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RELATIVE PURITY OVER NOETHERIAN RINGS

LADISLAV BICAN

(Communicated by Tibor Katriňák)

ABSTRACT. In this note we are going to show that if $M$ is a left module over a left noetherian ring $R$ of the infinite cardinality $\lambda \geq |R|$, then its injective hull $E(M)$ is of the same size. Further, if $M$ is an injective module with $|M| \geq (2^\lambda)^+$ and $K \leq M$ is its submodule such that $|M/K| \leq \lambda$, then $K$ contains an injective submodule $L$ with $|M/L| \leq 2^\lambda$. These results are applied to modules which are torsionfree with respect to a given hereditary torsion theory and generalize the results obtained by different methods in author's previous papers: [A note on pure subgroups, Contributions to General Algebra 12. Proceedings of the Vienna Conference, June 3–6, 1999, Verlag Johannes Heyn, Klagenfurt, 2000, pp. 105–107], [Pure subgroups, Math. Bohem. 126 (2001), 649–652].

In this paper $R$ denotes an associative ring with identity, which is usually left noetherian, and $R$-mod stands for the category of all unitary left $R$-modules. As usual, for a submodule $K$ of the module $M$ and for any element $x \in M$ the annihilator (left) ideal $(K : x)$ of $R$ consists of all elements $r \in R$ with $rx \in K$. Dualizing the notion of the injective envelope of a module ([7]) H. Bass [1] investigated the projective cover of a module and he characterized the class of so called perfect rings over which every module has a projective cover. By a projective cover of a module $M$ it is meant an epimorphism $\varphi: F \to M$ with $F$ projective and such that the kernel $K$ of $\varphi$ is superfluous in $F$ in the sense that the equality $K + L = F$ implies $L = F$ whenever $L$ is a submodule of $F$. 

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Recently, the general theory of covers has been studied intensively. If $G$ is an abstract class of modules (i.e. $G$ is closed under isomorphic copies) then a homomorphism $\varphi : G \to M$ with $G \in G$ is called a $G$-precover of the module $M$ if for each homomorphism $f : F \to M$ with $F \in G$ there is $g : F \to G$ such that $\varphi g = f$. A $G$-precover of $M$ is said to be a $G$-cover if every endomorphism $f$ of $G$ with $\varphi f = \varphi$ is the automorphism of $G$. It is well-known (see e.g. [11]) that an epimorphism $\varphi : F \to M$, $F$ projective, is a projective cover of the module $M$ if and only if it is a $P$-cover of $M$, where $P$ denotes the class of all projective modules. Denoting by $F$ the class of all flat modules, the Enochs' conjecture ([8]), whether every module over any associative ring with identity has an $F$-cover, has been recently solved in affirmative independently by E. Enochs and L. Bican with R. El Bashir in the common paper [4].

In the general theory of precovers several types of purities are used. In some cases (see e.g. [5], [6], [10], [11]) the existence of pure submodules in the kernels of some homomorphisms play an important role. Using the general theory of covers, in [6] the main result of this note appears as a corollary. However, the direct proof presented here is of some interest because the existence of non-zero pure submodules of “large” flat modules contained in submodules with “small” factors is sufficient for the existence of flat covers (see [6] and [4]).

In my previous paper [2] I proved that if $\lambda$ is an infinite cardinal, then for any torsionfree abelian group $F$ of the size $|F| \geq (2^\lambda)^+$ and any its subgroup $K$ such that $F/K$ is $p$-primary and $|F/K| \leq \lambda$, the subgroup $K$ contains a non-zero subgroup $L$ pure in $F$. This result was extended in [3] to the case when $F K$ is an arbitrary torsion group of the size at most $\lambda$, but the lower bound for the size of $F$ is $(\nu^{\aleph_0})^+$, where $\nu$ is the first cardinal with $\lambda_i < \nu$, and $\lambda_i$ are given by $\lambda_0 = \lambda$ and $\lambda_{i+1} = 2^{\lambda_i}$ for every $i = 0, 1, \ldots$.

The purpose of this note is to solve this problem for modules over left noetherian rings, which are torsionfree with respect to a given hereditary torsion theory for the category $R$-mod. As a consequence we obtain that in the abelian groups category the estimation $(2^\lambda)^+$ valid for the $p$-primary case is good enough for the general case. Moreover, we shall see that the submodule $L$ of $K$ can be found “large” in the sense that $|F/L| \leq 2^\lambda$. As a by-product we shall also prove that for any module $M$ over a left noetherian ring $R$ of the size $|M| \geq \max(|R|, \aleph_0)$ the injective envelope $E(M)$ of $M$ is of the same size as $M$.

**Theorem 1.** Let $R$ be a left noetherian ring, $\mu = \max(|R|, \aleph_0)$. If $M$ is an arbitrary module then

(i) $|E(M)| = |M|$ whenever $|M| \geq \mu$;

(ii) $|E(M)| \leq \mu$ whenever $|M| < \mu$.
Proof.

(i) Proving indirectly, let us suppose that $|E(M)| > |M|$ for some $M \in R$-mod with $|M| \geq \mu$. For each finite subset $L = \{a_1, \ldots, a_m\}$ we fix an order on $L$ and consequently we shall consider $L$ as a finite sequence. Now for each left ideal $I$ of $R$ we fix a finite set $G_I$ of generators of $I$ and to each element $x \in E(M)$ we associate the finite sequence $S_x = \{a_1 x, \ldots, a_m x\} \subseteq M$ in such a way that $I_x = (M : x)$ and $G_{I_x} = \{a_1, \ldots, a_m\}$ (as a sequence). Further, we define the equivalence relation $\sim_0$ on the set $\overline{M}_0$ by setting $x \sim_0 y$ if and only if $(M : x) = I_x = I_y = (M : y)$ and $S_x = S_y$ (as sequences, again). Obviously, there is at most $|M|$ different sequences of the form $S_x$ and consequently there exists an equivalence class $M_0$ under $\sim_0$ having $|E(M)|$ elements. Finally, we put $x_0 = 0$, we select an element $x_1 \in M_0$ arbitrarily and we shall continue by the induction. Assume that for some $n < \omega$ we have constructed the subsets $M_0, M_1, \ldots, M_n$ of $E(M)$ and the elements $x_0, x_1, \ldots, x_{n+1}$ in such a way that $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n$, $x_0 = 0$, $x_{i+1} \in M_i$, $|M_i| = |E(M)|$ for each $i = 0, 1, \ldots, n$ and $I_{x-x_i} = I_{y-x_i}$, $S_{x-x_i} = S_{y-x_i}$ for all $x, y \in M_i$ and all $i = 0, 1, \ldots, n$. Setting $\overline{M}_{n+1} = \{x \in M_n : x - x_{n+1} \notin M\}$ we obviously get the set of the size $|E(M)|$ and we define the equivalence relation $\sim_{n+1}$ on the set $\overline{M}_{n+1}$ by setting $x \sim_{n+1} y$ if and only if $I_{x-x_{n+1}} = I_{y-x_{n+1}}$ and $S_{x-x_{n+1}} = S_{y-x_{n+1}}$. By the same argument as in the case $n = 0$ we obtain the existence of an equivalence class $M_{n+1}$ under $\sim_{n+1}$ having $|E(M)|$ elements and we finally select an element $x_{n+2} \in M_{n+1}$ arbitrarily. To finish the proof, let $n \in \{1, 2, \ldots\}$ and $x \in M_n \subseteq M_{n-1}$ be arbitrary. Then $x \sim_n x_{n+1}$ yields $I_{x-x_n} = I_{x_{n+1}-x_n} = I$ and for $G_I = \{a_1, \ldots, a_m\}$ it is $a_i(x - x_n) = a_i(x_{n+1} - x_n)$, $i = 1, \ldots, m$. Thus, for each $r \in I$, $r = \sum_{i=1}^{m} r_i a_i$, we have $r(x - x_{n+1}) = r(x - x_n) - r(x_{n+1} - x_n) = \sum_{i=1}^{m} r_i a_i(x - x_n) - \sum_{i=1}^{m} r_i a_i(x_{n+1} - x_n) = 0$. On the other hand, $x - x_{n+1} \notin M$ and so there is an element $s \in I_{x-x_{n+1}} = (M : (x-x_{n+1}))$ with $0 \notin s(x-x_{n+1}) \in M$, $M$ being essential in $E(M)$. We have thus proved that $I_{x-x_n} \subseteq I_{x-x_{n+1}}$ for each $n = 1, 2, \ldots$, which contradicts the hypothesis that the ring $R$ is left noetherian.

(ii) If the ring $R$ is infinite, then $E(R) \oplus E(M) \cong E(R \oplus M)$ is of the size $\mu$ by (i), while for $R$ finite $E(R^{(\omega)} \oplus M)$ is of the size $\mu$ again, and the assertion follows easily. ∎
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For the sake of completeness we sketch the proof of the following technical lemma, the proof of which can be found in [5]. Recall, that two homomorphisms \( f : F \to M \) and \( g : G \to M \) are called \( M \)-equivalent, if there is an isomorphism \( \pi : F \to G \) such that \( g\pi = f \).

**Lemma 2.** Let \( F = \bigoplus_{\delta \in D} F_\delta \) be a direct sum of modules and let \( f : F \to M \) be an arbitrary homomorphism. Then there is a subset \( D' \subseteq D \) such that \( F = U \oplus V \), where \( U = \bigoplus_{\delta \in D'} F_\delta \), \( V \subseteq \text{Ker } f \) and for \( \delta, \varepsilon \in D' \), \( \delta \neq \varepsilon \), the restrictions \( f|_{F_\delta} \) and \( f|_{F_\varepsilon} \) are not \( M \)-equivalent.

**Proof.** Denoting \( f_\delta = f|_{F_\delta} \) for each \( \delta \in D \), we can define the equivalence relation \( \sim \) on the set \( D \) in such a way that we put \( \delta \sim \varepsilon \) if and only if \( f_\delta \) and \( f_\varepsilon \) are \( M \)-equivalent. Let \( D' \) be any representative set of equivalence classes under \( \sim \). If \( \delta \sim \varepsilon \), then there is an isomorphism \( \pi_{\varepsilon\delta} : F_\delta \to F_\varepsilon \) such that \( f_\varepsilon \pi_{\varepsilon\delta} = f_\delta \).

Setting \( G_{\varepsilon\delta} = \{ x - \pi_{\varepsilon\delta}(x) : x \in F_\delta \} \) and \( D'_\delta = \{ \varepsilon \in D : \varepsilon \sim \delta, \varepsilon \neq \delta \} \), it is a routine to check that \( F = \bigoplus_{\delta \in D'} \left( F_\delta \oplus \left( \bigoplus_{\varepsilon \in D'_\delta} G_{\varepsilon\delta} \right) \right) \), the inclusion \( V = \bigoplus_{\delta \in D'} \left( \bigoplus_{\varepsilon \in D'_\delta} G_{\varepsilon\delta} \right) \subseteq \text{Ker } f \) is obvious and the proof is complete. \( \square \)

**Theorem 3.** Let \( R \) be a left noetherian ring and let \( \lambda \geq \mu = \max(\|R\|, \aleph_0) \) be an arbitrary cardinal number. If \( M \) is an injective module with \( |M| \geq (2^\lambda)^+ \) and if \( K \) is its submodule such that \( |M/K| \leq \lambda \), then there exists a submodule \( L \subseteq K \) of \( M \) such that \( L \) is injective and \( |M/L| \leq 2^\lambda \).

**Proof.** Since \( R \) is left noetherian, we can decompose the module \( M \) into a direct sum \( M = \bigoplus_{\delta \in D} M_\delta \) of injective hulls of cyclic modules. By Lemma 2 there is a subset \( D' \subseteq D \) such that \( M = U \oplus V \), where \( V \subseteq \text{Ker } \pi, \pi : M \to M/K \) being the canonical projection, and \( U = \bigoplus_{\delta \in D'} M_\delta \), where \( \pi|_{M_\delta} \) and \( \pi|_{M_\varepsilon} \) are not \( M/K \)-equivalent whenever \( \delta, \varepsilon \in D', \delta \neq \varepsilon \). There is at most \( \mu \) left ideals of \( R \), the ring \( R \) being left noetherian, hence at most \( \mu \) different cyclic modules and consequently, by Theorem 1, at most \( (\lambda^\mu)^\mu \leq \lambda^\lambda = 2^\lambda \) homomorphisms from injective hulls of cyclic modules into \( M/K \). So, it suffices to put \( L = V \), since in this case we have \( L \subseteq K \) and \( |M/L| = |U| \leq 2^\lambda \). \( \square \)

Recall, that a hereditary torsion theory \( \sigma = (T, \mathcal{F}) \) for the category \( R\text{-mod} \) consists of two abstract classes \( T \) and \( \mathcal{F} \), the \( \sigma \)-torsion class and the \( \sigma \)-torsionfree class, respectively, such that \( \text{Hom}(T, F) = 0 \) whenever \( T \in T \) and \( F \in \mathcal{F} \), the class \( T \) is closed under submodules, factor-modules, extensions and arbitrary
direct sums, the class $\mathcal{F}$ is closed under submodules, extensions and arbitrary direct products and for each module $M$ there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

**Theorem 4.** Let $\sigma = (T, \mathcal{F})$ be a hereditary torsion theory for the category $R$-$\text{mod}$ over a left noetherian ring $R$. If $\lambda \geq \mu = \max(|R|, \aleph_0)$ is an arbitrary cardinal number and $F \in \mathcal{F}$ is an arbitrary module such that $|F| \geq (2^\lambda)^+$, then every submodule $K \leq F$ such that $|F/K| \leq \lambda$ contains a submodule $L$ with $F/L \in \mathcal{F}$ and $|F/L| \leq 2^\lambda$.

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{j} & E(F) \\
\pi \downarrow & & \downarrow g \\
F/K & \xrightarrow{i} & E(F/K)
\end{array}
\]

where $j$, $i$ are inclusions, $\pi$ is the canonical projection and the existence of a homomorphism $g$ making the square commutative follows from the injectivity of $E(F/K)$. Since $|E(F)| \geq (2^\lambda)^+$ and $|E(F/K)| \leq \lambda$ by Theorem 1, it follows from Theorem 3 that there exists an injective submodule $V \subseteq \text{Ker } g$ such that $|E(F)/V| \leq 2^\lambda$. Setting $L = V \cap F$, we obviously have $L \subseteq K$, $F/L \cong (F+V)/V \subseteq E(F)/V$ yields $|F/L| \leq 2^\lambda$ and $F/L \in \mathcal{F}$, $V$ being a direct summand of $E(F)$.

As a special case we obtain the following generalization of our previous results proved in [2] and [3].

**Corollary 5.** Let $\lambda$ be an infinite cardinal and let $F$ be a torsionfree abelian group such that $|F| \geq (2^\lambda)^+$. If $K \leq F$ is any subgroup with $|F/K| \leq \lambda$, then there exists a pure subgroup $L$ of $F$ contained in $K$ and such that $|F/L| \leq 2^\lambda$.

**Proof.** Obvious.

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**References**


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