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ON RESTRICTED DOMINATION IN GRAPHS

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ABSTRACT. The k-restricted domination number of a graph G is the minimum number d_k such that for any subset U of k vertices of G, there is a dominating set in G including U and having at most d_k vertices. Some new upper bounds in terms of order and degrees for this number are found.

1. Introduction

For a graph theory terminology not presented here, we follow Haynes, et al. [4]. All our graphs are finite and undirected with no loops or multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For any vertex v of G its *open neighborhood* $N(v,G)$ is $\{x \in V(G) : vx \in E(G)\}$, its *closed neighborhood* $N[v,G]$ is $N(v,G) \cup \{v\}$, and its *degree* $\deg(v,G)$ is $|N(v,G)|$. The minimum and maximum degrees of vertices in $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set $S \subseteq V(G)$ its *open neighborhood* $N(S,G)$ is $\bigcup_{v\in S} N(v,G)$, its *closed neighborhood* $N[S,G]$ is $N(S,G) \cup S$, and its *degree* $\deg(S,G)$ is $|N(S,G) \setminus S|$. The *k-set minimum degree* of G is the greatest integer $\delta_k(G)$ such that $\delta_k(G) \leq \deg(X,G)$ for all subsets X of $V(G)$ of cardinality k. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S,G\rangle$. The complement of a graph G is denoted by $\overline{G}$. A vertex in a graph G is said to *dominate* every vertex adjacent to it. A set D of vertices in G is a *dominating set* if every vertex in $V(G) \setminus D$ is dominated by at least one vertex in D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. Any dominating set with $\gamma(G)$ vertices is called a *γ-set*. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [4], [5].

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In this paper we study restricted domination in graphs. The concept of restricted domination was introduced by S a n c h i s [11]. We shall use the notation which was proposed by H e n n i n g [6]. Let U be a subset of vertices of a graph G. The *restricted domination number* $r(G, U, \gamma)$ *of* U is the minimum cardinality of a dominating set of G containing U. A smallest possible dominating set of G containing all the vertices in U is called a γ_U*-set*. The *k-restricted domination number* of G is the smallest integer $r_k(G, \gamma)$ such that $r_k(G, U, \gamma) \le r_k(G, \gamma)$ for all subsets U of $V(G)$ of cardinality k. In the case $k = 0$, the k-restricted domination number is the domination number. When $k = 1$ the k-restricted domination number is called the *domsaturation number* of a graph and is denoted by $\mathrm{ds}(G)$. Several results connecting ds and other graph-theoretic parameters are obtained by A r u m u g a m and K a l a [1].

2. Bounds in terms of order and degrees

The problem of determining $\gamma(G)$ for an arbitrary graph is NP-complete (G a r e y, et al. [3]). Various authors have investigated bounds on the domination number of a graph in terms of order and degrees. The earliest such result is due to O r e [8]. M c C u a i g and S h e p h e r d [7] investigated upper bounds on $\gamma(G)$ in the case $\delta(G) \ge 2$.

Theorem A. *Let G be a graph.*

(a) ([8]) *If $\delta(G) \ge 1$, then $\gamma(G) \le |V(G)|/2$.*

(b) ([7]) *If G is a connected graph of order at least 8 and $\delta(G) \ge 2$ then $\gamma(G) \le 2|V(G)|/5$.*

Similar results on the restricted domination number was established by H e n n i n g [6]:

Theorem B. ([6]) *Let G be a connected graph and $1 \le k \le |V(G)|$.*

(a) *If $\delta(G) \ge 1$, then $2r_k(G, \gamma) < |V(G)| + k$;*

(b) *If $\delta(G) \ge 2$, then $5r_k(G, \gamma) \le 2|V(G)| + 3k$.*

In this paper we obtain upper bounds on the restricted domination number, which are analogous to the following bounds on the domination number due to F l a c h and V o l k m a n n [2] and P a y a n [9]:

Theorem C. (F l a c h and V o l k m a n n [2]) *Let G be a graph, $\delta(G) > 1$, $A \subset V(G)$ and let the graph $G - N[A, G]$ have at least one isolated vertex. Then $2\gamma(G) \le |V(G)| + |A| + (1/\delta(G) - 1)\deg(A, G)$.*

Theorem D. (P a y a n [9]) *Let G be a graph of order at least two. Then $\gamma(G) \le \delta(\overline{G})(\Delta(\overline{G}) - 1)/(|V(G)| - 1) + 2$.*

We shall need the following lemma.

Lemma 2.1. *Let G be a graph, $\delta(G) \geq 1$, $\emptyset \neq X \subseteq V_0 \subseteq V(G)$ and $Z_0 \neq \emptyset$ be the set of isolated vertices of $G - V_0$. Let $D \subseteq N(Z_0, G)$ be minimal with respect to the property $Z_0 \subseteq N(D, G)$. Then:*

(a) ([2]) $2|D| \leq |Z_0| + |N(Z_0, G)| \,/\, \delta(G)$;

(b) $2r(G, X, \gamma) \leq 2r(\langle V_0, G\rangle, X, \gamma) + 2|D| + |V(G)| - |V_0| - |Z_0|$.

Proof.

(b) Let P be a γ_X-set of the graph $\langle V_0, G\rangle$ and Q be a γ-set of the graph $\langle V(G) - (V_0 \cup Z_0), G\rangle$. Then the set $S = P \cup Q \cup D$ is a dominating set of G and $X \subset S$. Hence $r(G, X, \gamma) \leq |S| \leq |P| + |Q| + |D|$ and from Theorem A it follows $r(G, X, \gamma) \leq r(\langle V_0, G\rangle, X, \gamma) + (|V(G)| - |V_0| - |Z_0|)/2 + |D|$. Hence we have the result. □

Theorem 2.2. *Let G be a graph, $\delta(G) \geq 1$, $\emptyset \neq X \subseteq V(G)$ and Z_0 be the set of isolated vertices of the graph $G - N[X, G]$.*

(i) *If $Z_0 = \emptyset$ then $2r(G, X, \gamma) \leq |V(G)| + |X| - \deg(X, G)$.*

(ii) *If $Z_0 \neq \emptyset$ then $2r(G, X, \gamma) \leq |V(G)| + |X| + \deg(X, G)/\delta(G) - \deg(X, G)$.*

Proof. Let $V_0 = N[X, G]$. Then $r(\langle V_0, G\rangle, X, \gamma) = |X|$ and $|V_0| = \deg(X, G) + |X|$.

(i): If $V_0 = V(G)$ then the result is obvious. Now, let $V_0 \neq V(G)$ and let M be a γ-set of $G - V_0$. Then $X \cup M$ is a dominating set of G. Hence by Theorem A, $r(G, X, \gamma) \leq |X| + |M| \leq |X| + (|V(G)| - |X| - \deg(X, G))/2$ and the result follows.

(ii): Let $z \in Z_0$. Since $\deg(z, G - N[X, G]) = 0$ and $\delta(G) \geq 1$, we have $\emptyset \neq N(z, G) \subseteq N[X, G]$. Let $y \in N(z, G)$. If $y \in X$ then $z \in N[X, G]$ a contradiction. Hence $y \in N(X, G) \setminus X$. So, we proved that $N(Z_0, G) \subseteq N(X, G) \setminus X$. From this and by Lemma 2.1 we have $2r(G, X, \gamma) \leq 2|X| + |Z_0| + |N(Z_0, G)|/\delta(G) + |V(G)| - |X| - \deg(X, G) - |Z_0| \leq |V(G)| + |X| + \deg(X, G)/\delta(G) - \deg(X, G)$. □

Corollary 2.3. *Let G be a graph, $\delta(G) \geq 1$ and $1 \leq k \leq |V(G)|$. Then $2r_k(G, \gamma) \leq |V(G)| + k + \delta_k(G)(1/\delta(G) - 1)$.*

Remark. Note that if $\delta(G) \geq 2$ and $|V(G)| < k + 5\delta_k(G) - 5\delta_k(G)/\delta(G)$, then the upper bound stated in Corollary 2.3 supersedes Henning's bound (see Theorem B (b)). In particular, for the Petersen graph $P_{5,2}$ which clearly has $r_2(P_{5,2}) = 4$ and $\delta_2(P_{5,2}) = 4$, from Corollary 2.3 it follows that $r_2(P_{5,2}) \leq 4$ whereas from Theorem B (b) — $r_2(P_{5,2}) \leq 5$. So, the bound stated in Corollary 2.3 is attainable.

Sampathkumar and Neeralagi [10] (see also [5, Chap. 10, pp. 291] defined a vertex x of a graph G to be *γ-totally free* if x belongs to no γ-set If x is a γ-totally free vertex of a graph G and X is a γ-set of G, then clearly $x \notin X$ and $X \cup \{x\}$ is $\gamma_{\{x\}}(G)$-set of G. Hence if x is a γ-totally free vertex of a graph G, then $1 + \gamma(G) = r(G, \{x\}, \gamma) = r_1(G, \gamma) = \mathrm{ds}(G)$. Now from Theorem 2.2 we have:

Corollary 2.4. *Let G be a graph, $\delta(G) \geq 1$ and let x be a γ-totally free vertex. Then $\gamma(G) + 1 = \mathrm{ds}(G) = r_1(G, \gamma) = r(G, \{x\}, \gamma) \leq \big(|V(G)| + 1 + \deg(x, G)(1/\delta(G) - 1)\big)/2$.*

Corollary 2.5. *Let G be a graph, $\delta(G) \geq 1$ and $\tau = \big(|V(G)| + 1 + \Delta(G) \cdot \cdot (1/\delta(G) - 1)\big)/2$.*

(i) *If G has a γ-totally free vertex of degree $\Delta(G)$ then $\gamma(G) + 1 = \mathrm{ds}(G)$ $r_1(G, \gamma) \leq \tau$.*

(ii) *If G has no γ-totally free vertex of degree $\Delta(G)$ then $\gamma(G) \leq \tau$ and $\mathrm{ds}(G)$ $r_1(G, \gamma) \leq \tau + 1$.*

(iii) ([2]) *$\gamma(G) \leq \tau$.*

We require one observation for the proof of the next theorem.

Observation 2.6. *Let G be a graph.*

(i) *If $\emptyset \neq X \subseteq V(G)$ then $X \cup \bigcap_{u \in X} N(u, \overline{G})$ is a dominating set of G and $\big|\bigcap_{u \in X} N(u, \overline{G})\big| \geq r(G, X, \gamma) - |X|$.*

(ii) *If $X \subseteq Y \subseteq V(G)$ then $r(G, X, \gamma) \leq r(G, Y, \gamma)$.*

Theorem 2.7. *Let G be a graph, $X \subseteq V(G)$, $A = V(G) - N[X, G]$ and B $\bigcap_{u \in X} N(u, G) \neq \emptyset$. Then*

$$r(G, X, \gamma) \leq (|A|(|V(G)| - 1) - \sum_{t \in A} \deg(t, G) + |B||X|)/(|A| + |B|) + 1.$$

Proof. If $A = \emptyset$, then we have to prove that $r(G, X, \gamma) \leq |X| + 1$, which is trivially true. So, we may assume $A \neq \emptyset$. Let A_1 $N[X, G] - (X$ B and let $M \subseteq E(\overline{G})$ be the set of all edges between A and B in $\overline{G}$. Note that $A = \bigcap_{q \in X} N\big(q, \overline{G}\big)$. Counting the number of edges from B to A in G, using Observation 2.6, we see that $|M| = \sum_{t \in B} |A \cap N(t, \overline{G})| = \sum_{t \in B} \big|\bigcap_{s \in X \cup \{t\}} N(s, \overline{G})$ $> \sum_{t \in B} \big(r(G, X \cup \{t\}, \gamma) - |X| - 1\big) \geq |B|(r(G, X, \gamma) - |X| - 1)$.

On the other hand, counting the number of edges from A to B in $\overline{G}$, we see that $|M| = \sum_{t\in A} |B \cap N(t,\overline{G})| = \sum_{t\in A} |(V(G) - (X \cup A \cup A_1)) \cap N(t,\overline{G})| \le \sum_{t\in A} |N(t,\overline{G})| - \sum_{t\in A} |X \cap N(t,\overline{G})| - \sum_{t\in A} |A \cap N(t,\overline{G})| \le \sum_{t\in A} |N(t,\overline{G}| - |X||A| - \sum_{t\in A} \big(r(G, X\cup\{t\},\gamma) - |X| - 1\big) \le \sum_{t\in A} |N(t,\overline{G})| - |X||A| - |A|(r(G,X,\gamma) - |X| - 1)$. Since $|N(t,\overline{G})| = |V(G)| - 1 - \deg(t,G)$, we have $|M| \le |A||V(G)| - \sum_{t\in A} \deg(t,G) - |A|r(G,X,\gamma)$.

Combining this we have $|B|(r(G,X,\gamma) - |X| - 1) \le |M| \le |A||V(G)| - \sum_{t\in A} \deg(t,G) - |A|r(G,X,\gamma)$. Hence we have the result. □

COROLLARY 2.8. *Let G be a graph of order $n \ge 2$ and let $\sigma = (n - \Delta(G) - 1) \cdot (n - \delta(G) - 2)/(n-1) + 2$.*

(i) *If G has a γ-totally free vertex of degree $\Delta(G)$ then $\gamma(G) + 1 = r_1(G,\gamma) = \mathrm{ds}(G) \le \sigma$;*

(ii) *If G has no γ-totally free vertex of degree $\Delta(G)$ then $\gamma(G) \le \sigma$ and $r_1(G,\gamma) = \mathrm{ds}(G) \le \sigma + 1$.*

Proof. If $\Delta(G) = 0$ then the result is obvious. So, we may assume $\Delta(G) \ge 1$. Let $x \in V(G)$ and $\deg(x,G) = \Delta(G)$. Let $X = \{x\}$, $A = V(G) - N[x,G]$ and $B = N(x,G)$. Clearly $|B| = \Delta(G)$ and $|A| = n - 1 - \Delta(G)$. Hence $\sum_{t\in A} \deg(t,G) \ge \delta(G)(n - 1 - \Delta(G))$. Now, from Theorem 2.7 we have: $r(G,\{x\},\gamma) \le ((n-1-\Delta(G))(n-1) - \delta(G)(n-1-\Delta(G)) + \Delta(G))/(n-1) + 1 = \sigma$. If x is γ-totally free then $\gamma(G) + 1 = \mathrm{ds}(G) = r_1(G,\gamma) = r(G,\{x\},\gamma)$, so we have (i). If x is not γ-totally free then $r(G,\{x\},\gamma) = \gamma(G) \le r_1(G,\gamma) = \mathrm{ds}(G) \le \gamma(G) + 1 = r\big(G,\{x\},\gamma\big) + 1 \le \sigma + 1$. The proof is completed. □

Remark. From Corollary 2.8 we immediately have Theorem D, because of $\delta(\overline{G}) = |V(G)| - \Delta(G) - 1$ and $\Delta\big(\overline{G}\big) = |V(G)| - \delta(G) - 1$.

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