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Strict topologies and Banach-Steinhaus type theorems

Marian Nowak

Abstract. Let $X$ be a completely regular Hausdorff space, $E$ a real Banach space, and let $C_b(X, E)$ be the space of all $E$-valued bounded continuous functions on $X$. We study linear operators from $C_b(X, E)$ endowed with the strict topologies $\beta_z$ ($z = \sigma, \tau, \infty, g$) to a real Banach space $(Y, \| \cdot \|_Y)$. In particular, we derive Banach-Steinhaus type theorems for $(\beta_z, \| \cdot \|_Y)$ continuous linear operators from $C_b(X, E)$ to $Y$. Moreover, we study $\sigma$-additive and $\tau$-additive operators from $C_b(X, E)$ to $Y$.

Keywords: vector-valued continuous functions, strict topologies, locally solid topologies, Dini-topologies, strong Mackey space, $\sigma$-additive operators, $\tau$-additive operators

Classification: 47A70, 47B38, 46E10

1. Introduction

We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on $L$ with respect to a dual pair $\langle L, K \rangle$. Given a Hausdorff locally convex space $(L, \xi)$ by $L'_\xi$ we will denote its topological dual. Recall that $(L, \xi)$ is said to be a strong Mackey space if every relatively countably $\sigma(L'_\xi, L)$-compact subset of $L'_\xi$ is $\xi$-equicontinuous (see [K, p. 196], [KO$_1$, p. 482]). Clearly, if $(L, \xi)$ is a strong Mackey space, then $\xi = \tau(L, L'_\xi)$.

Let $X$ be a completely regular Hausdorff space and $(E, \| \cdot \|_E)$ a real Banach space. Let $C_b(X, E)$ be the Banach space of all $E$-valued bounded continuous functions on $X$ provided with supremum norm $\| \cdot \|_\infty$. We will write $C_b(X)$ instead of $C_b(X, \mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. For a function $f \in C_b(X, E)$ we will write $\| f \|_E(x) = \| f(x) \|_E$ for all $x \in X$. Then $\| f \| \in C_b(X)$.

A subset $H$ of $C_b(X, E)$ is said to be solid whenever $\| f_1 \| \leq \| f_2 \|$ (i.e., $\| f_1(x) \|_E \leq \| f_2(x) \|_E$ for all $x \in X$) and $f_1 \in C_b(X, E)$, $f_2 \in H$ implies $f_1 \in H$. A linear topology $\xi$ on $C_b(X, E)$ is said to be locally solid if it has a local base at 0 consisting of solid sets (see [K], [KO$_2$], [NR]).

In the topological measure theory a number of locally convex topologies $\beta_z$ on $C_b(X, E)$, called strict topologies have been studied. Definitions of strict topologies $\beta_z$ base in some natural way on the topology of $X$, or perhaps its Stone-Čech compactification $\beta X$. Then $(C_b(X, E), \beta_z)'$ can be identified with some natural spaces $M_{\xi}(X, E')$ of $E'$-valued measures (see [F], [K], [KO$_1$], [KO$_2$], [KO$_3$], [KV], [NR]). In this paper we consider the strict topologies $\beta_\sigma, \beta_\tau, \beta_g$ and $\beta_\infty$ on
Note that in [F] and [K] the topologies $\beta_\sigma$ and $\beta_\tau$ are denoted by $\beta_1$ and $\beta$ respectively. It is well known that the strict topologies $\beta_z$ ($z = \sigma, \tau, g$ and $\infty$) are locally solid (see [K, Theorem 8.1] for $z = \sigma, \tau, \infty$ and [KO, Theorem 6] for $z = g$). Moreover, $(C_b(X,E),\beta_z)$ is a strong Mackey space for $z = \sigma$ (see [KO1, Corollary 6]); $z = \tau$ and $X$ paracompact (see [K, Theorem 6.1]); $z = \infty$ (see [K, Theorem 3.7]) and $z = g$ (see [KO2, Theorem 7]).

From now on $(Y, \| \cdot \|_Y)$ is a real Banach space, and let $Y'$ stand for its Banach dual. Let $\mathcal{L}(C_b(X,E),Y)$ stand for the space of all bounded (= $(\| \cdot \|_\infty, \| \cdot \|_Y)$-continuous) linear operators from $C_b(X,E)$ to $Y$. The strong operator topology (briefly SOT) is the locally convex topology on $\mathcal{L}(C_b(X,E),Y)$ defined by the family of seminorms $\{p_f : f \in C_b(X,E)\}$, where $p_f(T) = \|T(f)\|_Y$ for all $T \in \mathcal{L}(C_b(X,E),Y)$. The weak operator topology (briefly WOT) is the locally convex topology on $\mathcal{L}(C_b(X,E),Y)$ defined by the family of seminorms $\{p_{f,y'} : f \in C_b(X,E), y' \in Y'\}$, where $p_{f,y'}(T) = |\langle T(f), y' \rangle|$ for all $T \in \mathcal{L}(C_b(X,E),Y)$. In view of the Banach-Steinhaus theorem the space $\mathcal{L}(C_b(X,E),Y)$ provided with SOT is sequentially complete. By $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$ (for $z = \sigma, \tau, g$ and $\infty$) we will denote the subspace of $\mathcal{L}(C_b(X,E),Y)$ consisting of all those $T \in \mathcal{L}(C_b(X,E),Y)$ which are $(\beta_z, \| \cdot \|_Y)$-continuous.

In Section 2 we study topological properties of the spaces $\mathcal{L}_{\beta_z}(C_b(X,E),Y)$, where $z = \sigma, \tau, g$ and $\infty$. In particular, we derive Banach-Steinhaus type theorems for $(\beta_z, \| \cdot \|_Y)$-continuous linear operators from $C_b(X,E)$ to $Y$ (see Theorem 2.5 and Corollary 2.6 below). In Section 3 we consider $\sigma$-additive and $\tau$-additive operators from $C_b(X,E)$ to $Y$.

2. Linear operators on $C_b(X,E)$ with strict topologies

For a bounded linear operator $T : C_b(X,E) \longrightarrow Y$ let $T' : Y' \longrightarrow C_b(X,E)'$ denote its conjugate, i.e., $\langle f, T'(y') \rangle = \langle T(f), y' \rangle$ for $f \in C_b(X,E)$ and $y' \in Y'$.

**Proposition 2.1.** Let $T : C_b(X,E) \longrightarrow Y$ be a bounded linear operator and let $z = \sigma, g, \infty$ ($z = \tau$ and $X$ is paracompact). Then the following statements are equivalent:

(i) $T'(Y') \subset C_b(X,E)_{\beta_z}'$, i.e., $y' \circ T \in C_b(X,E)_{\beta_z}'$ for each $y' \in Y'$;

(ii) $T$ is $(\sigma(C_b(X,E), C_b(X,E)'_{\beta_z}), \sigma(Y,Y'))$-continuous;

(iii) $T$ is $(\beta_z, \| \cdot \|_Y)$-continuous.

**Proof:** (i)$\iff$(ii) General fact; see [AB, Proposition 9.26].

(ii)$\iff$(iii) It is known that $T$ is $(\sigma(C_b(X,E), C_b(X,E)'_{\beta_z}), \sigma(Y,Y'))$-continuous if and only if $T$ is $(\tau(C_b(X,E), C_b(X,E)'_{\beta_z}), \tau(Y,Y'))$-continuous (see [AB; Ex. 11, p. 149]). Since $\beta_z = \tau(C_b(X,E), C_b(X,E)'_{\beta_z})$ and $\tau(Y,Y')$ coincides with the $\| \cdot \|_Y$-topology, the proof is complete.

(iii)$\Longrightarrow$(i) It is obvious. \qed
Proposition 2.2. \( \mathcal{L}_{\beta_z}(C_b(X, E), Y) \) is a sequentially closed subspace of \( \mathcal{L}(C_b(X, E), Y) \) for WOT, where \( z = \sigma, g, \infty \) (resp. \( z = \tau \) and \( X \) is paracompact).

Proof: Let \( \{T_n\} \) be a sequence in \( \mathcal{L}_{\beta_z}(C_b(X, E), Y) \) such that \( T_n \rightarrow T \) for WOT, where \( T \in \mathcal{L}(C_b(X, E), Y) \). Given \( y'_o \in Y' \), for each \( f \in C_b(X, E) \) we get \( (y'_o \circ T_n)(f) = \lim (y'_o \circ T_n)(f) \), where \( y'_o \circ T_n \in C_b(X, E)'_{\beta_z} \) for \( n \in \mathbb{N}, \) and \( y'_o \circ T \in C_b(X, E)' \). It follows that \( (y'_o \circ T_n) \) is a \( \sigma(C_b(X, E)'_{\beta_z}, C_b(X, E)) \)-Cauchy sequence in \( C_b(X, E)'_{\beta_z} \). Since the space \( C_b(X, E)'_{\beta_z} \) is sequentially complete (see [KO3, Theorem 3]), there exists \( \Phi_o \in C_b(X, Y)'_{\beta_z} \) such that \( \Phi_o(f) = \lim (y'_o \circ T_n)(f) \) for each \( f \in C_b(X, E) \). Hence \( y'_o \circ T = \Phi_o \in C_b(X, E)'_{\beta_z} \), and by Proposition 2.1 we get \( T \in \mathcal{L}_{\beta_z}(C_b(X, E), Y) \).

\[ \square \]

Corollary 2.3. Let \( z = \sigma, g, \infty \) (resp. \( z = \tau \) and \( X \) is paracompact). Then

(i) \( \mathcal{L}_{\beta_z}(C_b(X, E), Y) \) is a sequentially closed subspace of \( \mathcal{L}(C_b(X, E), Y) \) for SOT;

(ii) the space \( (\mathcal{L}_{\beta_z}(C_b(X, E), Y), \text{SOT}) \) is sequentially complete.

Proof: (i) It follows from Proposition 2.2 because WOT \( \subset \) SOT.

(ii) It follows from (i) because the space \( (\mathcal{L}(C_b(X, E), Y), \text{SOT}) \) is sequentially complete.

The following general result will be of importance (see [SZ, Theorem 2]).

Proposition 2.4. Let \( K \) be a SOT-compact subset of \( \mathcal{L}_{\beta_z}(C_b(X, E), Y) \), where \( z = \sigma, g, \infty \) (resp. \( z = \tau \) and \( X \) is paracompact). If \( H \) is a \( \sigma(Y', Y) \)-closed and \( \| \cdot \|_{Y'} \)-equicontinuous subset of \( Y' \), then the set \( \bigcup \{ T'(H) : T \in K \} \) is a \( \sigma(C_b(X, E)'_{\beta_z}, C_b(X, E)) \)-compact subset of \( C_b(X, E)'_{\beta_z} \).

Now we are ready to prove Banach-Steinhaus type theorems for \( (\beta_z, \| \cdot \|_{Y'}) \)-continuous linear operators from \( C_b(X, E) \) to \( Y \).

Theorem 2.5. Let \( K \) be a SOT-compact subset of \( \mathcal{L}_{\beta_z}(C_b(X, E), Y) \), where \( z = \sigma, g, \infty \) (resp. \( z = \tau \) and \( X \) is paracompact). Then \( K \) is a \( (\beta_z, \| \cdot \|_{Y'}) \)-equicontinuous.

Proof: Since the closed unit ball \( B_{Y'} \) in \( Y' \) is \( \sigma(Y', Y) \)-closed and \( (\| \cdot \|_{Y'}) \)-equicontinuous (see [AB, Theorem 9.21]), by Proposition 2.4 the set \( Z = \bigcup \{ T'(B_{Y'}) : T \in K \} \) is \( \sigma(C_b(X, E)'_{\beta_z}, C_b(X, E)) \)-compact subset of \( C_b(X, E)'_{\beta_z} \). Since \( (C_b(X, E), \beta_z) \) is a strong Mackey space, the set \( Z = \{ y' \circ T : T \in K \} \) is \( (\beta_z, \| \cdot \|_{Y'}) \)-equicontinuous. Then for given \( \varepsilon > 0 \) there exists a neighbourhood \( V_{\varepsilon} \) of 0 for \( \beta_z \) such that for all \( f \in V_{\varepsilon} \),

\[
\sup_{T \in K} \| T(f) \|_{Y} = \sup \{ \| (y' \circ T)(f) \| : T \in K, y' \in Y' \} \leq \varepsilon.
\]

This means that \( K \) is \( (\beta_z, \| \cdot \|) \)-equicontinuous, as desired. \( \square \)
Corollary 2.6. Let $T_k : C_b(X, E) \longrightarrow Y$ be $(\beta_z, \| \cdot \|_Y)$-continuous linear operators for $z = \sigma, g, \infty$ (resp. $z = \tau$ and $X$ is paracompact) and $k \in \mathbb{N}$. Assume that $T(f) := \lim T_k(f)$ exists in $(Y, \| \cdot \|_Y)$ for each $f \in C_b(X, E)$. Then $T$ is $(\beta_z, \| \cdot \|_Y)$-continuous and the set $\{ T_k : k \in \mathbb{N} \}$ is $(\beta_z, \| \cdot \|_Y)$-equicontinuous.

PROOF: In view of the Banach-Steinhaus theorem $T$ is bounded. Hence by Corollary 2.3 $T$ is $(\beta_z, \| \cdot \|_Y)$-continuous. Then $T_k \longrightarrow T$ in $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$ for SOT, so $\{ T_k : k \in \mathbb{N} \} \cup \{ T \}$ is a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$. Hence by Theorem 2.5 the set $\{ T_k : k \in \mathbb{N} \}$ is $(\beta_z, \| \cdot \|_Y)$-equicontinuous. \hfill \Box

3. $\sigma$-additive and $\tau$-additive operators on $C_b(X, E)$

We start by recalling definitions of $\sigma$-Dini and Dini topologies on $C_b(X, E)$ (see [NR, Definition 5.1]). For a net $(u_\alpha)$ in $C_b(X)$ we will write $u_\alpha \downarrow 0$ whenever $u_\alpha(x) \downarrow 0$ for all $x \in X$.

Definition 3.1. Let $\xi$ be a locally solid topology on $C_b(X, E)$.

(i) $\xi$ is said to be a $\sigma$-Dini topology if, whenever $(f_n)$ is a sequence in $C_b(X, E)$ such that $\|f_n\| \downarrow 0$, then $f_n \longrightarrow 0$ for $\xi$.

(ii) $\xi$ is said to be a Dini topology if, whenever $(f_\alpha)$ is a net in $C_b(X, E)$ such that $\|f_\alpha\| \downarrow 0$, then $f_\alpha \longrightarrow 0$ for $\xi$.

We have (see [NR, Theorem 5.2]):

Proposition 3.1. $\beta_\sigma$ (resp. $\beta_\tau$) is the finest locally convex $\sigma$-Dini (resp. Dini) topology on $C_b(X, E)$.

Following [F, Definition 2.1] we can extend the definition of $\sigma$-additive and $\tau$-additive linear functionals on $C_b(X, E)$ to linear operators from $C_b(X, E)$ to $Y$.

Definition 3.2. Let $T : C_b(X, E) \longrightarrow Y$ be a bounded linear operator, and let $B_{\infty} = \{ g \in C_b(X, E) : \| g \|_\infty \leq 1 \}$.

(i) $T$ is said to be $\sigma$-additive if $\sup_{g \in B_{\infty}} \| T(u_n g) \|_Y \longrightarrow 0$ for each sequence $(u_n)$ in $C_b(X)$ such that $u_n \downarrow 0$.

(ii) $T$ is said to be $\tau$-additive if $\sup_{g \in B_{\infty}} \| T(u_\alpha g) \|_Y \longrightarrow 0$ for each net $(u_\alpha)$ in $C_b(X)$ such that $u_\alpha \downarrow 0$.

By $L_\sigma(C_b(X, E), Y)$ (resp. $L_\tau(C_b(X, E), Y)$) we will denote the set of all $\sigma$-additive (resp. $\tau$-additive) operators from $C_b(X, E)$ to $Y$.

Proposition 3.2. For a bounded linear operator $T : C_b(X, E) \longrightarrow Y$ the following statements are equivalent:

(i) $y' \circ T \in C_b(X, E)'_{y'_b}$ for each $y' \in Y'$;

(ii) $T$ is $(\beta_\sigma, \| \cdot \|_Y)$-continuous;

(iii) $T$ is $(\beta_\sigma, \| \cdot \|_Y)$-sequentially continuous;

(iv) $T$ is $\sigma$-additive.
Proof: (i) $\iff$ (ii) It follows from Proposition 2.1.

(ii) $\implies$ (iii) It is obvious.

(iii) $\implies$ (iv) Assume that $T$ is $(\beta_2, \| \cdot \|_Y)$-sequentially continuous, and let $u_n \downarrow 0$ in $C_b(X)$. Note that $\sup_{g \in B_\infty} \| T(u_n g) \|_Y \leq \| T \| \cdot \| u_n \|_\infty < \infty$. Let $\varepsilon > 0$ be given. Then for each $n \in \mathbb{N}$ there exists $g_n \in B_\infty$ such that

$$\sup_{g \in B_\infty} \| T(u_n g) \|_Y \leq \| T(u_n g_n) \|_Y + \frac{\varepsilon}{2}.$$ 

For a fixed $e_o \in S_E$ (= the closed unit sphere in $E$), we have $\|(u_n g(x))\|_E \leq \|(u_n \otimes e_o)(x)\|_E = \|u_n(x) e_o\|_E = u_n(x) \downarrow 0$ for all $x \in X$. Since $\beta_\sigma$ is a $\sigma$-Dini topology, we obtain that $u_n \otimes e_o \rightarrow 0$ for $\beta_\sigma$. Hence $u_n g \rightarrow 0$ for $\beta_\sigma$ because $\beta_\sigma$ is locally solid. It follows that $\| T(u_n g_n) \|_Y \rightarrow 0$. Choose $n_\varepsilon \in \mathbb{N}$ such that $\| T(u_n g_n) \|_Y \leq \frac{\varepsilon}{2}$ for $n \geq n_\varepsilon$. It follows that $\sup_{g \in B_\infty} \| T(u_n g) \|_Y \leq \varepsilon$ for $n \geq n_\varepsilon$, i.e., $T$ is $\sigma$-additive.

(iv) $\implies$ (i) It follows from [F, Theorem 2.3].

Hence we have

$$\mathcal{L}_{\beta_\sigma}(C_b(X, E), Y) = L_\sigma(C_b(X, E), Y).$$

Now we are in position to state a Banach-Steinhaus type theorem for $\sigma$-additive operators from $C_b(X, E)$ to $Y$.

**Theorem 3.3.** Let $\mathcal{K}$ be a SOT-compact subset of $L_\sigma(C_b(X, E), Y)$. Then $\mathcal{K}$ is uniformly $\sigma$-additive, i.e.,

$$\sup_{T \in \mathcal{K}} \left( \sup_{g \in B_\infty} \| T(u_n g) \|_Y \right) \rightarrow 0 \text{ whenever } u_n \downarrow 0 \text{ in } C_b(X).$$

Proof: By Theorem 2.5 the set $\mathcal{K}$ is $(\beta_\sigma, \| \cdot \|_Y)$-equicontinuous. Let $\varepsilon > 0$ be given. Then there exists a solid neighbourhood $V_\varepsilon$ of 0 for $\beta_\sigma$ such that $\| T(f) \|_Y \leq \varepsilon$ for all $f \in V_\varepsilon$ and each $T \in \mathcal{K}$. Now let $u_n \downarrow 0$ in $C_b(X)$, and let $e_o \in S_E$ be fixed. Then for each $n \in B_\infty$ we have $\|(u_n g(x))\|_E \leq \|(u_n \otimes e_o)(x)\|_E = u_n(x) \downarrow 0$ for all $x \in X$. Since $\beta_\sigma$ is a $\sigma$-Dini topology, there exists $n_\varepsilon \in \mathbb{N}$ such that $u_n \otimes e_o \in V_\varepsilon$ for $n \geq n_\varepsilon$. Hence for each $g \in B_\infty$, we get $u_n g \in V_\varepsilon$ for $n \geq n_\varepsilon$, because $V_\varepsilon$ is a solid subset of $C_b(X, E)$. It follows that for $n \geq n_\varepsilon$, $\sup_{T \in \mathcal{K}} \left( \sup_{g \in B_\infty} \| T(u_n g) \|_Y \right) \leq \varepsilon$. 

As an application of Corollary 2.6 and Theorem 3.3 we have

**Corollary 3.4.** Let $T_k : C_b(X, E) \rightarrow Y$ be $\sigma$-additive operators for $k \in \mathbb{N}$, and assume that $T(f) := \lim_k T_k(f)$ exists in $(Y, \| \cdot \|_Y)$ for each $f \in C_b(X, E)$. Then $T$ is a $\sigma$-additive operator and the set $\{ T_k : k \in \mathbb{N} \}$ is uniformly $\sigma$-additive, i.e., $\sup_k \left( \sup_{g \in B_\infty} \| T_k(u_n g) \|_Y \right) \rightarrow 0 \text{ whenever } u_n \downarrow 0 \text{ in } C_b(X)$.

Following the proofs of Proposition 3.2, Theorem 3.3 and Corollary 3.4 we can derive analogous results for $\tau$-additive operators from $C_b(X, E)$ to $Y$. 

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Proposition 3.5. Assume that $X$ is paracompact. Then for a bounded linear operator $T : C_b(X, E) \rightarrow Y$ the following statements are equivalent:

(i) $y' \circ T \in C_b(X, E)'_{\beta_T}$ for each $y' \in Y'$;
(ii) $T$ is $(\beta_T, \| \cdot \|_Y)$-continuous;
(iii) $T$ is $\tau$-additive.

Hence, if $X$ is paracompact, then

$$L_{\beta_T}(C_b(X, E), Y) = L_{\tau}(C_b(X, E), Y).$$

Theorem 3.6. Assume that $X$ is paracompact. Let $K$ be a SOT-compact subset of $L_{\tau}(C_b(X, E), Y)$. Then $K$ is uniformly $\tau$-additive, i.e.,

$$\lim_{\alpha \to 0} \sup_{T \in K} \left( \sup_{g \in B_\infty} \| T(u_\alpha g) \|_Y \right) = 0$$

whenever $u_\alpha \downarrow 0$ in $C_b(X)$.

Corollary 3.7. Assume that $X$ is paracompact. Let $T_k : C_b(X, E) \rightarrow Y$ be $\tau$-additive operators for $k \in \mathbb{N}$, and assume that $T(f) := \lim T_k(f)$ exists in $(Y, \| \cdot \|_Y)$ for each $f \in C_b(X, E)$. Then $T : C_b(X, E) \rightarrow Y$ is a $\tau$-additive operator and the set $\{T_k : k \in \mathbb{N}\}$ is uniformly $\tau$-additive, i.e.,

$$\sup_k \left( \sup_{g \in B_\infty} \| T_k(u_\alpha g) \|_Y \right) \rightarrow 0$$

whenever $u_\alpha \downarrow 0$ in $C_b(X)$.

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