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SEPARATING POINTS OF MEASURES ON EFFECT ALGEBRAS

ANNA AVALLONE

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ABSTRACT. We prove the existence of separating points for every countable family of nonatomic σ-additive modular measures on a σ-complete lattice ordered effect algebra.

Introduction

A question which has been studied in classical Measure Theory is to establish when a family $M$ of measures on a Boolean ring $R$ admits a separating point, i.e. a point $a \in R$ such that $\mu(a) \neq \nu(a)$ for all pair $(\mu, \nu)$ of different members of $M$. In particular, in [B-W] the authors proved that every countable family of atomless σ-additive measures on a σ-complete Boolean ring admits a dense $G_\delta$-set of separating points with respect to a suitable topology. In [A-B-C], it is proved that a similar result also holds for modular functions on orthomodular lattices. Then a natural question which arises is the validity of such result in the more general context of modular measures on effect algebras.

This paper is devoted to the answer, affirmative, to the previous question (see (3.1)).

The paper is organized as follows: in Section 2 we study some properties of modular measures on lattice ordered effect algebras and of their generated uniformities that are involved in the proof of the main result and in Section 3 we prove the main result.

We recall that effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 (see [B-F]) for modelling unsharp measurement in

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a quantum mechanical system. They are a generalization of many structures 
which arises in quantum physics (see [B-C]) and in Mathematical Economics (see 
[E-Z], [M], [P], [B-K]), in particular of orthomodular lattices in non-commutative 
measure theory and MV-algebras in fuzzy measure theory. For a study of effect 
geometries we refer to the bibliography of [D-P].

1. Preliminaries

An effect algebra \((L, \oplus, 0, 1)\) is a structure consisting of a set \(L\), two special 
elements 0 and 1, and a partially defined binary operation \(\oplus\) on \(L \times L\) satisfying 
the following conditions for every \(a, b, c \in L\):

1. If \(a \oplus b\) is defined, then \(b \oplus a\) is defined and \(a \oplus b = b \oplus a\).
2. If \(b \oplus c\) is defined and \(a \oplus (b \oplus c)\) are defined, then \(a \oplus b\) and \((a \oplus b) \oplus c\) 
   are defined and \(a \oplus (b \oplus c) = (a \oplus b) \oplus c\).
3. For every \(a \in L\), there exists a unique \(a^\perp \in L\) such that \(a \oplus a^\perp\) is 
   defined and \(a \oplus a^\perp = 1\).
4. If \(a \oplus 1\) is defined, then \(a = 0\).

In every effect algebra a dual operation \(\ominus\) to \(\oplus\) can be defined as follows: \(a \ominus c\) 
equals \(b\) if and only if \(b \oplus c\) exists and equals \(a\).

Moreover we can define a binary relation on \(L\) by \(a \leq b\) if and only if there 
exists \(c \in L\) such that \(c \oplus a = b\) and \(\leq\) is a partial ordering in \(L\) with 0 as 
smallest element. We say that two elements \(a, b \in L\) are orthogonal, and we 
write \(a \perp b\), if \(a \ominus b\) exists. Then \(a \perp b\) if and only if \(a \leq b^\perp\).

Moreover, for every \(a \in L\), we have \(a^\perp = 1 \ominus a\).

If \((L, \leq)\) is a lattice, we say that the effect algebra is a lattice ordered effect 
geometries or a D-lattice.

Effect algebras are a common generalization of orthomodular posets and 
MV-algebras. For a study, we refer to [D-P].

It is helpful to recall from [D-P] the following basic results.

**Proposition (1.1).** Assume \(a, b, c\) elements of an effect algebra \(L\).

1. If \(a \leq b \leq c\), then \(b \ominus a \leq c \ominus a\) and \((c \ominus a) \ominus (b \ominus a) = c \ominus b\).
2. \(a \ominus a = 0\), \(a \ominus 0 = a\) and \(a \ominus 0 = a\).
3. If \(a \leq b\), then \(b \ominus a = 0\) if and only if \(b = a\).
4. If \(a \leq b\), then \(b = a \ominus (b \ominus a)\).
5. If \(a \perp b\), then \(a \leq a \oplus b\) and \((a \oplus b) \ominus a = b\).
6. If \(a \leq b \leq c^\perp\), then \(a \ominus c \leq b \ominus c\).
7. If \(a \perp b\) and \(a \perp c\), then \(a \ominus b = a \ominus c\) if and only if \(b = c\).
8. If \(a \leq b \leq c\), then \(a \perp c \ominus b\) and \(a \ominus (c \ominus b) = c \ominus (b \ominus a)\).
9. If \(L\) is a D-lattice, \(a \leq c\) and \(b \leq c\), then \(c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)\).
In what follows, we denote by $L$ a D-lattice and by $(G, +)$ a topological Abelian group.

For $a, b \in L$ with $a \leq b$, we set $[a, b] = \{c \in L : a \leq c \leq b\}$. Moreover we set $\Delta = \{(a, b) \in L \times L : a = b\}$. We say that $L$ is $\sigma$-complete if every countable set in $L$ has supremum and infimum.

If $a_1, \ldots, a_n \in L$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, \ldots, a_n\}$ of $L$ is orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists. For a sequence $\{a_n\}$, we say that it is orthogonal if, for every positive integer $n$, $\bigoplus_{i \leq n} a_i$ exists. If, moreover, $\sup \bigoplus_{i \leq n} a_i$ exists, we set $\bigoplus_{i \leq n} a_i = \sup \bigoplus_{i \leq n} a_i$.

For $a, b \in L$, we set $a \Delta b = (a \lor b) \oplus (a \land b)$. By [D-P, (1.8.2)] we have $a \Delta b = ((a \lor b) \oplus a) \lor ((a \lor b) \oplus b)$. Hence, if $b \leq a$, we get $a \Delta b = a \oplus b$ since $a \land a = 0$.

A function $\mu : L \to G$ is said to be a measure if, for every $a, b \in L$, with $a \leq b$, $\mu(b \ominus a) = \mu(b) - \mu(a)$. It is easy to see that $\mu$ is a measure if and only if, for every $a, b \in L$, with $a \perp b$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. We say that $\mu$ is $\sigma$-additive if, for every orthogonal sequence $\{a_n\}$ in $L$ such that $\bigoplus_{n} a_n$ exists, $\mu\left(\bigoplus_{n} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$. $\mu$ is said to be modular if, for every $a, b \in L$, $\mu(a \lor b) + \mu(a \land b) = \mu(a) + \mu(b)$.

Every measure on an MV-algebra (and therefore every $T_\infty$-valuation on a clan of fuzzy sets in the sense of [B-K]) is a modular function, and every modular function on an orthomodular lattice is a measure.

### 2. D-uniformities and modular measures

An essential tool to obtain the main result of the paper is to use the topological structure generated by modular measures on D-lattices. For this reason the present section is devoted to the study of some properties of modular measures and of their generated uniformities, which are involved in the results of the next section.

By [F-T], every $G$-valued modular function $\mu$ on any lattice generates a lattice uniformity $\mathcal{U}(\mu)$, i.e. a uniformity which makes the lattice operations $\lor$ and $\land$ uniformly continuous. By [A-B, (3.2)], for a $G$-valued modular measure $\mu$ on a D-lattice $L$, $\mathcal{U}(\mu)$ is a D-uniformity, i.e. $\mathcal{U}(\mu)$ makes also the operation $\oplus$ and $\ominus$ of $L$ uniformly continuous, and a base of $\mathcal{U}(\mu)$ is the family of the sets $\{(a, b) \in L \times L : \mu([0, a \Delta b]) \subseteq W\}$, where $W$ is a neighbourhood of 0.
in $G$. Hence a base of neighbourhoods of 0 in $U(\mu)$ is the family consisting of the sets $\{a \in L : \mu([0,a]) \subseteq W\}$, where $W$ is a neighbourhood of 0 in $G$.

Moreover it is helpful to recall that, by [A-B, 4.1], a lattice uniformity $U$ on $L$ is a D-uniformity if and only if, for every $U \in U$, there exists $V \in U$ such that $V \ominus \Delta \subseteq U$ and $\Delta \ominus V \subseteq U$. Moreover, if $U$ is a D-uniformity, then for every $U \in U$ there exists $V \in U$ such that $V \oplus \Delta \subseteq U$. By [A-V, (2.4)], a base of $U$ is the family of sets

$$F^\Delta = \{(a,b) \in L \times L : a\Delta b \in F\},$$

where $F$ is a neighbourhood of 0 in $U$.

**Proposition (2.1).** If $\mu : L \to G$ is a modular measure and $U$ is a D-uniformity, then the following conditions are equivalent:

1. $\mu$ is uniformly continuous.
2. $\mu$ is continuous.
3. $U(\mu) \subseteq U$.

**Proof.** The equivalence of (1) and (3) holds because $U(\mu)$ is the weakest lattice uniformity which makes $\mu$ uniformly continuous (see [W2, 3.1]).

The equivalence of (1) and (2) has been proved in [A, 3.5].

A lattice uniformity $U$ on $L$ is said to be exhaustive if every monotone sequence in $L$ is a Cauchy sequence in $U$. A $G$-valued modular function $\mu$ on $L$ is said to be exhaustive if $U(\mu)$ is exhaustive.

**Proposition (2.2).** Let $U$ be a D-uniformity. Then the following conditions are equivalent:

1. $U$ is exhaustive.
2. Every increasing sequence in $L$ is Cauchy in $U$.
3. Every orthogonal sequence in $L$ converges to 0 in $U$.

**Proof.**

(1) is trivial.

(2) $\implies$ (3): Let $\{a_n\}$ be an orthogonal sequence in $L$ and $F$ a neighbourhood of 0 in $U$. For each positive integer $n$, set $b_n = \bigoplus_{i \leq n} a_i$. By (1.1)-5, $\{b_n\}$ is an increasing sequence and therefore $b_n \to 0$ in $U$. By (2.1), we can find a positive integer $\nu$ such that, for each $n > \nu$, $b_{n+1} \ominus b_n = b_{n+1} \Delta b_n \in F$. By (1.1)-4 we get $b_{n+1} = b_n \ominus (b_{n+1} \ominus b_n)$. On the other hand, we have $b_{n+1} = b_n \ominus a_n$. Therefore, by (1.1)-7, we obtain $a_n = b_{n+1} \ominus b_n \in F$ for each $n > \nu$. Hence $\{a_n\}$ converges to 0 in $U$. 

132
(3) \implies (1):

(i) As first step, we prove that, for every increasing sequence \( \{a_n\} \) in \( L \) and for every \( U \in \mathcal{U} \), there exists a positive integer \( \nu \) such that, for every \( n > \nu \), \( (a_n, a_{n+1}) \in U \).

Let \( \{a_n\} \) be an increasing sequence in \( L \) and \( U \in \mathcal{U} \). By (2.1), we can choose a neighbourhood \( F \) of 0 in \( U \) such that \( F^\Delta \subseteq U \). For each positive integer \( n \), set \( b_n = a_{n+1} \ominus a_n \). By [A-B, 2.4], \( \{b_n\} \) is an orthogonal sequence in \( L \). By (3), we can find a positive integer \( \nu \) such that, for each \( n > \nu \), \( b_n \in F \). Therefore we get \( (a_n, a_{n+1}) \in F^\Delta \subseteq U \).

(ii) We prove that every increasing sequence in \( L \) is Cauchy in \( U \).

Let \( \{a_n\} \) be an increasing sequence in \( L \) and suppose that \( \{a_n\} \) is not Cauchy. Then we can inductively obtain subsequences \( \{b_n\} \) and \( \{c_n\} \) of \( \{a_n\} \) and \( U \in \mathcal{U} \) such that, for each \( n \), \( b_n \leq c_n \leq b_{n+1} \) and \( (b_n, c_n) \notin U \). Set \( \{d_1, \ldots, d_n\} = \{b_1, c_1, b_2, c_2, \ldots\} \). Then \( \{d_n\} \) is an increasing sequence in \( L \) and, for each \( n \), we can find \( m > n \) such that \( (d_m, d_{m+1}) \notin U \), a contradiction with (i).

(iii) We prove that every decreasing sequence is Cauchy in \( U \).

Let \( \{a_n\} \) be a decreasing sequence in \( L \). Then \( \{a_n^\perp\} \) is an increasing sequence in \( L \). By (ii), \( \{a_n^\perp\} \) is Cauchy in \( U \). Let \( U, V \in \mathcal{U} \) be such that \( \Delta \ominus V \subseteq U \) and choose a positive integer \( \nu \) such that, for each \( n > \nu \), \( (a_n^\perp, a_{n+1}^\perp) \in V \). Then, for each \( n > \nu \), we obtain \( (a_n, a_{n+1}) = (1, 1) \ominus (a_n^\perp, a_{n+1}^\perp) \in \Delta \ominus V \subseteq U \). \( \square \)

**Corollary (2.3).** Let \( \mu \colon L \to G \) be a modular measure. Then \( \mu \) is exhaustive if and only if, for every orthogonal sequence \( \{a_n\} \) in \( L \), \( \{\mu(a_n)\} \) converges to 0 in \( \mathcal{U} \).

**Proof.** It is trivial that the condition is necessary since \( \mu \) is continuous with respect to \( \mathcal{U}(\mu) \).

Conversely, suppose that \( \mu(a_n) \to 0 \) for every orthogonal sequence \( \{a_n\} \) in \( L \). Suppose that there exists an orthogonal sequence \( \{a_n\} \) in \( L \) which does not converge to 0 in \( \mathcal{U}(\mu) \). Then we can find a neighbourhood \( W \) of 0 in \( G \) such that, for each positive integer \( \nu \), there exists \( n > \nu \) such that \( \mu([0, a_n]) \) is not contained in \( W \). Hence we can obtain an orthogonal sequence \( \{b_n\} \) in \( L \) with \( \mu(b_n) \notin W \) for each \( n \), a contradiction. Then \( \mu \) is exhaustive by (2.2). \( \square \)

A lattice uniformity \( \mathcal{U} \) is said to be **nonatomic** if, for every \( a \in L \) and every neighbourhood \( F \) of 0 in \( \mathcal{U} \), there exists an orthogonal finite family \( \{a_1, \ldots, a_n\} \) in \( L \) with \( \bigoplus_{i \leq n} a_i = a \) and \( a_i \in F \) for each \( i \leq n \).

A modular function \( \mu \colon L \to G \) is said to be nonatomic if \( \mathcal{U}(\mu) \) is nonatomic.
PROPOSITION (2.4). Let $U$ be a $D$-uniformity. Then the following conditions are equivalent:

(1) $U$ is nonatomic.
(2) For every neighbourhood $F$ of 0 in $U$, there exists an orthogonal finite family $\{a_1, \ldots, a_n\}$ in $L$ such that $\bigoplus_{i<n} a_i = 1$ and $a_i \in F$ for each $i \leq n$.
(3) $U$ is chained, i.e. for every $a, b \in L$ with $a \leq b$ and every $U \in U$, there exist $a_1, \ldots, a_{n-1}$ in $L$ such that $a = a_0 \leq a_1 \leq \cdots \leq a_n = b$ and $(a_{i-1}, a_i) \in U$ for each $i \leq n$.

Proof.

(1) $\implies$ (2) is trivial. 

(2) $\implies$ (3): 

(i) As first step, we prove (3) with $a = 0$ and $b = 1$. Let $V, V' \in U$ be such that $\Delta \oplus V \subseteq U$. Choose an orthogonal family $\{a_1, \ldots, a_n\}$ in $L$ with $\bigoplus_{i<n} a_i = 1$ and $(0, a_i) \in V$ for each $i \leq n$. Set $b_0 = 0$ and, for each $i \in \{1, \ldots, n\}$, $b_i = \bigoplus_{j<i} a_j$. Then we have $0 = b_0 \leq b_1 \leq \cdots \leq b_n = 1$ and, for each $i \in \{1, \ldots, n\}$,

$$(b_{i-1}, b_i) = \left( \bigoplus_{i<n-1} a_i, \bigoplus_{i<n-1} a_i \right) \oplus (0, a_i) \in \Delta \oplus V \subseteq U.$$ 

(ii) Now we prove (3). Let $a, b \in L$ with $a \leq b$ and $U \in U$. Choose $V, V' \in U$ with $V' \wedge \Delta \subseteq V$ and $V \vee \Delta \subseteq U$. By (i), we can find $b_1, \ldots, b_{n-1}$ in $L$ with $0 = b_0 \leq b_1 \leq \cdots \leq b_{n-1} = 1$ and $(b_{i-1}, b_i) \in V'$ for each $i \leq n$. Set, for each $i \leq n$, $c_i = (b_i \wedge b) \vee a$. Then we have $a = c_0 \leq c_1 \leq \cdots \leq c_n = b$ and, for each $i \leq n$,

$$(c_{i-1}, c_i) = (b_{i-1} \wedge b, b_i \wedge b) \vee (a, a) \in (V' \wedge \Delta) \vee \Delta \subseteq V \vee \Delta \subseteq U.$$ 

(3) $\implies$ (1): Let $a \in L$ and $F$ be a neighbourhood of 0 in $U$. Choose $U, V \in U$ such that $V \ominus \Delta \subseteq U$ and $U(0) \subseteq F$. Choose $a_1, \ldots, a_{n-1}$ in $L$ such that $0 = a_0 \leq a_1 \leq \cdots \leq a_n = a$, and $(a_{i-1}, a_i) \in V$ for each $i \in \{1, \ldots, n\}$. Set $b_0 = 0$ and, for each $i \in \{1, \ldots, n\}$, $b_i = a_i \ominus a_{i-1}$. By [A-B, (2.1)], $\{b_0, \ldots, b_n\}$ is orthogonal and $b_0 \oplus \cdots \oplus b_n = a$. Moreover we have, for each $i \leq n$, $(0, b_i) = (a_{i-1}, a_i) \ominus (a_{i-1}, a_i) \in V \ominus \Delta \subseteq U$, from which $b_i \in F$. □

We recall that, for a $G$-valued modular measure $\mu$ on $L$, a base of neighbourhoods of 0 in $U(\mu)$ is the family of sets $\{a \in L : \mu([0,a]) \subset W\}$, where $W$ is a neighbourhood of 0 in $G$. From this fact and (2.4), we immediately obtain the following result (see also [B, 2.6]).
COROLLARY (2.5). Let \( \mu: L \to G \) be a modular measure. Then the following conditions are equivalent:

1. \( \mu \) is nonatomic.
2. For every \( a \in L \) and every neighbourhood \( W \) of 0 in \( G \), there exists an orthogonal family \( \{ a_1, \ldots, a_n \} \) in \( L \) such that \( \bigoplus_{i \leq n} a_i = a \) and \( \mu([0,a_i]) \subseteq W \) for each \( i \leq n \).
3. For every neighbourhood \( W \) of 0 in \( G \), there exists an orthogonal family \( \{ a_1, \ldots, a_n \} \) in \( L \) such that \( 0 \leq a_i = 1 \) and \( \mu([0,a_i]) \subseteq W \) for each \( i \leq n \).

A modular function \( \mu: L \to G \) is said to be atomless if, for every \( a \in L \) with \( \mu(a) \neq 0 \), there exists \( b < a \) such that \( \mu(b) \neq 0 \) and \( \mu(b) \neq \mu(a) \).

We prove that, as for measures on Boolean algebras, nonatomic and atomless are equivalent for \( \sigma \)-additive modular measures.

We need the following definitions and result.

A lattice \( L' \) is said to be dense in itself if, for every \( a, b \in L' \) with \( a < b \), there exists \( c \in L' \) with \( a < c < b \), and atomless if, for every \( a \in L' \) with \( a \neq 0 \), there exists \( b \in L' \) with \( 0 < b < a \).

LEMMA (2.6). Let \( L' \) be an effect algebra. Then \( L' \) is dense in itself if and only if \( L' \) is atomless.

Proof. If \( L' \) is dense in itself, then it is trivial that \( L' \) is atomless.

Conversely, suppose that \( L' \) is atomless. Let \( a, b \in L' \) with \( a < b \), and set \( c = b \ominus a \). By (1.1)-3, \( c \neq 0 \). Choose \( d \in L' \) with \( 0 < d < c \). Then \( d \perp a \) by (1.1)-1. Moreover, by (1.1)-5 and 7, we get \( d \ominus a \geq a \) and \( d \ominus a \neq a \) and, by (1.1)-4, we have \( b = a \ominus c \). Therefore \( d \ominus a \leq b \) by (1.1)-6, and \( d \ominus a \neq b \), otherwise we would have \( c = b \ominus a = (d \ominus a) \ominus a = d \) by (1.1)-5. Hence \( L' \) is dense in itself. \( \square \)

In the next result we use that, by [A-B, 2.4], a modular measure \( \mu: L \to G \) is \( \sigma \)-additive if and only if \( \mu \) is \( \sigma \)-order-continuous (\( \sigma \)-o.c.), i.e., for every increasing (decreasing) sequence \( \{ a_n \} \) in \( L \) such that \( a = \sup a_n \) (\( a = \inf a_n \)) exists, \( \mu(a_n) \to \mu(a) \).

PROPOSITION (2.7). Suppose that \( G \) is metrizable, \( L \) is \( \sigma \)-complete and \( \mu: L \to G \) is a \( \sigma \)-additive modular measure. Then \( \mu \) is nonatomic if and only if \( \mu \) is atomless.
Proof. Suppose that $\mu$ is nonatomic. Let $a \in L$ with $\mu(a) \neq 0$. Since $G$ is Hausdorff, we can find a neighbourhood $W$ of 0 in $G$ such that $\mu(a) \not\in W$. Choose an orthogonal family $\{a_1, \ldots, a_n\}$ with $\bigoplus_{i \leq n} a_i = a$ and $\mu([0, a_i]) \subseteq W$ for each $i \leq n$. Since $\mu(a) = \sum_{i \leq n} \mu(a_i)$, we can find $j \leq n$ with $\mu(a_j) \neq 0$. Since $\mu(a_j) \in W$, we get $\mu(a_j) \neq \mu(a)$.

Now suppose that $\mu$ is atomless. By (2.4) we have to prove that $\mathcal{U}(\mu)$ is chained. Set $\hat{L} = L/N(\mu)$ and denote by $\hat{\mathcal{U}}$ the quotient uniformity generated by $\mathcal{U}(\mu)$. By [W2, 5.8], if $(\hat{L}, \hat{\mathcal{U}})$ is complete and $\hat{\mathcal{U}}$ is exhaustive, then $\mathcal{U}(\mu)$ is chained if and only if $\hat{L}$ is dense in itself. Now, since $\mu$ is atomless, $\hat{L}$ is atomless and therefore dense in itself by (2.6). Moreover, since $\mu$ is $\sigma$-o.c. by [A-B, (2.4)] and $L$ is $\sigma$-complete, then $\mathcal{U}(\mu)$ is exhaustive by [W2, 3.5] and [W1, 8.1.2]. Finally, since $\hat{\mathcal{U}}$ has a countable base because $G$ is metrizable, by [W1, 8.1.4, 6.3] we obtain that $(\hat{L}, \hat{\mathcal{U}})$ is complete. Therefore $\mathcal{U}(\mu)$ is chained, i.e. $\mu$ is nonatomic. □

**Proposition (2.8).** Let $\{\mathcal{U}_\alpha : \alpha \in A\}$ be a family of nonatomic lattice uniformities on $L$. Then $\mathcal{U} = \sup_{\alpha \in A} \mathcal{U}_\alpha$ is nonatomic.

Proof. Let $a \in L$ and $F$ a neighbourhood of 0 in $\mathcal{U}$. Choose $U \in \mathcal{U}$, $\alpha_1, \ldots, \alpha_n \in A$ and $U_i \in \mathcal{U}_{\alpha_i}$ such that $U(0) \subseteq F$ and $\bigcap_{i \leq n} U_i \subseteq U$. By [W1, 1.1.3], for each $i \leq n$ we can find $V_i \in \mathcal{U}_{\alpha_i}$ such that $V_i \subseteq U_i$ and $[a \wedge b, a \lor b] \times [a \wedge b, a \lor b] \subseteq V_i$ whenever $(a, b) \in V_i$. (*)

We prove by induction the existence of an orthogonal family $\{a_1, \ldots, a_s\}$ in $\bigcap_{i \leq n} V_i(0) \subseteq U(0) \subseteq F$ with $a = \bigoplus_{i \leq r} a_i$.

By induction, suppose that there exists an orthogonal family $\{a_1, \ldots, a_s\}$ in $\bigcap_{i \leq n-1} V_i(0)$ with $a = \bigoplus_{i \leq s} a_i$. For each $i \leq s$, we can find an orthogonal family $\{b_{j_i} : j \leq r_i\}$ in $V_n(0)$ with $\bigoplus_{j \leq r_i} b_{j_i} = a_i$. Since $b_{j_i} \leq a_i$, by (*) we get $b_{j_i} \in \bigcup_{j \leq n} V_j(0)$ and, if we denote by $c_1, \ldots, c_t$ the elements of $\{b_{j_i} : j \leq r_i, i \leq s\}$, we obtain as before that $\bigoplus_{i \leq t} c_i$ exists and $\bigoplus_{i \leq s} c_i = a$. Hence $\mathcal{U}$ is nonatomic. □

**Corollary (2.9).** If $\mu : L \rightarrow G$ and $\nu : L \rightarrow G$ are nonatomic modular measures, then $\lambda = \mu + \nu$ is nonatomic, too.

Proof. Recall that, for a modular measure $\mu$, a base of $\mathcal{U}(\mu)$ is the family consisting of the sets $\{(a, b) \in L \times L : \mu([0, a \Delta b]) \subseteq W\}$, where $W$ is a neighbourhood of 0 in $G$. From this, we immediately obtain that $\mathcal{U}(\lambda) \subseteq \mathcal{U}(\mu) \lor \mathcal{U}(\nu)$. By (2.8), we obtain that $\mathcal{U}(\lambda)$ is nonatomic and then $\lambda$ is nonatomic, too. □
3. Separating points of modular measures

If $M$ is a family of $G$-valued measures on $L$, we say that an element $a \in L$ is a \textit{separating point} for $M$ if $\mu(a) \neq \nu(a)$ for every $\mu, \nu \in M$ with $\mu \neq \nu$.

The aim of this section is to prove the following result.

**Theorem (3.1).** Let $M$ be a family of $G$-valued nonatomic modular measures on $L$. Suppose that $M$ is finite, or that $M$ is countable, $L$ is $\sigma$-complete and every $\mu \in M$ is $\sigma$-additive. Then the set of the separating points of $M$ is a dense $G_\delta$-set with respect to $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \mu \in M\}$ (and therefore is non-empty).

To prove (3.1), we need the following notations and results.

For a modular measure $\mu : L \to G$, we set

$$I(\mu) = \{a \in L : \mu([0,a]) = \{0\}\},$$

$$N(\mu) = \{(a,b) \in L \times L : \mu([0, a \wedge b]) = \{0\}\}.$$

By [A-B, (4.2)], we have $N(\mu) = \bigcap\{U : U \in \mathcal{U}(\mu)\}$ and $I(\mu) = \bigcap\{F : F$ neighbourhood of $0$ in $\mathcal{U}(\mu)\}$ (see notations before (2.1)). Moreover, by [W2, 2.5, 3.1], we also have $N(\mu) = \{(a,b) \in L \times L : \mu$ is constant on $[a \wedge b, a \vee b]\}$.

**Proposition (3.2).** Let $\mu : L \to G$ be a modular measure and $\mathcal{U}$ a $D$-uniformity. Then the following conditions are equivalent:

1. The set $\{a \in L : \mu(a) \neq 0\}$ is dense in $(L, \mathcal{U})$.
2. $I(\mu)$ is not open in $(L, \mathcal{U})$.

\textbf{Proof.} Set $I_0 = \{a \in L : \mu(a) = 0\}$.

(1) $\Rightarrow$ (2): Suppose that $I(\mu)$ is open. We prove that $I_0$ is open, a contradiction with (1).

Let $a \in I_0$ and set $U_a = \{b \in L : a \wedge b \in I(\mu)\}$. Since $I(\mu)$ is open in $(L, \mathcal{U})$, by [A-V, (2.4)] we get that $U_a$ is a neighbourhood of $a$ in $\mathcal{U}$. We prove that $U_a \subseteq I_0$. Let $b \in U_a$ and $U, V \in \mathcal{U}(\mu)$ be such that $\Delta \oplus V \subseteq U$. By (1.1)-4, we have $a \vee b = (a \wedge b) \oplus (a \Delta b)$. Then, since $a \Delta b \in I(\mu) = \bigcap\{F : F$ neighbourhood of $0$ in $\mathcal{U}(\mu)\}$, we get

$$(a \wedge b, a \vee b) = (a \wedge b, a \wedge b) \oplus (0, a \Delta b) \in \Delta \oplus V \subseteq U.$$ 

Since $U \in \mathcal{U}(\mu)$ is arbitrary, we get $(a \wedge b, a \vee b) \in N(\mu)$. Then $\mu$ is constant on $[a \wedge b, a \vee b]$. In particular, we have $\mu(b) = \mu(a) = 0$, since $a \in I_0$. Therefore $b \in I_0$.

(2) $\Rightarrow$ (1): Let $U \in \mathcal{U}$. Choose symmetric $V, V', V'' \in \mathcal{U}$ such that $V \subseteq V' \subseteq V''$ $V' \vee \Delta \subseteq U$, $V'' \oplus \Delta \subseteq U$ and $[a \wedge b, a \vee b] \times [a \wedge b, a \vee b] \subseteq V$ whenever $(a, b) \in V$ (see [W1, 1.1.3]).
(i) As first step, we prove that there exists \( d \in V(0) \) such that \( \mu(d) \neq 0 \).

Suppose that \( \mu(d) = 0 \) for each \( d \in V(0) \). By the choice of \( V \), we get \([0, d] \subseteq V(0)\) for each \( d \in V(0) \) and therefore \( V(0) \subseteq I(\mu) \). We prove that \( I(\mu) \) is open in \((L, \mathcal{U})\), a contradiction with (2).

Let \( a \in I(\mu) \) and set \( U_a = \{ b \in L : a \Delta b \in V(0) \} \). By [A-V, (2.4)], \( U_a \) is a neighbourhood of \( a \) in \( \mathcal{U} \). We prove that \( U_a \subseteq I(\mu) \).

Let \( b \in U_a \) and \( c \leq b \). Since \( a \Delta b = ((a \lor b) \ominus a) \lor ((a \lor b) \ominus b) \) and \( a \Delta b \in V(0) \subseteq I(\mu) \), we get \((a \lor b) \ominus a \in I(\mu) \). Since \((a \lor b) \ominus a \leq (a \lor b) \ominus a \), we have \( \mu((a \lor c) \ominus a) = 0 \). On the other hand, \( \mu((a \lor c) \ominus a) = \mu(a \lor c) - \mu(a) = \mu(a) + \mu(c) - \mu(a \land c) - \mu(a) = \mu(c) - \mu(a \land c) = 0 \), since \( a \in I(\mu) \). Since \( c \leq b \) is arbitrary, we get \( b \in I(\mu) \).

(ii) Now we prove that \( L \setminus I_0 \) is dense in \((L, \mathcal{U})\). Let \( a \in I_0 \). By (i), we can choose \( d \in V(0) \) such that \( \mu(d) \neq 0 \). Set \( b = (a \lor d) \ominus a \). Then \( \mu(b) = \mu(a \lor d) - \mu(a) = \mu(d) - \mu(a \land d) \). Since \( \mu(d) \neq 0 \), we get either
\[
\mu(b) \neq 0 \quad \text{or} \quad \mu(a \land d) \neq 0.
\]

Now by (1.1)-8 we get that \( a \perp b \) and \( a \oplus b = (a \lor d) \ominus (a \ominus a) = a \lor d \).

Hence \( \mu(a \lor d) = \mu(a \oplus b) = \mu(a) + \mu(b) = \mu(b) \) since \( a \in I_0 \). Moreover, if we set \( c = a \ominus (a \land d) \), we have \( \mu(c) = \mu(a) - \mu(a \land d) = -\mu(a \land d) \). By (*) we get either \( a \lor d \notin I_0 \) or \( c \notin I_0 \). On the other hand, since \([0, d] \subseteq V(0) \subseteq V'(0) \subseteq V''(0)\), we have \((a, a \lor d) \in \Delta \lor V' \subseteq U \) and, by (1.1)-4, \((a, c) = (a \land d, 0) \oplus (c, c) \in V'' + \Delta \subseteq U \). Then, in any case, we obtain an element \( e \notin I_0 \) such that \( (a, e) \in U \).

\[ \square \]

**Corollary (3.3).** Let \( \mathcal{U} \) be a \( D \)-uniformity on \( L \) and \( M \) a family of \( G \)-valued \( \mathcal{U} \)-continuous modular measures on \( L \). Then the following assertions are true:

1. If \( M \) is finite, then \( A_M = \{ a \in L : \mu(a) \neq 0 \} \) is a dense open set in \((L, U)\) if and only if, for every \( \mu \in M \), \( I(\mu) \) is not open in \((L, U)\).

2. If \( M \) is countable and \((L, U)\) is a Baire space, then \( A_M \) is a dense \( G_\delta^* \)-set in \((L, U)\) if and only if, for every \( \mu \in M \), \( I(\mu) \) is not open in \((L, U)\).

**Proof.** First suppose that \( A_M \) is dense in \((L, U)\). Then, for every \( \mu \in M \), the set \( A_\mu = \{ a \in L : \mu(a) \neq 0 \} \) is dense in \((L, U)\). Therefore, by (3.2), \( I(\mu) \) is not open in \((L, U)\).

Conversely, suppose that, for every \( \mu \in M \), \( I(\mu) \) is not open in \((L, U)\). Then, by (3.2), for every \( \mu \in M \), \( A_\mu \) is dense in \((L, U)\). Moreover, for every \( \mu \in M \), \( A_\mu \) is open in \( \mathcal{U}(\mu) \) since \( \mu \) is continuous with respect to \( \mathcal{U}(\mu) \), and therefore \( A_\mu \) is also open in \( \mathcal{U} \) by (2.1). Then, if \( M \) is finite, \( A_M \) is a finite intersection of dense open sets and therefore is a dense open set in \((L, U)\). If \( M \) is countable,
then $A_M$ is a countable intersection of dense open sets, hence is open and dense in $(L,\mathcal{U})$, because $(L,\mathcal{U})$ is a Baire space.

**Theorem (3.4).** Let $M$ be a family of $G$-valued nonatomic non-zero modular measures on $L$. Then the following assertions are true:

1. If $M$ is finite, then the set $A_M = \{ a \in L : \mu(a) \neq 0 \text{ for every } \mu \in M \}$ is a dense open set with respect to $\mathcal{U} = \sup \{ \mathcal{U}(\mu) : \mu \in M \}$.
2. If $M$ is countable, $L$ is $\sigma$-complete and every $\mu \in M$ is $\sigma$-additive, then $A_M$ is a dense $G_\delta$-set with respect to $\mathcal{U} = \sup \{ \mathcal{U}(\mu) : \mu \in M \}$.

**Proof.** We prove that, for every $\mu \in M$, $I(\mu)$ is not open in $(L,\mathcal{U})$.

Suppose that there exists $\mu \in M$ such that $I(\mu)$ is open in $(L,\mathcal{U})$. Let $a \in L$. Since $\mathcal{U}$ is nonatomic by (2.8), we can find an orthogonal family $\{a_1, \ldots, a_n\}$ in $I(\mu)$ with $a = \bigoplus a_i$. Therefore $\mu(a) = \sum_{i=1}^{n} \mu(a_i) = 0$. Since $a$ is arbitrary, we get $\mu = 0$, a contradiction. Now the conclusion follows by (3.3), since, if $L$ is $\sigma$-complete and every $\mu \in M$ is $\sigma$-additive, then $(L,\mathcal{U})$ is a Baire space by [W2, 3.5] and [W1, 3.1.5].

Now we can prove (3.1).

**Proof of (3.1).** We apply (3.4) to the family $M' = \{ \mu - \nu : \mu, \nu \in M, \mu \neq \nu \}$, recalling that the elements of $M'$ are nonatomic by (2.9).

**References**


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