is a $d$-algebra, which is not a $BCK/BCI$-algebra, and the binary operation $\circ$ defined above is a companion operation on $X$.

A $d$-algebra $X$ is said to be a companion $d$-algebra if it has a companion operation.

**Proposition 2.3.** Let $(X; *, 0)$ be a $d$-algebra. If $X$ has a companion operation $\circ$, then it is unique.

**Proof.** Assume the binary operations $\circ_i$ and $\circ_2$ are companion operations on $X$. Then $((x \circ_i y) * x) * y = 0$ for any $x, y \in X$ ($i = 1, 2$). By (VII) we obtain

$$
(x \circ_1 y) * (x \circ_2 y) = 0. 
$$

(1)

Interchange $\circ_1$ with $\circ_2$. Then

$$
(x \circ_2 y) * (x \circ_1 y) = 0. 
$$

(2)

By (III) we obtain $\circ_1 = \circ_2$. Hence the operation $\circ$ is unique. □

**Example 2.4.** Every $BCK$-algebra with condition (S) is a companion $d$-algebra.

Example 2.2 is a companion $d$-algebra which is not a $BCK/BCI$-algebra. This means that a companion $d$-algebra is a generalization of a $BCK/BCI$-algebra with condition (S).

**Proposition 2.5.** Let $(X; *, \circ, 0)$ be a companion $d$-algebra. Then for any $x, y, z \in X$, we have

(i) if $x * z = 0$, then $x * (z \circ y) = 0$,

(ii) $x * (x \circ y) = 0$,

(iii) $x \circ 0 = x$. 

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(i) Since \((x * z) * y = 0 * y = 0\), by (VII), \(x * (z \odot y) = 0\).

(ii) Put \(z := x\) in (i).

(iii) We claim that if \(x * 0 = 0\), then \(x = 0\). In fact, since \(0 * x = 0\), by (III) we have \(x = 0\). Since \(X\) is a companion \(d\)-algebra, \(((x \odot 0) * x) * 0 = 0\) and so \((x \odot 0) * x = 0\). If we put \(y := 0\) in (ii), then \(x * (x \odot 0) = 0\). By (III) we have \(x \odot 0 = x\). \(\square\)

**Theorem 2.6.** Let \((X; *, \odot, 0)\) be a companion \(d\)-algebra. Let \(\diamond\) be a binary operation on \(X\) such that

\[
(x * y) * z = x * (y \odot z).
\]  

Then \(X\) is a companion \(d\)-algebra and \(\diamond\) is exactly the operation \(\odot\).

**Proof.** By applying (3) and (I), we have

\[
((x \diamond y) * x) * y = (x \diamond y) * (x \diamond y)
\]

by (3) \[
= 0,
\]

by (I) proving the condition (VI). Let \(z \in X\) with \((z * x) * y = 0\). Then by (3), \(z * (x \odot y) = (z * x) * y = 0\), proving the condition (VII). Hence \(\diamond\) is a companion operation, which is unique by Proposition 2.3. \(\square\)

Given a \(d\)-algebra \((X; *, 0)\), we define a partial binary relation \(<\) by \(x < y\) \(\iff x * y = 0\), where \(x, y \in X\).

**Proposition 2.7.** If \((X; *, \odot, 0)\) is a bounded companion \(d\)-algebra, i.e., there is an element \(1 \in X\) such that \(x * 1 = 0\) for any \(x \in X\), then \(x \odot 1 = 1\) for any \(x \in X\).

**Proof.** Since \(u * x \leq 1\) for any \(u \in X\), \((u * x) * 1 = 0\). By applying (VII) we have \(u \leq x \odot 1\), for any \(u \in X\), which implies \(1 = x \odot 1\). \(\square\)

A \(d\)-algebra \((X; *, 0)\) is said to be **positive implicative** if \((x * y) * z = (x * z) * (y * z)\) for any \(x, y, z \in X\).

**Proposition 2.8.** Let \((X; *, \odot, 0)\) be a companion \(d\)-algebra.

(i) \(0 \leq x \odot y\), \(x \leq x \odot y\), for any \(x, y \in X\),

(ii) if \(X\) is positive implicative, then \(y \leq x \odot y\) for any \(x, y \in X\).

**Proof.**

(i) Since \((0 * x) * y = 0\), \(0 \leq x \odot y\). From \((x * x) * y = 0 * y = 0\), we obtain \(x \leq x \odot y\).

(ii) Since \(X\) is positive implicative, \((y * x) * y = (y * y) * (x * y) = 0 * (x * y) = 0\) and hence \(y \leq x \odot y\). \(\square\)
**Theorem 2.9.** Let \((X; *, \odot, 0)\) be a companion \(d\)-algebra. Assume that \(x * 0 = x\) for any \(x \in X\).

1. \(X\) is positive implicative,
2. if \(x \leq y\), then \(x \odot y = y\),
3. \(x \odot x = x\) for any \(x, y \in X\).

Then (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii).

**Proof.**

(i) \(\Rightarrow\) (ii). If \(x \leq y\), then

\[
0 = ((x \odot y) * x) * y
= [(x \odot y) * y] * (x * y) \quad [X: \text{positive implicative}]
= [(x \odot y) * y] * 0 \quad [x * y = 0]
= (x \odot y) * y, \quad [x * 0 = x]
\]

which means that \(x \odot y \leq y\). By applying Proposition 2.8-(ii), we have \(x \odot y = y\).

(ii) \(\Rightarrow\) (iii). Let \(y := x\) in (ii). □

**Definition 2.10.** ([NJK]) Let \((X; *, 0)\) be a \(d\)-algebra and \(\emptyset \neq I \subseteq X\). \(I\) is called a \(d\)-subalgebra of \(X\) if \(x * y \in I\) whenever \(x \in I\) and \(y \in I\). \(I\) is called a BCK-ideal of \(X\) if it satisfies:

1. \((D_0)\) \(0 \in I\),
2. \((D_1)\) \(x * y \in I\) and \(y \in I\) imply \(x \in I\).

\(I\) is called a \(d\)-ideal of \(X\) if it satisfies \((D_1)\) and

3. \((D_2)\) \(x \in I\) and \(y \in X\) imply \(x * y \in I\), i.e., \(I * X \subseteq I\).

**Definition 2.11.** Let \((X; *, \odot, 0)\) be a companion \(d\)-algebra and \(\emptyset \neq I \subseteq X\). \(I\) is called a \(\odot\)-subalgebra if \(x \odot y \in I\) for any \(x, y \in I\).

In Example 2.2, the set \(I_1 := \{0, 1\}\) is a \(\odot\)-subalgebra of \(X\), while \(I_2 := \{0, 1, 2\}\) is not a \(\odot\)-subalgebra of \(X\).

**Theorem 2.12.** Let \((X; *, \odot, 0)\) be a companion \(d\)-algebra. If \(I\) is a BCK-ideal of \(X\), then \(I\) is a \(\odot\)-subalgebra of \(X\).

**Proof.** If \(X\) is a companion \(d\)-algebra, then \(((x \odot y) * x) * y = 0 \in I\) for any \(x, y \in I\). Since \(I\) is a BCK-ideal of \(X\) and \(y \in I\), \((x \odot y) * x \in I\). Moreover, since \(x \in I\), we obtain \(x \odot y \in I\), proving the theorem. □

The converse of Theorem 2.12 need not be true in general. For example, \(J := \{0, 1, 3\}\) is a \(\odot\)-subalgebra of \(X\), but not a BCK-ideal of \(X\), since \(2 * 3 = 0 \in J\), \(3 \in J\), but \(2 \notin J\) in Example 2.2.
PROPOSITION 2.13. Let \((X;*,\odot,0)\) be a companion d-algebra and let \(I\) be a BCK-ideal of \(X\). If \(x \odot y \in I\), then \(x \in I\) where \(x, y \in X\).

Proof. By Proposition 2.5-(ii), \(x \odot (x \odot y) = 0 \in I\). Since \(x \cdot y \in I\) and \(I\) is a BCK-ideal of \(X\), we have \(x \in I\). □

COROLLARY 2.14. Let \((X;*,\odot,0)\) be a companion d-algebra and let \(I\) be a BCK-ideal of \(X\). If \(x \odot y = y \odot x \in I\), then \(x, y \in I\) where \(x, y \in X\).

COROLLARY 2.15. Let \((X;*,\odot,0)\) be a companion d-algebra and let \(I\) be a BCK-ideal of \(X\). Then \(x \in I \iff x \odot x \in I\).

Proof. It follows immediately from Theorem 2.12 and Proposition 2.13. □

3. dsu condition

In a d-algebra \(X\), the identity \((x \ast y) \ast x = 0\) does not hold in general.

DEFINITION 3.1. ([NJK]) A d-algebra \(X\) is called a d*-algebra if it satisfies the identity \((x \ast y) \ast x = 0\) for all \(x, y \in X\).

Clearly, a BCK-algebra is a d*-algebra, but the converse need not be true.

Example 3.2. Let \(X := \{0, 1, 2, \ldots\}\) and let the binary operation \(\ast\) be defined as follows:

\[
x \ast y := \begin{cases} 
0 & \text{if } x \leq y, \\
1 & \text{otherwise.}
\end{cases}
\]

Then \((X, \ast, 0)\) is a d-algebra which is not a BCK-algebra (see [NK3, Example 2.8]). We can easily see that \((X, \ast, 0)\) is a d*-algebra.

THEOREM 3.3. ([NJK]) In a d*-algebra, every BCK-ideal is a d-ideal.

The following corollary is obvious.

COROLLARY 3.4. ([NJK]) In a d*-algebra, every BCK-ideal is a d-subalgebra.

For companion d-algebras the condition \((x \ast y) \ast (x \cdot y) = 0\) is also one which is not unusual, since in ‘usual’ circumstances we expect the difference to be smaller than the usual (dsu condition).

DEFINITION 3.5. Let \((X;*,\odot,0)\) be a companion d-algebra. \(X\) is said to have a dsu condition if

\[
(x \ast y) \ast (x \odot y) = 0
\]

for any \(x, y \in X\).
**Proposition 3.6.** Let \((X; *, 0, O)\) be a companion \(d\)-algebra having the dsu condition. If \(I\) is a BCK-ideal of \(X\), then it is a \(d\)-subalgebra of \(X\).

**Proof.** By Theorem 2.12, \(x \odot y \in I\) for any \(x, y \in I\). Since \(X\) has the dsu condition, \((x \ast y) \ast (x \odot y) = 0 \in I\) and \(I\) is a BCK-ideal of \(X\), we obtain \(x \ast y \in I\).

Let \((X; *, 0)\) be a \(d\)-algebra and \(x \in X\). Define \(x \ast X := \{x \ast a : a \in X\}\). \(X\) is said to be edge ([NK3]) if for any \(x\) in \(X\), \(x \ast X = \{x, 0\}\).

**Lemma 3.7.** ([NJK]) If \((X; *, 0)\) is a edge \(d\)-algebra, then \((x \ast (x \ast y)) \ast y = 0\) and \(x \ast 0 = x\) for any \(x, y \in X\).

**Theorem 3.8.** Let \((X; *, 0, O)\) be a companion edge \(d^*\)-algebra. If
\[
(z \ast (x \odot y)) \ast ((z \ast x) \ast y) = 0,
\]
then \(X\) has a dsu condition.

**Proof.** Let \(z := x \ast y\) in (5). Then
\[
0 = ((x \ast y) \ast (x \odot y)) \ast ((x \ast y) \ast x) \ast y)
= ((x \ast y) \ast (x \odot y)) \ast (0 \ast y)
= ((x \ast y) \ast (0 \ast y)) \ast 0
= ((x \ast y) \ast (x \odot y)),
\]
proving the theorem.

**Proposition 3.9.** Let \((X; *, 0, O)\) be a companion edge \(d\)-algebra. If
\[
(z \ast (x \odot y)) \ast ((x \ast z) \ast y) = 0,
\]
then \(X\) has a dsu condition.

**Proof.** Let \(z := x \ast y\) in (6). Then by Lemma 3.7
\[
0 = ((x \ast y) \ast (x \odot y)) \ast ((x \ast (x \ast y)) \ast y)
= ((x \ast y) \ast (x \odot y)) \ast 0
= ((x \ast y) \ast (x \odot y)),
\]
proving the proposition.

**4. Completeness**

A companion \(d\)-algebra \((X; *, 0, O)\) is said to be complete if for any \(x \in X\), there exists an \(x^*\) in \(X\) such that \(x \odot x^* = x\). Note that such an \(x^*\) need not be unique. For such an example, we find \(2 \odot 0 = 2 \odot 1 = 2, 3 \odot 1 = 3 \odot 2 = 3\) in Example 2.2.
PROPOSITION 4.1. Let \((X;*,0,O)\) be a companion d-algebra. If we define a partial binary relation \(\preceq\) by
\[
x \preceq y \iff (x \odot z) * (y \odot z) = 0 \quad \text{for all } z \in X,
\]
then \(\preceq\) is reflexive and anti-symmetric.

Proof. Clearly, \(\preceq\) is reflexive. If \(x \preceq y, y \preceq x\), then \((x \odot z) * (y \odot z) = 0 = (y \odot z) * (x \odot z)\) for any \(z \in X\). By applying (III) we have
\[
x \odot z = y \odot z
\]
for any \(z \in X\). Since \(X\) is complete, there exist \(x^*, y^* \in X\) such that \(x = x \odot x^*, y = y \odot y^*\). If we let \(z := x^*\) and \(z := y^*\) in (8), respectively, then \(x = x \odot x^* = y \odot x^*, y = y \odot y^* = x \odot y^*\). Thus by Proposition 2.5-(ii), \(x \odot y = x \odot (x \odot y^*) = 0\) and \(y \odot x = y \odot (y \odot x^*) = 0\) and hence \(x = y\), proving the proposition. \(\square\)

For any BCK/BCI-algebras the following transitivity condition holds:
\[
\text{if } x \odot y = 0 \text{ and } y \odot z = 0, \text{ then } x \odot z = 0
\]
(see [MeJu, Theorem 1.2-(b)]). This condition does not hold in d-algebra in general.

Example 4.2. Let \(X := \{0, a, b, c\}\) be a set with the following tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X;*,0)\) is a d-algebra, which is not a BCK/BCI-algebra (see [NJK]), and \(a \odot b = 0 = b \odot c\), but \(a \odot c = a \neq 0\).

Thus, if a d-algebra satisfies the transitivity condition, then the natural order \(\leq\) given by \(x \leq y\) if and only if \(x \odot y = 0\) is a partial order.

PROPOSITION 4.3. Let \((X;*,0,O)\) be a complete companion d-algebra. If \(X\) satisfies the transitivity condition, then \((X;\preceq)\) is a poset.

PROPOSITION 4.4. Let \((X;*,0,O)\) be a complete companion d-algebra. If \(x \preceq y\), then \(x \leq y\) in \(X\).

Proof. If \(x \preceq y\), then \((x \odot \alpha) * (y \odot \alpha) = 0\) for any \(\alpha \in X\). This implies \((x \odot 0) * (y \odot 0) = 0\) and hence \(x \odot y = 0\) by Proposition 2.5-(iii). \(\square\)

The converse of Proposition 4.4 need not be true in general.
Example 4.5. Let $X := \{0, a, b, c, d, 1\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
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<td>a</td>
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<td>b</td>
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<tr>
<td>d</td>
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<tr>
<td>1</td>
<td>1</td>
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<td>a</td>
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<table>
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<tr>
<th>$\odot$</th>
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<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
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<td>c</td>
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<td>1</td>
<td>c</td>
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</tr>
</tbody>
</table>

Then $(X; *, \odot, 0)$ is a companion $d$-algebra, which is not a $BCK/BCI$-algebra, since $(c * b) * d = a \neq 0 = (c * b) * b$. We know that $a \leq b$, but $a \odot c = d$ and $b \odot c = b$, and $d$ and $b$ are incomparable. Hence $a \ll b$ does not hold.

The converse of Proposition 4.4 holds for $BCK/BCI$-algebras (see [Hu, $BCI$-algebras, p. 98, Theorem 8]).

5. Pogroupoid and subcompanion operators

In [Ne], J. N e g g e r s defined a groupoid $S(\cdot)$ to be a pogroupoid if

1. $x \cdot y \in \{x, y\}$;
2. $x \cdot (y \cdot x) = y \cdot x$;
3. $(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot z$

for all $x, y, z \in S$. For a given pogroupoid $S(\cdot)$ he defined an associated partial order $po(S)$ by $x \leq y$ iff $y \cdot x = y$ and he then demonstrated that $po(S)$ is a poset. On the one hand, for a given poset $S(\leq)$ he also defined a binary operation on $S$ by $y \cdot x = y$ if $x \leq y$, $y \cdot x = x$ otherwise, and proved that $S(\cdot)$ is a pogroupoid. Thus, denoting this pogroupoid by pogr($S$), it may be shown that pogr($po(S)$) = $S(\cdot)$ and po(pogr($S$)) = $S(\leq)$ provide a natural isomorphism between the category of pogroupoids and the category of posets.

Given a poset $P(\leq)$ it is $A$-free if there is no full-subposet $X(\leq)$ of $P(\leq)$ which is order isomorphic to the poset $A(\leq)$. If $C_n$ denotes a chain of length $n$ and if $n$ denotes an antichain of cardinal number $n$, while + denotes the disjoint union of posets, then the poset $(C_2 + 1)$ (or $C_2 + C_1$) has Hasse-diagram:
and may be represented as \( \{ p \leq q, p \circ r, q \circ r \} \), where \( a \circ b \) denotes the relation of not being comparable (i.e., \( a \circ b \) iff \( a \leq b \) and \( b \leq a \) are both false) (see [NK2]). J. Neggers and H. S. Kim [NK1] proved that the pogroupoid \( S(\cdot) \) is a semigroup if and only if \( S(\cdot) = \text{pogr}(P) \) where \( P(\leq) \) is \( (C_2 + 1) \)-free as a poset.

Given a \( d \)-algebra \( (X; \ast, 0) \), we define a binary operation \( \star \) on \( X \) by

\[
x \star y = y \ast x = y \quad \text{if } x \ast y = 0,
x \ast y = y, \quad y \ast x = x \quad \text{otherwise.}
\]

The operation \( \star \) described above is said to be a pogroupoid. Even though the derived digraph from a \( d \)-algebra may have no \( (C_2 + 1) \)-full subposet, its derived algebra \( (X; \ast) \) need not be a pogroupoid.

**Example 5.1.** Consider a \( d \)-algebra \( (X; \ast, 0) \) with the following left table:

\[
\begin{array}{cccc}
\ast & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & b \\
b & b & b & 0 & 0 \\
c & c & c & c & c \\
\end{array}
\]

Then \( (X; \ast, 0) \) is a \( d \)-algebra, which is not a \( BCK/BCI \)-algebra. It is easy to see that its derived digraph has no \( (C_2 + 1) \)-full subposet, but \( (X; \ast) \) is not a pogroupoid, since \( (c \ast b) \ast a = c \ast a = a \), while \( (c \ast b) \ast (b \ast a) = c \ast b = c \).

**Proposition 5.2.** Let \( (X; \ast, 0) \) be a \( d^* \)-algebra. Then \( ((x \ast y) \ast x) \ast y = 0 \) for any \( x, y \in X \).

**Proof.** It follows immediately from the definition of pogroupoid. \( \Box \)

**Proposition 5.3.** Let \( (X; \ast, 0) \) be a \( d^* \)-algebra. Assume \( (y \ast x) \ast y = 0 \) provided \( x \ast y = 0 \). Then \( ((y \ast x) \ast x) \ast y = 0 \) for any \( x, y \in X \).

**Proof.** If \( x \ast y = 0 \), then \( y \ast x = y \) and hence \( ((y \ast x) \ast x) \ast y = (y \ast x) \ast y = 0 \). If \( x \ast y \neq 0 \), then \( y \ast x = x \) and \( ((y \ast x) \ast x) \ast y = (x \ast x) \ast y = 0 \ast y = 0 \), proving the proposition. \( \Box \)

There exists an example of non-\( d^* \)-algebra satisfying \( (y \ast x) \ast y = 0 \) when \( x \ast y = 0 \). The \( d \)-algebra \( X \) in Example 5.1 is such an algebra, since \( (a \ast c) \ast a = b \ast a \neq 0 \). Propositions 5.2 and 5.3 hold for any \( BCK/BCI/BCH \)-algebras. The notion of a subcompanion operation is a generalized concept of Proposition 5.2.
**PROPOSITION 5.4.** Let \((X; *, 0, O)\) be a companion \(d\)-algebra. If \((X; *)\) is a pogroupoid, then \((x * y) * (x * y) = 0\) for any \(x, y \in X\).

**Proof.** Since \(X\) is a \(d^*\)-algebra, by Proposition 5.2, \(((x * y) * x) * y = 0\) for any \(x, y \in X\). Since \(\circ\) is a companion operation, by (VII), \((x * y) * (x \circ y) = 0\).

Let \((X; *, 0, O)\) be a \(d\)-algebra. If we define \(x * y = 0\), then \(*\) is a (trivial) subcompanion operation on \(X\).

Let \((X; \circ, 0)\) be a \(d\)-algebra and \(\circ_i\) be a binary operation on \(X\) \((i = 1, 2)\). Define a relation:

\[\circ_1 \leq \circ_2 \iff (x \circ_1 y) * (x \circ_2 y) = 0\]

for any \(x, y \in X\). Then it is reflexive and anti-symmetric. Let \(\text{Bin}(X) := \{\circ : \circ\) binary operation on \(X\}\). Define a binary operation \(\oplus\) on \(\text{Bin}(X)\) by

\[x(\circ_1 \oplus \circ_2)y := (x \circ_1 y) * (x \circ_2 y)\]

Denote by \(\circ_a, a \in X,\) the binary operation \(x \circ_a y := a\) for any \(x, y \in X\).

**THEOREM 5.5.** If \((X; *, 0)\) is a \(d\)-algebra, then \((\text{Bin}(X), \oplus, \circ_0)\) is also a \(d\)-algebra and the mapping \(a \mapsto \circ_a\) is an injection of \((X; *, 0)\) into \((\text{Bin}(X); \oplus, \circ_0)\) which is a \(d\)-morphism.

**Proof.** Clearly, \(\text{Bin}(X)\) satisfies the conditions (I) and (III). For any \(\circ \in \text{Bin}(X)\) and for any \(x, y \in X, x(\circ_0 \oplus \circ)y = (x \circ_0 y) * (x \circ y) = 0 * (x \circ y) = 0 = x \circ_0 y,\) which means that \(\circ_0 \oplus \circ = \circ_0,\) proving that \((\text{Bin}(X), \oplus, \circ_0)\) is a \(d\)-algebra. We claim that \(\circ_a \ast \circ_b = \circ_{a \ast b}\) for any \(a, b \in X.\) In fact, \(x(\circ_a \ast \circ_b)y = (x \circ_a y) * (x \circ_b y) = a \ast b = x \circ_{a \ast b}y\) for any \(x, y \in X.\) If we define a map \(\varphi : X \rightarrow \text{Bin}(X)\) by \(\varphi(a) := \circ_a,\) then \(\varphi(a \ast b) = \circ_{a \ast b} = \circ_a \oplus \circ_b = \varphi(a) \oplus \varphi(b)\) for any \(a, b \in X,\) proving the theorem.

**THEOREM 5.6.** Let \((X; *, 0, O)\) be a companion \(d\)-algebra. If we define a binary operation \(\square\) by

\[x(\circ_1 \square \circ_2)y := (x \circ_1 y) \circ (x \circ_2 y)\]

for any \(x, y \in X,\) then \((\text{Bin}(X); \oplus, \square, \circ)\) is also a companion \(d\)-algebra containing \((X; *, 0, O)\) via the identification \(a \mapsto \circ_a.\)

**Proof.** Since \(X\) is a companion \(d\)-algebra, \(x[((\circ_1 \square \circ_2) \oplus \circ_1) \circ_2]y = [(x \circ_1 y) \oplus (x \circ_2 y)] * (x \circ_1 y) * (x \circ_2 y) = 0\) for any \(x, y \in X.\) Since the proof of (VII) is similar, we omit it.

**PROPOSITION 5.7.** If \(d\)-algebra \((X; *, 0)\) has the transitivity condition, then \((\text{Bin}(X), \oplus, \circ_0)\) has also the transitivity condition.

**Proof.** Straightforward.
PROPOSITION 5.8. Let \((X; *, 0)\) be a d-algebra. If \(\diamond \in \text{Bin}(X)\) is commutative with \(x \ast (x \diamond y) = 0\) for all \(x, y \in X\), then \((x \ast y) \ast (x \diamond y) = 0\).

Proof. For any \(x, y \in X\), either \(x \ast y = x\) or \(x \ast y = y\). If \(x \ast y = x\), then \((x \ast y) \ast (x \diamond y) = x \ast (x \diamond y) = 0\). If \(x \ast y = y\), since \(\diamond\) is commutative, \((x \ast y) \ast (x \diamond y) = y \ast (x \diamond y) = y \ast (y \diamond x) = 0\), proving the proposition.

A d-algebra \((X; *, 0)\) is said to be a d-chain if \(x \ast y \neq 0\), then \(y \ast x = 0\). \(x, y \in X\).

Note that \(\text{Bin}(X)\) need not be a d-chain, even though \(X\) is a d-chain. Consider a BCK/BCI/d-algebra \(A' := \{0, a, b\}\) with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; *, 0)\) is a d-chain. Define maps \(f: X \to X\) by \(f(0) = b, f(a) = a, f(b) = 0\), \(g: X \to X\) by \(g(0) = 0, g(a) = a, f(b) = b\). If we define binary operations on \(X\) by \(x \circ y := f(x), x \circ g y := g(x)\), for all \(x \in X\), then \((0 \circ f a) \ast (0 \circ g a) = f(0) \ast g(0) = b \ast 0 = b \neq 0\) and \((b \circ g 0) \ast (b \circ f 0) = g(b) \ast f(b)\). \((b \ast 0 = b \neq 0\). Hence \(\circ f \ast \circ g \neq \circ 0 \neq \circ g \ast \circ f\), showing that \(\text{Bin}(X)\) is not a d-chain.

PROPOSITION 5.9. Let \((X; *, \circ, 0)\) be a d-algebra and \(*\) be a pogroupoid operation on \(X\). Then \(X\) is a d-chain if and only if \(x \ast y = y \ast x\) for all \(x, y \in X\).

Proof. Let \(X\) be a d-chain. If \(x \ast y = 0\), then \(x \ast y = y \ast x = y\). If \(x \ast y \neq 0\), then \(y \ast x = 0\), since \(X\) is a d-chain, and hence \(x \ast y = y \ast x = x\). Conversely, assume that there are \(x, y \in X\) such that \(x \ast y \neq 0 \neq y \ast x\). Then \(y = x \ast y = y \ast x = x\), a contradiction.

THEOREM 5.10. Let \((X; *, \circ, 0)\) be a companion d-algebra. If the companion operation is the pogroupoid operation, then the algebra \((X; *, 0)\) is a d-chain and companion operation is commutative.

Proof. Assume that \((X; *, 0)\) is not a d-chain. Then there are \(x, y \in X\) such that \(x \ast y \neq 0 \neq y \ast x\). This means that \(x \ast y = y\) and \(x \ast (x \ast y) = x \ast y \neq 0\). By Proposition 2.5-(ii), we have \(0 = x \ast (x \circ y) = x \ast (x \ast y)\), a contradiction. Hence, \((X; *, 0)\) is a d-chain. When \((X; *, 0)\) is a d-chain, at least one of \(x \ast y, y \ast x\) is zero, and hence by definition of \(*\), the companion operation is commutative.
COROLLARY 5.11. Let \((X; *, 0, 0)\) be a companion \(d\)-algebra. If the companion operation is the pogroupoid operation, then \((X; *, 0)\) is a \(d^*\)-algebra.

Proof. By Theorem 5.10, the situation \(x * y \neq 0, y * x \neq 0\) does not occur. If \(x * y = 0\), then \(x \circ y = x * y = y\) and hence \((y * x) * y = ((x \circ y) * x) * y = 0, (x * y) * x = 0 * x = 0\). The case \(y * x = 0\) is the same case to the above case. □

Consider the following example. Let \(X := \{0, a, b, c\}\) be a set with

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; *, 0)\) is a \(d\)-chain, but \((b * c) * b = a * b = b \neq 0\), i.e., \(X\) is not a \(d^*\)-algebra. Note that \(X\) is not a companion \(d\)-algebra, since \(a \circ c\) is not defined.

REFERENCES


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