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MODAL OPERATORS ON BOUNDED COMMUTATIVE RESIDUATED ℓ-MONOIDS

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ABSTRACT. Bounded commutative residuated lattice ordered monoids ($R\ell$-monoids) are a common generalization of, e.g., Heyting algebras and BL-algebras, i.e., algebras of intuitionistic logic and basic fuzzy logic, respectively. Modal operators (special cases of closure operators) on Heyting algebras were studied in [MacNAB, D. S.: *Modal operators on Heyting algebras*, Algebra Universalis **12** (1981), 5–29] and on MV-algebras in [HARLENDEROVÁ, M.—RACHŮNEK, J.: *Modal operators on MV-algebras*, Math. Bohem. **131** (2006), 39–48]. In the paper we generalize the notion of a modal operator for general bounded commutative $R\ell$-monoids and investigate their properties also for certain derived algebras.

Commutative residuated lattice ordered monoids ($R\ell$-monoids) are duals to commutative $DR\ell$-monoids which were introduced by Swamy [16] as a common generalization of Abelian lattice ordered groups and Brouwerian algebras. By [11], [12], [13], also algebras of logics behind fuzzy reasoning can be considered as particular cases of bounded commutative $R\ell$-monoids. Namely from this point of view, MV-algebras, an algebraic counterpart of the Łukasiewicz infinite-valued propositional logic, are precisely bounded commutative $R\ell$-monoids satisfying the double negation law. Further, BL-algebras, an algebraic semantics of the Hájek basic fuzzy logic, are just bounded commutative $R\ell$-monoids isomorphic to subdirect products of linearly ordered commutative $R\ell$-monoids. Heyting algebras (duals to Brouwerian algebras), i.e. algebras of intuitionistic logic, are characterized as bounded commutative $R\ell$-monoids with idempotent multiplication.

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Modal operators (special cases of closure operators) on Heyting algebras were introduced and studied by Macnab in [10]. Analogously, modal operators on MV-algebras were introduced in [7] recently.

In this paper we define modal operators for arbitrary bounded commutative $R\ell$-monoids and we study their properties in the class of normal $R\ell$-monoids in particular.

For concepts and results relating to MV-algebras, BL-algebras and Heyting algebras see for instance [3], [6], [1].

Definition 1. A *bounded commutative $R\ell$-monoid* is an algebra $M = (M;\ \cdot,\ \ ,\ \wedge, \to, 0, 1)$ of type $\langle 2,2,2,2,0,0 \rangle$ satisfying the following conditions.

(i) $(M;\ \odot, 1)$ is a commutative monoid.

(ii) $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ if and only if $x \leq y \to z$, for any $x, y, z \in M$.

(iv) $x \odot (x \to y) = x \wedge y$, for any $x, y \in M$.

Bounded commutative $R\ell$-monoids are special cases of residuated lattices, more precisely (see for instance [4]), they are exactly commutative integral generalized BL-algebras in the sense of [2] and [8].

In what follows, by an *$R\ell$-monoid* we will mean a bounded commutative $R\ell$-monoid.

Let us define on any $R\ell$-monoid M the unary operation of negation "$^-$" by $x^- := x \to 0$ for any $x \in M$. Further, we put $x \oplus y := (x\ \ \odot y^-)^-$ for any $x, y \in M$.

Algebras of the above mentioned propositional logics can be characterized in the class of all $R\ell$-monoids as follows: An $R\ell$-monoid M is

a) a BL-algebra ([13]) if and only if M satisfies the identity of pre-linearity $(x \to y) \vee (y \to x) = 1$;

b) an MV-algebra ([11], [12]) if and only if M fulfills the double negation law $x^{--} = x$;

c) a Heyting algebra ([16]) if and only if the operations " $\cdot$ " and "$\wedge$" coincide on M.

Lemma 1. ([16], [15]) *In any bounded commutative $R\ell$-monoid M we have for any $x, y \in M$:*

(1) $x \leq y \iff x \to y = 1$.

(2) $x \odot y \leq x \wedge y \leq x, y$.

(3) $x \leq y \implies x \odot z \leq y \odot z$.

(4) $x \leq y \implies z \to x \leq z \to y,\ y \to z \leq x \to z$.

(5) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(6) $(x \to y) \odot (y \to z) \leq x \to z$.

(7) $1^{--} = 1$, $0^{--} = 0$.

(8) $x \leq x^{--}$, $x^{-} = x^{---}$.

(9) $x \leq y \implies y^{-} \leq x^{-}$.

(10) $(x \vee y)^{-} = x^{-} \wedge y^{-}$.

(11) $(x \wedge y)^{--} = x^{--} \wedge y^{--}$.

(12) $(x \odot y)^{-} = y \to x^{-} = y^{--} \to x^{-} = x \to y^{-} = x^{--} \to y^{-}$.

(13) $(x \odot y)^{--} \geq x^{--} \odot y^{--}$.

(14) $(x \to y)^{--} = x^{--} \to y^{--}$.

Remark 2. It is obvious that $x \oplus z \leq y \oplus z$ holds for any $x, y, z \in M$ such that $x \leq y$. Further by [14, Lemma 2.11], $x^{--} \oplus y^{--} = x \oplus y$ for any $x, y \in M$, hence also $x \oplus y = x^{--} \oplus y = x \oplus y^{--} = x^{--} \oplus y^{--}$.

Definition 2. Let M be an $R\ell$-monoid. A mapping $f\colon M \longrightarrow M$ is called a *modal operator* on M if, for any $x, y \in M$,

1. $x \leq f(x)$;
2. $f(f(x)) = f(x)$;
3. $f(x \odot y) = f(x) \odot f(y)$.

If, moreover, for any $x, y \in M$,

4. $f(x \oplus y) = f(x \oplus f(y))$,

then f is called a *strong modal operator* on M.

Proposition 3. *If f is a modal operator on an $R\ell$-monoid M and $x, y \in M$, then*

(i) $x \leq y \implies f(x) \leq f(y)$;

(ii) $f(x \to y) \leq f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$;

(iii) $f(x) \leq (x \to f(0)) \to f(0)$;

(iv) $f(x) \odot x^{-} \leq f(0)$;

(v) $x \oplus f(0) \geq f(x^{--}) \geq f(x)$;

(vi) $f(x \vee y) = f(x \vee f(y)) = f(f(x) \vee f(y))$.

Proof.

(i) $x \leq y \implies f(x \wedge y) = f(x) \implies f(y \odot (y \to x)) = f(x) \implies f(y) \odot f(y \to x) = f(x) \implies f(x) \leq f(y)$.

(ii) By (i), $f(x) \odot f(x \to y) = f(x \odot (x \to y)) = f(x \wedge y) \leq f(y)$. This implies

$$f(x \to y) \leq f(x) \to f(y).$$

From this we get

$$\begin{aligned}f(f(x) \to f(y)) &\leq f\left(f(x)\right) \to f\left(f(y)\right) = f(x) \to f(y) \leq x \to f(y)\\ &\leq f(x \to f(y)) \leq f(x) \to f(f(y))\\ &= f(x) \to f(y) \leq f(f(x) \to f(y)),\end{aligned}$$

therefore

$$f(x) \to f(y) = f\left(f(x) \to f(y)\right) = x \to f(y) = f\left(x \to f(y)\right).$$

(iii) By use of (ii) and (i), we have

$$\begin{aligned}f(x) \odot (f(x) \to f(0)) = f(x) \wedge f(0) = f(0) &\implies f(x) \leq (f(x) \to f(0)) \to f(0)\\ &\implies f(x) \leq (x \to f(0)) \to f(0).\end{aligned}$$

(iv) By (ii), we obtain

$$0 \leq f(0) \implies x^- = x \to 0 \leq x \to f(0) = f(x) \to f(0),$$

thus

$$f(x) \odot x^- \leq f(x) \odot (f(x) \to f(0)) = f(x) \wedge f(0) = f(0).$$

(v) According to Remark 2, Lemma 1(12), (8) and the part (ii) consecutively,

$$\begin{aligned}x \oplus f(0) &= x^{--} \oplus f(0) = \left(x^{---} \odot f(0)^-\right)^- = x^{---} \to f(0)^{--}\\ &= x^- \to f(0)^{--} = f\left(x^- \to f(0)^{--}\right) \geq f\left(x^- \to f(0)\right) \geq f(x^- \to 0)\\ &= f(x^{--}) \geq f(x).\end{aligned}$$

Hence

$$x \oplus f(0) \geq f(x^{--}) \geq f(x).$$

(vi) $f(x \vee y) \leq f\left(x \vee f(y)\right) \leq f\left(f(x) \vee f(y)\right) = f\left(f(x \vee y)\right) = f(x \vee y)$. □

Remark 4. By the definition of a modal operator and Proposition 3(i) every modal operator on an $R\ell$-monoid M is a closure operator on the lattice $(M; \vee, \wedge)$.

Remark 5. M. Galatos and C. Tsinakis introduced in [5] the notion of a *nucleus* of a residuated lattice L as a closure operator γ on L satisfying $\gamma(a)\gamma(b) \leq \gamma(ab)$, to represent generalizations of MV-algebras (dropping integrality, commutativity and the existence of bounds) by means of ℓ-groups and nuclei of negative cones of ℓ-groups. From this point of view, a modal operator f on an $R\ell$-monoid M is a nucleus of M satisfying $f(x) \odot f(y) \geq f(x \odot y)$.

PROPOSITION 6. *If f is a strong modal operator on an $R\ell$-monoid M and $x, y \in M$, then*

(vii) $f(x \oplus y) = f\left(f(x) \oplus f(y)\right)$;

(viii) $x \oplus f(0) = f(x^{--})$.

Proof. Let us suppose that f is a strong modal operator. Then

(vii) $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus f(y))$;

(viii) By Proposition 3(v), $f(x \oplus f(0)) = f(x \oplus 0) = f(x^{--})$ implies $f(x^{--}) = f(x \oplus f(0)) \geq x \oplus f(0) \geq f(x^{--})$. □

THEOREM 7. *Let M be an $R\ell$-monoid and $f \colon M \longrightarrow M$ be a mapping. Then f is a modal operator on M if and only if for any $x, y \in M$ it is satisfied:*

(a) $x \to f(y) = f(x) \to f(y)$;

(b) $f(x) \odot f(y) \geq f(x \odot y)$.

Proof. Let a mapping f fulfil conditions (a) and (b).

1. For any $x \in M$ we have $x \to f(x) = f(x) \to f(x) = 1$. Therefore $x \leq f(x)$.
2. For all $x \in M$ it holds $1 = f(x) \to f(x) = f(f(x)) \to f(x)$. This implies $f(f(x)) \leq f(x)$. Therefore, by 1, $f(f(x)) = f(x)$.
3. For any $x, y \in M$ it is true $x \odot y \leq f(x \odot y) \implies y \leq x \to f(x \odot y) = f(x) \to f(x \odot y) \implies y \odot f(x) \leq f(x \odot y) \implies f(x) \leq y \to f(x \odot y) = f(y) \to f(x \odot y) \implies f(x) \odot f(y) \leq f(x \odot y) \implies f(x) \odot f(y) = f(x \odot y)$.

The converse implication is obvious. □

COROLLARY 8. *If M is an $R\ell$-monoid and $f \colon M \longrightarrow M$ is a mapping, then f is a nucleus of M if and only if f satisfies (a) of Theorem 7 and it is isotone.*

Remark 9. If M is a Heyting algebra and $x, y \in M$, then $f(x) \odot f(y) = f(x) \wedge f(y) \geq f(x \wedge y) = f(x \odot y)$. Therefore, by Theorem 7, f is a modal operator on M iff it satisfies condition (a) (see also [10]).

We say that an $R\ell$-monoid M is *normal* if M satisfies the identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

Remark 10. By [15, Proposition 5], every BL-algebra and every Heyting algebra is normal, hence the variety of normal $R\ell$-monoids is considerably wide.

Let M be an $R\ell$-monoid. For arbitrary element $a \in M$ we denote by $\varphi_a \colon M \longrightarrow M$ the mapping such that $\varphi_a(x) = a \oplus x$ for every $x \in M$.

Denote by

$$I(M) = \{a \in M : \ a \odot a = a\}$$

the set of all multiplicative idempotents in an $R\ell$-monoid M. It is obvious that $0, 1 \in I(M)$. By [9, Lemma 2.8.3], $a \odot x = a \wedge x$ holds for any $a \in I(M)$, $x \in M$. Further, if M is a normal $R\ell$-monoid and $a \in I(M)$, then also $a^{--} \in I(M)$.

THEOREM 11. *If M is a normal $R\ell$-monoid and $a \in M$, then φ_a is a strong modal operator on M if and only if $a^{-}, a^{--} \in I(M)$.*

P r o o f.

a) Let $a, x, y \in M$, $a^-, a^{--} \in I(M)$.

1. $\varphi_a(x) = a \oplus x = (a^- \odot x^-)^- \geq x^{--} \geq x$.
2. $\varphi_a(\varphi_a(x)) = a \oplus (a \oplus x) = a \oplus (a^- \odot x^-)^- = (a^- \odot (a \quad \cdot x^-)^{--})^-$ $(a^- \odot (a^- \odot x^-))^- = ((a^- \odot a^-) \odot x^-)^- = (a^- \quad \cdot x^-)^- \quad a \oplus x = \varphi_a(x)$.
3. We first prove that $a \oplus x = (a \vee x)^{--}$.

   By Lemma 1(10), we obtain $a \oplus x - (a^- \odot x^-)^- - (a^- \wedge x^-)$ $((a \vee x)^-)^- - (a \vee x)^{--}$.

   We will now prove condition 3 from the definition of a modal operator. We have

$$\begin{aligned}
\varphi_a(x) \odot \varphi_a(y) &= (a \oplus x) \odot (a \oplus y) = (a^{--} \oplus x) \odot (a^{--} \oplus y) \\
&= (a^{--} \vee x)^{--} \odot (a^{--} \vee y)^{--} = ((a^{--} \vee x) \ \cdot \ (a^{--} \vee y))^{--} \\
&= ((a^{--} \odot a^{--}) \vee (x \odot a^{--}) \vee (a^{--} \odot y) \vee (x \ \cdot \ y))^{--} \\
&= (a^{--} \vee (x \odot y))^{--} = a^{--} \oplus (x \odot y) = a \oplus (x \ \cdot \ y) \\
&= \varphi_a(x \odot y).
\end{aligned}$$

4. According to [14, Proposition 2.10], $(M; \oplus)$ is a commutative semigroup.

   For this reason

$$\begin{aligned}
\varphi_a(x \oplus y) &= a \oplus (x \oplus y) = a^{--} \oplus (x \oplus y) = (a \oplus a) \oplus (x \oplus y) \\
&= a \oplus (x \oplus (a \oplus y)) = \varphi_a(x \oplus \varphi_a(y)).
\end{aligned}$$

b) Let φ_a be a strong modal operator on M. Then on account of condition 3, we have $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$. Then for $x = y = 0$ we obtain $a \oplus (0 \cdot 0) -$ $(a \oplus 0) \odot (a \oplus 0)$, hence $a \oplus 0 = (a \oplus 0) \odot (a \oplus 0)$. Since $a \oplus 0 - a^-$ (see [14, Lemma 2.11]), we conclude that $a^{--} = a^{--} \odot a^{--}$, which yields $a^{--} \in I(M)$

From condition 4 we have $a \oplus (x \oplus y) = a \oplus (x \oplus (a \oplus y))$. Then for $x = y = 0$ it follows that $a^{--} = a \oplus 0 = a \oplus (0 \oplus 0) = a \oplus (0 \oplus (a \oplus 0)) - (a \oplus 0) \oplus a^{--} -$ $a^{--} \oplus a^{--}$, thus $a^{--} = a^{--} \oplus a^{--}$. From this $a^{--} = (a^- \ \cdot \ a^-)^-$, hence $a^- = (a^- \odot a^-)^{--}$. Since M is normal, we obtain $a^- = a^- \ \cdot \ a^-$ and so $a^- \in I(M)$. □

Remark 12. If M is an MV-algebra and $a \in M$, then $a \in I(M)$ if and only if $a^-, a^{--} \in I(M)$. Concurrently, by [7], in any MV-algebra it is true that φ_a is a modal operator on M if and only if φ_a is a strong modal operator on M (namely if and only if $a \in I(M)$). The question, whether φ_a is a modal operator on M if and only if it is a strong modal operator also for any normal $R\ell$-monoid M, remains open.

Corollary 13. *Let M be a normal $R\ell$-monoid and f be a modal operator on M such that $f(x) = f(x^{--})$ for all $x \in M$. Then f is strong if and only if $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$.*

Proof. Suppose that a modal operator f on M satisfies the condition $f(x) = f(x^{--})$ for every $x \in M$. Then by Proposition 6 and Theorem 11, f is strong if and only if $f = \varphi_a$ for some $a \in M$ such that $a^-, a^{--} \in I(M)$.

If f is strong and $x \in M$, then $f(x) = f(x^{--}) = f(0) \oplus x$. Hence $f = \varphi_{f(0)}$ and we have $f(0), f(0)^- \in I(M)$.

For any modal operator f we have $f(0)^{--} \in I(M)$. In fact, $f(0)^{--} = f(0 \odot 0)^{--} = (f(0) \odot f(0))^{--} = f(0)^{--} \odot f(0)^{--}$. Hence, if $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$, then by Theorem 11, f is strong. □

Corollary 14. *Specially for MV-algebras, we obtain* (see [7])*: If M is an MV-algebra and f is a modal operator on M, then f is strong if and only if $f = \varphi_{f(0)}$.*

Let M be an $R\ell$-monoid and $a \in M$. Consider mappings $\psi_a \colon M \longrightarrow M$ and $\chi_a \colon M \longrightarrow M$ such that $\psi_a(x) := a \to x$ and $\chi_a(x) := (x \to a) \to a$ for every $x \in M$. These mappings are significant modal operators in Heyting algebras (see [10]). We will now deal with the mappings ψ_a and χ_a in arbitrary $R\ell$-monoids.

Proposition 15. *If M is an $R\ell$-monoid and $a \in I(M)$, then for any $x, y \in M$*

$$x \to \psi_a(y) = \psi_a(x) \to \psi_a(y).$$

Proof. By the definition of ψ_a, $x \to \psi_a(y) = x \to (a \to y)$ and $\psi_a(x) \to \psi_a(y) = (a \to x) \to (a \to y)$. At the same time, by Lemma 1(5), $(a \to x) \to (a \to y) = ((a \to x) \odot a) \to y = (a \wedge x) \to y = (a \odot x) \to y = x \to (a \to y)$, whence the assertion follows. □

From Theorem 7 and Proposition 15 we obtain as an immediate consequence the following claim.

Corollary 16. *Let M be an $R\ell$-monoid and $a \in I(M)$. Then ψ_a is a modal operator on M if and only if for any $x, y \in M$*

$$\psi_a(x) \odot \psi_a(y) \geq \psi_a(x \odot y).$$

Lemma 17. *If M is an $R\ell$-monoid and $a \in M$, then for any $x, y \in M$*

$$x \to \chi_a(y) \leq \chi_a(x) \to \chi_a(y).$$

Proof. By the definition of χ_a and by Lemma 1(5), $x \to \chi_a(y) = x \to ((y \to a) \to a) = (y \to a) \to (x \to a)$, $\chi_a(x) \to \chi_a(y) = ((x \to a) \to a) \to ((y \to a) \to a)$. Since by [14, Lemma 2.3], $(y \to a) \to (x \to a) \leq ((x \to a) \to a) \to ((y \to a) \to a)$, we have $x \to \chi_a(y) \leq \chi_a(x) \to \chi_a(y)$. □

For any $R\ell$-monoid M, let us denote by $B(M)$ the set of all elements from M having the complement in the lattice $(M; \vee, \wedge, 0, 1)$. Note that $0, 1 \in B(M)$. If $a \in B(M)$ then its complement a' is equal to the element a^-. By [9, Lemma 2.8.8], $B(M) \subseteq I(M)$.

Proposition 18. *Let M be an $R\ell$-monoid and $a \in B(M)$. Then for any $x, y \in M$*

$$x \to \chi_a(y) = \chi_a(x) \to \chi_a(y).$$

Proof. Let $a \in B(M)$, $x, y \in M$. Then

$$x \to \chi_a(y) = x \to ((y \to a) \to a) = (y \to a) \to (x \to a),$$

$$\begin{aligned} \chi_a(x) \to \chi_a(y) &= ((x \to a) \to a) \to ((y \to a) \to a) \\ &= (y \to a) \to (((x \to a) \to a) \to a), \end{aligned}$$

$$x \to a = x \to a^{--} = (x \odot a^-)^-,$$

$$\begin{aligned} (x \to a) \to a &= ((x \to a) \odot a^-)^- = ((x \odot a^-)^- \odot a^-)^- = ((x \wedge a^-)^- \wedge a^-)^- \\ &= ((x \wedge a^-) \vee a)^{--} = ((x \vee a) \wedge (a^- \vee a))^{--} = (x \vee a)^{--} = x \oplus a, \end{aligned}$$

$$\begin{aligned} ((x \to a) \to a) \to a &= (((x \to a) \to a) \odot a^-)^- = ((x \oplus a) \odot a^-)^- = (x \oplus a) \to a \\ &= (x \vee a)^{--} \to a^{--} = ((x \vee a) \to a)^{--} = ((x \vee a) \cdot a^-)^- \\ &= ((x \odot a^-) \vee (a \odot a^-))^- = (x \odot a^-)^- = x \to a. \end{aligned}$$

Hence

$$\begin{aligned} \chi_a(x) \to \chi_a(y) &= (y \to a) \to (((x \to a) \to a) \to a) \\ &= (y \to a) \to (x \to a) = x \to \chi_a(y). \end{aligned}$$

□

Corollary 19. *Let M be an $R\ell$-monoid and $a \in B(M)$. Then χ_a is a modal operator on M if and only if for any $x, y \in M$*

$$\chi_a(x) \odot \chi_a(y) \geq \chi_a(x \odot y).$$

Let M be an $R\ell$-monoid and f be a modal operator on M. Then $\mathrm{Fix}(f) = \{x \in M : f(x) = x\}$ will denote the set of all fixed elements of the operator f. By the definition of a modal operator it is obvious that $\mathrm{Fix}(f) = \mathrm{Im}(f)$.

Since f is a closure operator on the lattice $(M; \vee, \wedge)$, we infer that $(\mathrm{Fix}(f); \vee_F, \wedge)$, where $y \vee_F z = f(y \vee z)$ and "$\wedge$" is the restriction of the corresponding operation from M on $\mathrm{Fix}(f)$, is a lattice.

Theorem 20. *If f is a modal operator on an $R\ell$-monoid M, then* $\mathrm{Fix}(f)$ *is closed under the operations "$\odot$" and "$\to$" and* $\mathrm{Fix}(f) = \big(\mathrm{Fix}(f); \cdot, \vee_F, \wedge, \to, f(0), 1\big)$ *is an $R\ell$-monoid.*

Proof.

(i) Let $x, y \in \mathrm{Fix}(f)$. Then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, thus $x \odot y \in \mathrm{Fix}(f)$.

(ii) $(\mathrm{Fix}(f); \vee_F, \wedge, f(0), 1)$ is a bounded lattice.

(iii) If $y, z \in \mathrm{Fix}(f)$, then by Proposition 3 we have $y \to z = f(y) \to f(z) = f(f(y) \to f(z)) = f(y \to z)$, hence $y \to z \in \mathrm{Fix}(f)$.

Therefore, if $x, y, z \in \mathrm{Fix}(f)$, then $x \odot y, y \to z \in \mathrm{Fix}(f)$ and for this reason $x \odot y \leq z$ holds in $\mathrm{Fix}(f)$ if and only if $x \leq y \to z$.

(iv) By foregoing, $\mathrm{Fix}(f)$ also satisfies the identity $x \odot (x \to y) = x \wedge y$. □

Remark 21. The above theorem strengthens general Lemma 3.3 of [5] proved for any residuated lattices in our special case of bounded (commutative) $R\ell$-monoids.

THEOREM 22. *Let M be an $R\ell$-monoid, $a \in I(M)$ and*

$$I(a) := [0, a] = \{x \in M : \ 0 \leq x \leq a\}.$$

For any $x, y \in I(a)$ we set $x \odot_a y = x \odot y$ and $x \to_a y := (x \to y) \wedge a$. Then $I(a) = (I(a); \odot_a, \vee, \wedge, \to_a, 0, a)$ is an $R\ell$-monoid.

Proof.

(i) If $x, y \in I(a)$, then $x \odot y \in I(a)$ and $x \odot a = x \wedge a = x$, hence $(I(a); \odot_a, a)$ is a commutative monoid.

(ii) Obviously, $(I(a); \vee, \wedge, 0, a)$ is a bounded lattice.

(iii) Let $x, y \in I(a)$. It holds that $x \to y$ is the greatest element $z \in M$ such that $x \odot z \leq y$. Therefore $(x \to y) \wedge a$ is the greatest element in $I(a)$ with this property. That means, $x \odot_a z \leq y$ if and only if $z \leq (x \to y) \wedge a = x \to_a y$ for every $z \in I(a)$.

(iv) For any $x, y \in I(a)$ we have $x \odot_a (x \to_a y) = x \odot ((x \to y) \wedge a) = x \odot (x \to y) \odot a = (x \wedge y) \wedge a = x \wedge y$. □

Remark 23. If for any $x, y \in I(a)$ we denote by x^{-_a} the negation of an element x and by $x \oplus_a y$ the sum of elements x and y in the $R\ell$-monoid $I(a)$, then it holds

$$x^{-_a} = x^- \wedge a, \qquad x \oplus_a y = (x \oplus y) \wedge a.$$

Indeed

$$x^{-_a} = x \to_a 0 = (x \to 0) \wedge a = x^- \wedge a,$$

$$\begin{aligned} x \oplus_a y &= (x^{-_a} \odot y^{-_a})^- \wedge a = (x^- \odot a \odot y^- \odot a)^- \wedge a = (x^- \odot y^- \odot a)^- \wedge a \\ &= (a \to (x^- \odot y^-)^-) \odot a = a \wedge (x^- \odot y^-)^- = a \wedge (x \oplus y). \end{aligned}$$

Now, let M be arbitrary $R\ell$-monoid (still bounded and commutative), $a \in I(M)$ and let f be a modal operator on M. Let us consider a mapping $f^a \colon I(a) \longrightarrow I(a)$ such that $f^a(x) = f(x) \wedge a$ $(= f(x) \odot a)$, for every $x \in I(a)$.

Theorem 24. *Let M be an $R\ell$-monoid, $a \in I(M)$ and f be a modal and strong modal, respectively, operator on M. Then f^a is a modal and strong modal, respectively, operator on the $R\ell$-monoid $I(a)$.*

Proof. Assume $x, y \in I(a)$.

1. $x \le a$ and $x \le f(x)$, hence $x \le a \wedge f(x) = f^a(x)$.
2. $f^a(f^a(x)) = f(f(x) \wedge a) \wedge a = f(f(x) \odot a) \wedge a = (f(f(x)) \odot f(a)) \wedge a$
$$= f(x) \wedge f(a) \wedge a = f(x) \wedge a = f^a(x).$$
3. $f^a(x \odot y) = f(x \odot y) \wedge a = f(x) \odot f(y) \odot a \odot a = (f(x) \wedge a) \odot (f(y) \wedge a)$
$$= f^a(x) \odot f^a(y).$$
4. Let f be strong. Then
$$\begin{aligned} f^a(x \oplus_a f^a(y)) &= f^a((x \oplus (f(y) \wedge a)) \wedge a) = f((x \oplus (f(y) \wedge a)) \wedge a) \wedge a \\ &= f(x \oplus (f(y) \wedge a)) \wedge f(a) \wedge a = f(x \oplus f(f(y) \wedge a)) \wedge a \\ &= f(x \oplus ((f(f(y)) \wedge f(a))) \wedge a = f(x \oplus (f(y) \wedge f(a))) \wedge a \\ &= f(x \oplus f(y \wedge a)) \wedge a = f(x \oplus f(y)) \wedge a = f(x \oplus y) \wedge a \\ &= f^a(x \oplus y). \end{aligned}$$
□

Theorem 25.

a) *Let M be an $R\ell$-monoid, let f be a modal operator on M and $\hat{f} = f|_{I(M)}$. Then $I(M)$ is a subalgebra of the reduct $(M; \odot, \vee, \wedge, 0, 1)$ and $\hat{f}$ is a mapping of $I(M)$ into $I(M)$ satisfying conditions* 1, 2, 3 *from the definition of a modal operator.*

b) *Let M be a normal $R\ell$-monoid and let $x^- \in I(M)$ for each $x \in I(M)$. Then $I(M)$ is closed also under the operation "$\oplus$". Moreover, if f is a strong modal operator on M, then $\hat{f}$ satisfies condition* 4 *from the definition of a strong modal operator.*

c) *Let M be a BL-algebra. Then $I(M)$ is a subalgebra of the algebra M which is a Heyting algebra. If f is a modal operator on M, then $\hat{f}$ is a modal operator on the Heyting algebra $I(M)$. If $x^- \in I(M)$ holds for each $x \in I(M)$ and f is a strong modal operator on M, then $\hat{f}$ is a strong modal operator on $I(M)$.*

Proof.

a) Let M be an $R\ell$-monoid and $x, y \in I(M)$. Then
$$(x \odot y) \odot (x \odot y) = (x \odot x) \odot (y \odot y) = x \odot y,$$
thus $x \odot y = x \wedge y \in I(M)$. Further,
$$(x \vee y) \odot (x \vee y) = (x \odot x) \vee (y \odot x) \vee (x \odot y) \vee (y \odot y) = x \vee y \vee (x \odot y) = x \vee y,$$
therefore also $x \vee y \in I(M)$.

Obviously, $0, 1 \in I(M)$.

Finally, if f is a modal operator on M, then for each $x \in I(M)$ we have

$$f(x) = f(x \odot x) = f(x) \odot f(x).$$

It follows that $f(x) \in I(M)$. Therefore $\hat{f}$ is a mapping of $I(M)$ into $I(M)$ satisfying conditions 1–3.

b) If $x^- \in I(M)$ holds for every $x \in I(M)$, then (similarly to the third part of the proof of Theorem 11) for any $x, y \in I(M)$ we obtain $x \oplus y = (x \vee y)^{--}$, and hence provided M is normal we have

$$\begin{aligned}(x \oplus y) \odot (x \oplus y) &= (x \vee y)^{--} \odot (x \vee y)^{--} = ((x \vee y) \odot (x \vee y))^{--} = (x \vee y)^{--} \\ &= x \oplus y,\end{aligned}$$

therefore $x \oplus y \in I(M)$.

At the same time it is obvious that if f is a strong modal operator on M, then $\hat{f}$ fulfills condition 4 as well.

c) By [13], an $R\ell$-monoid M is a BL-algebra if and only if M is isomorphic to a subdirect product of $R\ell$-chains ($=BL$-chains). Let now a BL-algebra M be a subdirect product of BL-chains M_α, $\alpha \in \Gamma$. If $a \in M$, then $a = (a_\alpha; \alpha \in \Gamma) \in I(M)$ if and only if $a_\alpha \in I(M_\alpha)$ for each $\alpha \in \Gamma$. Let $x = (x_\alpha; \alpha \in \Gamma)$, $y = (y_\alpha; \alpha \in \Gamma) \in I(M)$. Then $x_\alpha \to y_\alpha = 1$ for $y_\alpha \geq x_\alpha$ and $x_\alpha \to y_\alpha = y_\alpha$ for $x_\alpha > y_\alpha$. Whence $(x_\alpha \to y_\alpha; \alpha \in \Gamma) \in I(M)$ and it is equal to the element $x \to y$. By [13], furthermore, $I(M)$ is a Heyting algebra.

Then it is clear that $\hat{f}$ is a modal operator on $I(M)$ for any modal operator f on M. Moreover, by [15, Proposition 5], every BL-algebra is a normal $R\ell$-monoid. Therefore, if $x^- \in I(M)$ for each $x \in I(M)$, then $\hat{f}$ is a strong modal operator on the Heyting algebra $I(M)$ for every strong modal operator f on M. □

Remark 26. For any $a \in M$, also mappings $\pi_a : M \longrightarrow M$ (in our notation) defined by $\pi_a(x) = a \vee x$ for each $x \in M$ were introduced and studied for Heyting algebras in [10]. Evidently, if M is an arbitrary $R\ell$-monoid, then π_a satisfies conditions 1 and 2 from the definition of a modal operator on M. This begs the question if π_a fulfills condition 3 from this definition as well and in which cases $\pi_a = \varphi_a$ holds, respectively.

a) If M is a Heyting algebra then $x \odot y = x \wedge y$ for any $x, y \in M$. From the distributivity of the lattice $(M; \vee, \wedge)$ it follows that condition 3 is satisfied for any $a \in M$. At the same time, $a \oplus x = (a \vee x)^{--}$, hence π_a need not generally be equal to φ_a. For example, $\pi_0(x) = x$, $\varphi_0(x) = x^{--}$.

b) If M is an MV-algebra, then $a \vee x = a \oplus x$ holds for any $a \in I(M)$ and $x \in M$, and $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$. Therefore, we have $\varphi_a = \pi_a$ for each $a \in I(M)$ and hence, for each $a \in I(M)$, moreover π_a is a strong modal operator on M .

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