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VECTOR VALUED PARANORMED
STATISTICALLY CONVERGENT
DOUBLE SEQUENCE SPACES

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ABSTRACT. In this article we introduce the vector valued paranormed sequence spaces $2c(q,p)$, $2c_0(q,p)$, $(2c)^B(q,p)$, $(2c_0)^B(q,p)$, $(2c)^R(q,p)$ and $(2c_0)^R(q,p)$ defined over a seminormed space $(X,q)$. We study their different properties like completeness, solidness, symmetry, convergence freeness etc. We prove some inclusion results.

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1. Introduction

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast [2] and Schoenberg [11] independently. Later on it was further investigated by Fridy and Orhan [3], Salát [10], Rath and Tripathy [9], Tripathy [13], Tripathy and Sen [15] and many others. The idea depends on the notion of density of subsets of $\mathbb{N}$. Throughout the paper, $\chi_E$ denotes the characteristic function of $E$. A subset $E$ of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists.

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Throughout the paper, \( w(q), \ell_\infty(q), c(q), c_0(q), c(q), c_0(q) \) denote the class of all, bounded, convergent, null, statistically convergent and statistically null \( X \)-valued sequence spaces respectively, where \( X \) is a seminormed space, seminormed by \( q \).

A sequence \((x_k) \in \mathcal{C}(q)\) if for every \( \varepsilon > 0 \), there exists \( L \in X \) such that 
\[
\delta\left(\{ k \in \mathbb{N} : q(x_k - L) \geq \varepsilon \}\right) = 0.
\]

We write stat-lim \( x_k = L \).

Two sequences \((x_k)\) and \((y_k)\) are said to be equal for allmost all \( k \) (in short a.a.k.) if 
\[
\delta\left(\{ k \in \mathbb{N} : x_k \neq y_k \}\right) = 0.
\]

The studies on paranormed sequences were initiated by Nakano [7] and Simmons [12] at the initial stage. Later on they were studied by Maddox [5], Nanda [7], Tripathy and Sen [15] and many others.

Let \( p = (p_k) \) be a sequence of positive real numbers and \( H = \sup_k p_k < \infty \). Then for \((a_k)\) and \((b_k)\) two sequences of complex terms, we have the following well known inequality
\[
|a_k + b_k|^{p_k} \leq C\left(|a_k|^{p_k} + |b_k|^{p_k}\right), \quad \text{where} \quad C = \max(1, 2^{H-1}).
\]

On generalizing the sequence space \( c \) and \( c_0 \), Tripathy and Sen [15] introduced the following sequence spaces of complex terms:
\[
c(p) = \left\{(x_k) \in w : \text{stat-lim} |x_k - L|^{p_k} = 0 \right\} \quad \text{for some} \quad L \in C
\]
and
\[
c_0(p) = \left\{(x_k) \in w : \text{stat-lim} |x_k|^{p_k} = 0 \right\}.
\]

The spaces \( \ell_\infty(p), c(p), c_0(p), c(p) \cap \ell_\infty(p) \) and \( c_0(p) \cap \ell_\infty(p) \) are paranormed by 
\[
g(x) = \sup_k |x_k|^{p_k}, \quad \text{where} \quad M = \max(1, H).
\]

2. Definitions and preliminaries

Some works on double sequences is done by Hardy [4] and Moricz [6], Tripathy [14] and others. A double sequence \( (a_{nk}) \) is said to be convergent to \( L \) in Pringsheim’s sense if
\[
\lim_{n,k \to \infty} a_{nk} = L,
\]
where \( n \) and \( k \) tend to \( \infty \) independent of each other. The notion of regular convergence for double sequence was introduced by Hardy [4]. A double sequence \( (a_{nk}) \) is said to be regular convergent if it converges in the Pringsheim’s sense and the following limits exist.
\[
\lim_{n \to \infty} a_{nk} - L_k \quad \text{for each} \quad k \in \mathbb{N}
\]
and
\[
\lim_{k \to \infty} a_{nk} = J_n \quad \text{for each} \quad n \in \mathbb{N}.
\]
The notion of asymptotic density for subsets of $\mathbb{N} \times \mathbb{N}$ was introduced by Tripathy [14]. A subset $E$ of $\mathbb{N} \times \mathbb{N}$ is said to have density $\rho(E)$ if

$$
\rho(E) = \lim_{p,q \to \infty} \frac{1}{pq} \sum_{n \leq p, k \leq q} \chi_E(n, k)
$$

exists.

Tripathy [14] introduced the notion of statistically convergent double sequences. A double sequence $(a_{nk})$ is said to be statistically convergent to $L$ in Pringsheim’s sense if for every $\varepsilon > 0$, $\rho\{((n,k) : |a_{nk} - L| \geq \varepsilon)\} = 0$.

A double sequence $(a_{nk})$ is said to be regularly statistically convergent if it is statistically convergent in the Pringsheim’s sense and the following statistical limits exist

$$
\text{stat- lim}_{n \to \infty} a_{nk} = L_k \quad \text{for each} \quad k \in \mathbb{N}
$$

and

$$
\text{stat- lim}_{k \to \infty} a_{nk} = J_n \quad \text{for each} \quad n \in \mathbb{N}.
$$

Throughout the article $2w(q)$, $2\ell(q)$, $2c(q)$, $2c_0(q)$, $2c^R(q)$, $2c_0^R(q)$, $2c_0^R(q)$, $2c_B(q)$, $2c_0^B(q)$, $2c(q)$, $(2c_0^R(q))$, $(2c_0)^R(q)$, $(2c)^B(q)$, $(2c_0)^B(q)$, $(2c_0)^B(q)$ denote the spaces of all, bounded, convergent in Pringsheim’s sense, null in Pringsheim’s sense, regularly convergent, regularly null, bounded and convergent in Pringsheim’s sense, bounded and null in Pringsheim’s sense, statistically convergent in Pringsheim’s sense, statistically null in Pringsheim’s sense, regularly statistically convergent, regularly statistically null, bounded regularly convergent, bounded regularly null, bounded statistically convergent in Pringsheim’s sense, bounded null in Pringsheim’s sense $X$-valued double sequences respectively, where $X$ is a seminormed space, seminormed by $q$.

Let $(a_{nk})$ and $(b_{nk})$ be two double sequences, then we say that $a_{nk} = b_{nk}$ for almost all $n$ and $k$ (in short a.a.n & k) if $\rho\{((n,k) : a_{nk} \neq b_{nk})\} = 0$.

Let $p = (p_{nk})$ be a double sequence of positive real numbers. The notion of paranormed double sequences was introduced by Colak and Turkmenoglu [1] and further investigated by Turkmenoglu [16].

A double sequence space $E$ is said to be solid if $\langle a_{nk}\alpha_{nk} \rangle \in E$, whenever $\langle a_{nk} \rangle \in E$ and for all sequences $\langle \alpha_{nk} \rangle$ of scalars with $|\alpha_{nk}| \leq 1$ for all $n,k \in \mathbb{N}$.

A double sequence space $E$ is said to be symmetric if $\langle a_{\pi(n)\pi(k)} \rangle \in E$, whenever $\langle a_{nk} \rangle \in E$, where $\pi(n)$, $\pi(k)$ are permutations of $\mathbb{N}$.

A double sequence space $E$ is said to be monotone if it contains the canonical preimages of all its step spaces.

181
A double sequence space $E$ is said to be convergence free if $\langle b_{nk} \rangle \in E$, whenever $\langle a_{nk} \rangle \in E$ and $b_{nk} = \theta$, whenever $a_{nk} = \theta$, where $\theta$ is the zero element of $X$.

The zero double sequence is denoted by $2\theta = \langle \theta \rangle$ and the zero single sequence by $\theta = (\theta, \theta, \theta, \ldots)$. We introduce the following paranormed double sequence spaces.

$$2(c)(q,p) = \{\langle a_{nk} \rangle \in 2w : \text{stat-lim}(q(a_{nk} - L))^{p_{nk}} = 0 \text{ for some } L \in X \},$$

$$2(\tilde{c}_0)(q,p) = \{\langle a_{nk} \rangle \in 2w : \text{stat-lim}(q(a_{nk}))^{p_{nk}} = 0 \},$$

$\langle a_{nk} \rangle \in (2\tilde{c})^R(q,p)$ if $\langle a_{nk} \rangle \in (2\tilde{c})(q,p)$ and the following statistical limits hold:

$$\text{stat- lim}_{n \to \infty} (q(a_{nk} - L_k))^{p_{nk}} = 0 \text{ for each } k \in \mathbb{N}, \quad (1)$$

$$\text{stat- lim}_{k \to \infty} (q(a_{nk} - J_n))^{p_{nk}} = 0 \text{ for each } n \in \mathbb{N}. \quad (2)$$

We have $\langle a_{nk} \rangle \in (2\tilde{c}_0)^R(q,p)$ if $\langle a_{nk} \rangle \in 2\tilde{c}_0(q,p) \cap 2\ell_{\infty}(q,p)$ and equation (1) and (2) hold with $L_k = J_n = \theta$ for each $n, k \in \mathbb{N}$.

$$2\ell_{\infty}(q,p) = \{\langle a_{nk} \rangle : \sup_{n,k} (q(a_{nk}))^{p_{nk}} < \infty \}.$$

We define $2m(q,p) = 2\tilde{c}(q,p) \cap 2\ell_{\infty}(q,p)$, $2m_0(q,p) = 2\tilde{c}_0(q,p) \cap 2\ell_{\infty}(q,p)$, $2m^R(q,p) = (2\tilde{c})^R(q,p) \cap 2\ell_{\infty}(q,p)$ and $2m_0^R(q,p) = (2\tilde{c}_0)^R(q,p) \cap 2\ell_{\infty}(q,p)$. Let $p = (p_{nk})$ be a sequence of positive real numbers. Then the double sequence $\langle a_{nk} \rangle$ is said to be strongly $(p)$-Cesaro summable to $L$, i.e. $\langle a_{nk} \rangle \in 2w(p)(q)$ if

$$\lim_{u,v \to \infty} \frac{1}{uv} \sum_{n=1}^{u} \sum_{k=1}^{v} (q(a_{nk} - L))^{p_{nk}} = 0.$$

The following results will be used for establishing some results of this article.

**Lemma 1.** If a sequence space is solid, then it is monotone.

We procure the following result of Tripathy [14]. He proved it for $X = C$.

**Lemma 2.** (Tripathy [14, Theorem 1]) The following are equivalent:

(i) The double sequence $\langle a_{nk} \rangle$ is statistically convergent to $L$.

(ii) The double sequence $\langle a_{nk} - L \rangle$ is statistically convergent to $0$.

(iii) There exists a sequence $\langle b_{nk} \rangle \in 2c$ such that $a_{nk} = b_{nk}$ for a.a.n & k.

(iv) There exists a subset $M = \{(n_i, k_j) \in \mathbb{N} \times \mathbb{N} : i, j \in \mathbb{N}\}$ of $\mathbb{N} \times \mathbb{N}$ such that $\rho(M) = 1$ and $\langle a_{n_i, k_j} \rangle \in 2c$.

(v) There exists two sequences $\langle x_{nk} \rangle$ and $\langle y_{nk} \rangle$ such that $a_{nk} = x_{nk} + y_{nk}$ for all $n, k \in \mathbb{N}$, where $\langle x_{nk} \rangle$ converges to $L$ and $\langle y_{nk} \rangle \in 2\tilde{c}_0$. 

182
3. Main results

In this section we prove the results of this article. The proof of the following result is a routine verification.

**Theorem 1.** Let \( \langle p_{nk} \rangle \in 2\ell_\infty \), then the class of sequences \( 2\bar{c}(q,p) \), \( 2\bar{c}_0(q,p) \), \( (2\bar{c})^R(q,p) \), \( (2\bar{c}_0)^R(q,p) \), \( 2m(q,p) \), \( 2m_0(q,p) \), \( 2m^R(q,p) \) and \( (2m_0)^R(q,p) \) are linear spaces.

We prove the following decomposition theorem.

**Theorem 2.** The following are equivalent:

(i) The double sequence \( \langle a_{nk} \rangle \in 2\bar{c}(q,p) \), i.e. there exists \( L \in X \) such that \( \text{stat-lim}(q(a_{nk} - L))^{p_{nk}} = 0 \).

(ii) The double sequence \( \langle a_{nk} - L \rangle \in 2\bar{c}_0(q,p) \).

(iii) There exists a sequence \( \langle b_{nk} \rangle \in 2c(q,p) \) such that \( a_{nk} = b_{nk} \) for a.a. \( n, k \).

(iv) There exists a subset \( M = \{(n_i, k_j) \in \mathbb{N} \times \mathbb{N} : i, j \in \mathbb{N}\} \) of \( \mathbb{N} \times \mathbb{N} \) such that \( \rho(M) = 1 \) and \( \langle a_{n_i, k_j} \rangle \in 2c(q,t) \), where \( t = (p_{n_i, k_j}) \).

(v) There exists two sequences \( \langle x_{nk} \rangle \) and \( \langle y_{nk} \rangle \) such that \( a_{nk} = x_{nk} + y_{nk} \) for all \( n, k \in \mathbb{N} \), where \( \text{stat-lim}(q(a_{nk} - L))^{p_{nk}} = 0 \) and \( \langle y_{nk} \rangle \in 2\bar{c}_0(q,p) \).

**Proof.** Let \( z_{nk} = (q(a_{nk} - L))^{p_{nk}} \) for all \( n, k \in \mathbb{N} \). Then \( \text{stat-lim} z_{nk} = 0 \) and the result follows from Lemma 2.

**Theorem 3.** Let \( 0 < \inf p_{nk} \leq \sup p_{nk} < \infty \), then the spaces \( Z(q,p) \) for \( Z = (2\bar{c})^{BR} \), \( (2\bar{c}_0)^{BR} \), \( (2\bar{c})^B \), \( 2\ell_\infty(q,p) \) and \( (2\bar{c}_0)^B \) are paranormed spaces (not necessarily totally), paranormed by

\[
g(\langle a_{nk} \rangle) = \sup_{n,k} (q(a_{nk}))^{p_{nk}},
\]

where \( H = \max(1, \sup_{n,k} p_{nk}) \).

**Proof.** Clearly \( g(2\bar{\theta}) = 0 \), \( g(-A) = g(A) \), where \( A = \langle a_{nk} \rangle \) and \( g(A + B) \leq g(A) + (B) \). Now we verify the continuity of scalar multiplication.

Let \( A \to 2\bar{\theta} \), then \( g(A) \to 0 \). We have for a given scalar \( \lambda \),

\[
g(\lambda A) = \sup_{n,k} (q(\lambda a_{nk}))^{p_{nk}} \leq \max(1, |\lambda|) \cdot g(A) \to 0 \quad \text{as} \quad A \to 2\bar{\theta}.
\]

183
Next let \( \lambda \to 0 \). Without loss of generality, let \( |\lambda| < 1 \). Then for a given \( A \) \( \langle a_{nk} \rangle \), we have

\[
g(\lambda A) = \sup_{n,k} \left( g(\lambda a_{nk}) \right)^{p_{nk}} \leq |\lambda|^h \cdot g(A) \to 0 \quad \text{as} \quad \lambda \to 0,
\]

where \( h = \inf_{n,k} p_{nk} > 0 \).

The case when \( \lambda \to 0 \) and \( A \to 2^\partial \) implies \( g(\lambda A) \to 0 \) follows similarly.

Hence the spaces are paranormed by \( g \).

**Theorem 4.** Let \( p = \langle p_{nk} \rangle \in 2\ell_* \). Then the spaces \( Z(q,p) \) for \( Z = 2c, 2c_0, (2c_0)^R, (2c)^R, (2c)^B, (2c_0)^B, (2c)^B \) and \( (2c_0)^B \) are sequence algebras.

**Proof.** Consider the space \( 2c(q,p) \). Let \( \langle a_{nk} \rangle, \langle b_{nk} \rangle \in 2c(q,p) \). Then there exists \( K_1, K_2 \subset \mathbb{N} \times \mathbb{N} \) with \( \rho(K_1) = \rho(K_2) = 1 \) such that

\[
\lim_{n,k \to \infty, (n,k) \in K_1} (q(a_{nk} - L))^p_{nk} = 0 \quad \text{and} \quad \lim_{n,k \to \infty, (n,k) \in K_2} (q(a_{nk} - \xi))^p_{nk} = 0
\]

for some \( L, \xi \in X \).

Let \( K = K_1 \cap K_2 \), then \( \rho(K) = 1 \). Now it follows that

\[
\lim_{n,k \to \infty, (n,k) \in K} (q(a_{nk}b_{nk} - L\xi))^p_{nk} = 0.
\]

Thus \( \langle a_{nk}b_{nk} \rangle \in 2c(q,p) \).

Similarly it can be shown that the other spaces are also sequence algebras.

**Theorem 5.** The spaces \( 2c_0(q,p), (2c_0)^B(q,p), (2c_0)^R(q,p) \) and \( (2c_0)^{BR}(q,p) \) are solid. Hence are monotone.

**Proof.** Let \( \langle a_{nk} \rangle \in 2c_0(q,p) \) or \( (2c_0)^B(q,p) \) or \( (2c_0)^R(q,p) \) or \( (2c_0)^{BR}(q,p) \).

Let \( \langle a_{nk} \rangle \) be a double sequence of scalars with \( |a_{nk}| \leq 1 \) for all \( n, k \in \mathbb{N} \). Then the solidness of the above spaces follows from the following inequality

\[
(q(a_{nk}a_{nk}))^p_{nk} \leq (q(a_{nk}))^p_{nk} \quad \text{for all} \quad n, k \in \mathbb{N}.
\]

The rest follows from Lemma 1.

**Corollary 6.** The spaces \( 2c(q,p), (2c)^B(q,p), (2c)^R(q,p) \) and \( (2c)^{BR}(q,p) \) are not monotone, as such are not solid.

**Proof.** The proof follows from the following example and Lemma 1.

**Example 1.** Let \( X = \ell_* \) and \( p_{nk} - 1 \) for all \( n, k \in \mathbb{N} \). Let \( \langle a_{nk} \rangle \in 2c(q,p) \) be defined by \( a_{nk} = e = (1, 1, 1, 1, - - - ) \) for all \( n, k \in \mathbb{N} \). Let \( a_{nk} \) \( (a^i_{nk}) \) and \( q((a^i_{nk})) = \sup_{i \geq 2} |a^i_{nk}| \) for all \( n, k \in \mathbb{N} \). Consider the \( J \)th step space
VECTOR VALUED STATISTICALLY CONVERGENT DOUBLE SEQUENCE SPACES

Let \((b_{nk}) \in \((2c)^{BR}(q,p))\) defined by \(b_{nk} = a_{nk}\) for \(n\) even and all \(k \in \mathbb{N}\) and \(b_{nk} = \theta\), otherwise. Then \(b_{nk} \notin (2c)^{BR}(q,p)\). Hence \((2c)^{BR}(q,p)\) is not monotone.

From this example, it follows that the other spaces are not monotone, too.

**Theorem 7.** The spaces \(Z(q,p)\) for \(Z = 2c, (2c0)^R, (2c)^R, (2c)^{BR}, (2c0)^{BR}, (2c)^B\) and \((2c0)^B\) are not convergence free.

**Proof.** The proof follows from the following example.

**Example 2.** Let \(X = \ell_\infty, q((x_i)) = \sup_{i \geq 2} |x_i|, p_{nk} = 1\) for \(k\) odd and for all \(n \in \mathbb{N}\) and \(p_{nk} = 2\) otherwise. Consider the sequence \(\langle a_{nk} \rangle \in 2c(q,p)\) defined by \(a_{1k} = \theta = a_{n1}\) for all \(n, k \in \mathbb{N}\). \(a_{nk} = (2, 2, 2, 2, 2, \ldots)\), otherwise.

Consider \(\langle b_{nk} \rangle\) defined as

\[
\begin{align*}
b_{1k} &= \theta = b_{n1} \quad \text{for all } n, k \in \mathbb{N}, \\
b_{nk} &= e \quad \text{for all } k \text{ even and all } n > 1, \\
&= 2e \quad \text{otherwise.}
\end{align*}
\]

Then \(\langle a_{nk} \rangle \in 2c(q,p)\), but \(\langle b_{nk} \rangle \notin 2c(q,p)\). Hence \(2c(q,p)\) is not convergence free. This example shows that the spaces \(Z(q,p)\) for \(Z = (2c)^R, (2c)^{BR}, (2c)^B\) are not convergence free, too.

**Example 3.** Let \(p_{nk} = 1\) for all \(n, k \in \mathbb{N}\), \(X = C\) and \(q(x) = |x|\).

Consider the double sequence \(\langle a_{nk} \rangle\) defined as

\[
a_{nk} = \begin{cases} 
0 & \text{for } n \text{ even and for all } k \in \mathbb{N}, \\
\frac{1}{n} & \text{otherwise.}
\end{cases}
\]

Consider the sequence \(\langle b_{nk} \rangle\) defined as

\[
b_{nk} = \begin{cases} 
0 & \text{for } n \text{ even and for all } k \in \mathbb{N}, \\
1 & \text{otherwise.}
\end{cases}
\]

Then clearly \(\langle a_{nk} \rangle \in Z(q,p)\), but \(\langle b_{nk} \rangle \notin Z(q,p)\) for \(Z = 2c0, (2c0)^R, (2c0)^{BR}, (2c0)^B\).

Hence the spaces are not convergence free.

**Proposition 8.** The spaces \(Z(q,p)\) for \(Z = 2c, (2c0)^R, (2c)^R, (2c)^{BR}, (2c0)^{BR}, (2c)^B\) and \((2c0)^B\) are not symmetric.

**Proof.** The proof follows from the following example.
Example 4. Let \( p_{nk} = 1 \) for \( n \) even and all \( k \in \mathbb{N} \) and \( p_{nk} = 2 \) otherwise. Let \( X = \mathbb{C} \), and \( q(x) = |x| \). Consider the sequence \( \langle a_{nk} \rangle \) defined by \( a_{n1} = 1 = a_{1k} \), for all \( n = i^2 = k \), \( i \in \mathbb{N} \), and \( a_{nk} = 0 \), otherwise.

Then \( \langle a_{nk} \rangle \in Z(q,p) \) for \( Z = 2\tilde{c}, 2\tilde{c}_0, (2\tilde{c})^R, (2\tilde{c})^{BR}, (2\tilde{c}_0)^{BR}, (2\tilde{c})^B \) and \((2\tilde{c}_0)^B\). Consider its rearranged sequence \( \langle b_{nk} \rangle \) defined by

\[
\begin{align*}
  b_{nk} &= \begin{cases} 
    1 & \text{for all } n \text{ even and all } k \in \mathbb{N}, \\
    0 & \text{otherwise.}
  \end{cases}
\end{align*}
\]

Then \( \langle b_{nk} \rangle \notin Z(q,p) \) for \( Z = 2\tilde{c}, 2\tilde{c}_0, (2\tilde{c})^R, (2\tilde{c})^{BR}, (2\tilde{c}_0)^{BR}, (2\tilde{c})^B \) and \((2\tilde{c}_0)^B\). Hence the spaces \( Z(q,p) \) for \( Z = 2\tilde{c}, 2\tilde{c}_0, (2\tilde{c})^R, (2\tilde{c})^{BR}, (2\tilde{c}_0)^{BR}, (2\tilde{c})^B \) and \((2\tilde{c}_0)^B\) are not symmetric.

**Theorem 9.** For two sequences \( p = \langle p_{nk} \rangle \) and \( t = \langle t_{nk} \rangle \) we have \((2\tilde{c}_0)^B(q,p) \supset (2\tilde{c}_0)^B(q,t)\) if and only if \( \liminf_{n,k \to \infty} \left( \frac{p_{nk}}{t_{nk}} \right) > 0 \) where \( K \subset \mathbb{N} \times \mathbb{N} \) such that \( \rho(K) = 1 \).

**Proof.** Suppose that

\[
\liminf_{n,k \to \infty} \left( \frac{p_{nk}}{t_{nk}} \right) > 0. \tag{3}
\]

Then there exists \( \alpha > 0 \) such that \( p_{nk} > \alpha t_{nk} \) for sufficiently large pair \((n,k) \in K\). Let \( \langle a_{nk} \rangle \in (2\tilde{c}_0)^B(q,t) \), then for \( \varepsilon > 0 \) we have \( (q(a_{nk}))^{q_{nk}} < \varepsilon \) for all \((n,k) \in L \subset \mathbb{N} \times \mathbb{N} \), where \( L = \{(n,k) \in \mathbb{N} \times \mathbb{N} : (q(a_{nk}))^{q_{nk}} < \varepsilon \} \), such that \( \rho(L) = 1 \).

Let \( J = K \cap L \). Then \( \rho(J) = 1 \).

Now \( (q(a_{nk}))^{p_{nk}} \leq \left( (q(a_{nk}))^{t_{nk}} \right)^{\alpha} \). This implies \( \langle a_{nk} \rangle \in (2\tilde{c}_0)^B(q,t) \).

Next let \( (2\tilde{c}_0)^B(q,p) \supset (2\tilde{c}_0)^B(q,t) \), but there is no \( K \subset \mathbb{N} \times \mathbb{N} \) with \( \rho(K) = 1 \) such that (3) holds. Then there exists \( \{(n_i,k_j) : i,j \in \mathbb{N} \} \subset \mathbb{N} \times \mathbb{N} \) with \( \rho(\{(n_i,k_j) : i,j \in \mathbb{N} \}) \neq 0 \) such that \( \rho_{i,n_k} < q_{n_k,k_j} \). Define the sequence \( \langle a_{nk} \rangle \) by

\[
a_{nk} = \begin{cases} 
  \frac{1}{i} & \text{if } k = k_j, n = n_i, \\
  0 & \text{otherwise.}
\end{cases}
\]

Then \( \langle a_{nk} \rangle \in (2\tilde{c}_0)^B(q,t) \). But \( (a_{n_i,k_j})^{p_{n_i,k_j}} > \exp\left( -\log \frac{1}{i} \right) \) Hence we arrive at a contradiction. \( \Box \)

**Theorem 10.** Let \( 0 < \inf_{n,k} p_{nk} \leq \sup_{n,k} p_{nk} < \infty \). Then

\[
2w(p)(q) \cap 2\ell_\infty(q,p) = 2\tilde{c}(q,p) \cap 2\ell_\infty(q,p).
\]
VECTOR VALUED STATISTICALLY CONVERGENT DOUBLE SEQUENCE SPACES

Proof. Let $A = (a_{nk}) \in 2w(p)(q) \cap 2\ell_\infty(q,p)$ and $H = \sup_{n,k} p_{nk}, r = \max\{1, H\}$. Then taking $b_{nk} = (q(a_{nk} - L))^p_{nk}$ for all $n, k \in \mathbb{N}$, we have the result, which follows from [14, Theorem 4] of Tripathy.

The following result is a consequence of Theorem 10.

**Corollary 11.** For any two sequences of real numbers $p = (p_{nk})$ and $t = (t_{nk})$ satisfying the condition in the hypothesis of Theorem 10, we have

$$2w(p)(q) \cap 2\ell_\infty(q,p) = 2w(t)(q) \cap 2\ell_\infty(q,t).$$

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