Remarks on several types of convergence of bounded sequences

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Abstract. In this paper we analyze relations among several types of convergences of bounded sequences, in particulars among statistical convergence, $I\nu$-convergence, $\varphi$-convergence, almost convergence, strong $p$-Cesàro convergence and uniformly strong $p$-Cesàro convergence.

1. Introduction

Generalized approach to convergence was presented in [B, p.99] by means of the notion of a filter $\mathcal{F}$ of subsets of positive integer numbers $\mathbb{N}$. The same approach we can obtain by means of a dual notion of filter, what is an ideal $\mathcal{I}$ (i.e. ideal is an additive and hereditary class of sets). A sequence of real numbers $x = (x_n)_{n=1}^{\infty}$ is said to be $I$-convergent to $L$ provided that for every $\varepsilon > 0$ the set $A_\varepsilon$ belongs to $\mathcal{I}$, where $A_\varepsilon = \{ n \in \mathbb{N}; |x_n - L| \geq \varepsilon \}$. We write $I - \lim x_n = L$ (see [KŠW]). The notion of $I$-convergence is in certain sense equivalent to the notion of $\mu$-statistical convergence (see [C1]).

The aim of this paper is investigate relations among different types of convergences of real sequences. There are convergences defined by means of densities on the set $\mathbb{N}$ (i.e. density is a finitely additive measure) on one side. Let $\nu$ be a density then $I_{\nu} = \{ A \subset \mathbb{N}; \nu(A) = 0 \}$ is an associated ideal to the density $\nu$ which generates the $I_{\nu}$-convergence. In this paper we will use the asymptotic density and the uniform density, sometime called Banach density. On the other side, there are convergences that cannot be defined by means of any ideal $\mathcal{I}$. We take into consideration almost convergence, strong $p$-Cesàro convergence, uniformly strong $p$-Cesàro convergence and $\varphi$-convergence (for all definitions see Section 2).

The paper consists of four sections with the new results in Sections 3, where for each studied type of convergence we assign a set of all in this way convergent
sequences, these sets create linear subspaces of the linear space of all bounded sequences. We will study their mutual position and describe their structure from the point of view of topological properties, for instance separability and porosity. In the Section 4 we give an additional information for uniform density and for not so much known \( \varphi \)-convergence.

2. Definitions and basic properties

We recall some known notions. Denote by \( m \) the linear normed space of all bounded sequences \( x = (x_n)_{n=1}^{\infty} \) of real numbers with the supremum norm \( ||x|| = \sup_{n \in \mathbb{N}} |x_n| \). Let \( A \subset \mathbb{N} \). If \( m, n \in \mathbb{N} \) by \( A(m, n) \) we denote the cardinality of the set \( A \cap [m, n] \).

(i) If there exists the limit \( \lim_{n \to \infty} \frac{A(1,n)}{n} = d(A) \), then \( d(A) \) is said to be the asymptotic density of \( A \).

(ii) The following limits exist
\[
\lim_{n \to \infty} \min_{m=0,1,\ldots,n} \frac{A(m+1,m+n)}{n} = u(A),
\lim_{n \to \infty} \max_{m=0,1,\ldots,n} \frac{A(m+1,m+n)}{n} = \pi(A),
\]
and they are called the lower and upper uniform density of the set \( A \), respectively. If \( u(A) = \pi(A) = u(A) \) then \( u(A) \) is called the uniform density of \( A \), (see \([BF, BF1]\)). It is clear that if there exists \( u(A) \), then also there exists \( d(A) \) and \( u(A) = d(A) \).

The converse is not true (for instance Example 2 in the Section 3).

(iii) Put \( \mathcal{I} = \mathcal{I}_d = \{ A \subset \mathbb{N}; d(A) = 0 \} \), then \( \mathcal{I}_d \)-convergence coincides with the statistical convergence, which was introduced by H. Fast (1951)\([F]\) (see also \([C, Fr, P, S, \mathring{S}]\)). If \( x = (x_n)_{n=1}^{\infty} \) converges statistically to \( L \) then we write \( \lim \text{stat} x_n = L \) and \( \lim \text{stat} x_n = \mathcal{I}_d \)-lim \( x_n \). By \( m_0 \) we denote the set of all bounded statistical convergent sequences (see \([\mathring{S}]\)).

(iv) In the case if \( \mathcal{I} = \mathcal{I}_u = \{ A \subset \mathbb{N}; u(A) = 0 \} \) we obtain \( \mathcal{I}_u \)-convergence. If \( x = (x_n)_{n=1}^{\infty} \) is \( \mathcal{I}_u \)-convergent to \( L \) we write \( \mathcal{I}_u \)-lim \( x_n = L \). By \( m_1 \) we denote the set of all bounded \( \mathcal{I}_u \)-convergent sequences.

Further we recall the notions of strong \( p \)-Cesàro convergence, uniformly strong \( p \)-Cesàro convergence that is generalization of notion of strong almost convergence (see \([M]\)) and almost convergence.

(v) We say that a bounded sequence \( x = (x_n)_{n=1}^{\infty} \) is almost convergent or fast convergent to a number \( L \) if \( \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} x_{n+i} = L \), uniformly in \( n \) what is equivalent to condition that every Banach limit \(^1\) of \( x \) is equal to \( L \) (see \([MO, KN p. 216]\), \([P, p. 59-62]\)). By \( F \) we denote the set of all almost convergent sequences.

(vi) A sequence \( x = (x_n)_{n=1}^{\infty} \) is said to be strong \( p \)-Cesàro convergent \((0 < p < \infty)\) to a number \( L \) if \( \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} |x_i - L|^p = 0 \) (see \([C]\)). A sequence \( x = (x_n)_{n=1}^{\infty} \) is said to be uniformly strong \( p \)-Cesàro convergent \((0 < p < \infty)\) to a number \( L \) if \( \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} |x_{n+i} - L|^p = 0 \) uniformly in \( n \). This notion was introduced in \([BS]\) and it is a generalization of a notion of strong almost convergence in \([M]\). As usual \( x = (x_n)_{n=1}^{\infty} \) is Cesàro summable if there exists \( (C, 1) - \lim x_n = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = L \). By \( w_p \) resp. \( uw_p \) denote the set of all strong \( p \)-Cesàro convergent sequences.

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\(^1\) A Banach limit is a bounded linear functional on the space \( m \) of all bounded sequences \( x = (x_n)_{n=1}^{\infty} \) such that the sequence \( x_n = 1 \) has the Banach limit 1, and \( x \) and shifted \( x' = (x_{n+1})_{n=1}^{\infty} \) have the same Banach limit (if exists).
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uniformly strong $p$-Cesàro convergent sequences, respectively. It is immediate that $uw_p \subset w_p$ ($0 < p < \infty$) and Example 2 shows that the inclusion is strict. As usual by $c$ resp. $c_1$ denote the set of all convergent sequences, Cesàro sumable, respectively.

Notion $\varphi$-convergence had been introduced by I.J. Schoenberg (1959)[S].

(vii) A sequence $x = (x_n)_{n=1}^{\infty}$ is said to be $\varphi$-convergent to a number $L$ if
$$\lim_{n \to \infty} \frac{1}{n} \sum_{d|n} \varphi(d) x_d = L,$$
where $\varphi(n)$ is Euler function i.e. $\varphi(n)$ is the number of elements from $\{1, 2, \ldots, n\}$ coprime to $n$ and $\sum_{d|n}$ is sum over the positive divisors $d$ of $n$. By $c_\varphi$ we denote the set of all $\varphi$-convergent sequences.

The notion of porosity in a metric space is introduced in conformity with the definition of porosity in line (see [T, p. 183-190] and [Z]) as follows.

(viii) Let $(X, d)$ be a metric space and $Y \subset X$, $x \in X$, $\delta > 0$, then symbol $\gamma(x, \delta, Y)$ denotes the supremum of all $t > 0$ for which there exists $y \in X$ such that $B(y, t) \subset B(x, \delta) \setminus Y$. Here $B(x, \delta)$ denotes a ball centered at $x \in X$ with the radius $\delta > 0$. If there exist no such $t > 0$, then $\gamma(x, \delta, Y) = 0$. The numbers
$$p(x, Y) = \liminf_{\delta \to 0^+} \frac{\gamma(x, \delta, Y)}{\delta}, \quad p(x, Y) = \limsup_{\delta \to 0^+} \frac{\gamma(x, \delta, Y)}{\delta},$$
are called the lower and upper porosity of the set $Y$ at $x$. We say that $Y$ is porous or very porous at $x$ if $p(x, Y) > 0$ or $\overline{p}(x, Y) > 0$, respectively. If for each $x \in X$ we have $\overline{p}(x, Y) > 0$ or $p(x, Y) > 0$, then $Y$ is said to be porous or very porous in $X$, respectively. Obviously every porous set in $X$ is nowhere dense in $X$. If $\overline{p}(x, Y) \geq c > 0$ or $p(x, Y) \geq c > 0$ then $Y$ is called $c$-porous or very $c$-porous at $x$, respectively. If $Y$ is $c$-porous or very $c$-porous at $x$ for each $x \in X$, then $Y$ is called $c$-porous or very $c$-porous in $X$, respectively. In the case, that the number $p(x, Y) = \lim_{\delta \to 0^+} \frac{\gamma(x, \delta, Y)}{\delta}$ exists, it is called the porosity of $Y$ at $x$.

3. Results

In this section we analyse relations among types of convergence of bounded sequences defined above and describe the structure of the following spaces

c - the set of all convergent sequences,
c_\varphi - the set of all $\varphi$-convergent sequences,
m_1 - the set of all $\mathcal{T}_n$-convergent sequences,
$uw_p$ - the set of all uniformly strong $p$-Cesàro convergent sequences ($0 < p < \infty$),
w_p - the set of all strong $p$-Cesàro convergent sequences ($0 < p < \infty$),
m_0 - the set of all statistical convergent sequences,
F - the set of all almost convergent sequences,
c_1 - the set of all Cesàro sumable sequences,
m - the set of all bounded sequences,

equipped by the sup-norm from point of view of topological properties as subspaces of all bounded sequences $x = (x_n)_{n=1}^{\infty}$ of real numbers with the same norm.

I.J. Maddox (1974)[M1] has been shown that $m_0 = w_p$, $0 < p < \infty$ and $m_0 \subset c_1$ (see also [C, S]). In [BS] can be found that $m_1 \subset F$ and $m_1 = uw_p \subset m_0$ ($0 < p < \infty$). The inclusion $c_\varphi \subset m_0$ was proved in [S].
In this paper we obtain the following relations:

\[ c \subset m_1 = uwp \subset m_0 = wp \subset c_1 \subset m, (0 < p < \infty), \]

\[ c \subset m_1 \subset F \subset c_1 \subset m, \]

\[ c \subset c_\varphi \subset m_0 \subset m_1 \subset m. \]  

(1)

Following example shows, that all inclusions in (1) are strict.

**Example 1** Let \( P \) be the set of all primes. Define \( x_n = 1 \) for \( n \in P \) and \( x_n = 0 \) otherwise. For the reason that \( u(P) = 0 \) (see [BF1]), we have that \( x = (x_n)_{n=1}^\infty \) is \( \mathcal{I}_u \)-convergent to 0 but as we can see it is not convergent. Therefore \( c \neq m_1 \).

**Example 2** It is easy to see that for the set \( A = \bigcup_{k=1}^\infty ([10^k + 1, 10^k + k] \cap \mathbb{N}) \) we have \( d(A) = 0, \overline{u}(A) = 0, \overline{m}(A) = 1 \). Put \( x_n = 1 \) for \( n \in A \) and \( x_n = 0 \) for \( n \notin A \). Then \( \mathcal{I}_d - \lim x_n = 0 \) but \( x = (x_n)_{n=1}^\infty \) is not \( \mathcal{I}_u \)-convergent. Therefore \( m_1 \neq m_0 \).

**Example 3** Let \( x = (x_n)_{n=1}^\infty \) be the sequence defined by \( x_n = 1 \) if \( n \) is even and \( x_n = 0 \) if \( n \) is odd. The sequence \( x \) is Cesàro summable to 1/2 but it is not statistically convergent. Therefore \( m_0 \neq c_1 \).

Example 3 simultaneously shows that \( m_1 \neq F \) and also Example 2 shows that \( F \neq c_1 \). To show that \( c \neq c_\varphi \), we use the following P. Erdős’ solution (see [E]) of the problem 6090, AMM 1976, p. 385 proposed by T. Šalát and O. Strauch.

**Example 4** Let \( P = \{p_1 < p_2 < \cdots < p_k < \cdots\} \) be the set of all primes. Put \( A = \{p_1, p_1 p_2, \ldots, p_1 p_2 \cdots p_k, \ldots\} \) and define \( x_n = 1 \) for \( n \in A \) and \( x_n = 0 \) otherwise. Then the sequence \( x = (x_n)_{n=1}^\infty \) is \( \varphi \)-convergent to zero but it is not convergent.

Example 1 simultaneously shows that \( c_\varphi \neq m_0 \). To show that \( x = (x_n)_{n=1}^\infty \) is not \( \varphi \)-convergent we use [S, p.366, Th.2].

In [MO] is shown that almost convergence and statistical convergence are not compatible neither in the case of bounded sequences. Moreover Example 2 shows that \( m_0 \setminus F \neq \emptyset \) and Example 3 simultaneously shows that \( F \setminus m_0 \neq \emptyset \). On the basis of (1) the following question arise, what is mutual position between \( m_1 \) and \( c_\varphi \) or \( c_\varphi \) and \( F \), respectively. First of all we show that \( \mathcal{I}_u \)-convergence and \( \varphi \)-convergence are not compatible. Example 1 shows that \( m_1 \setminus c_\varphi \neq \emptyset \) and the following Example 5 shows that \( c_\varphi \setminus m_1 \neq \emptyset \).

**Example 5** Directly by E. Kováč [Ko, Th.6.5]: Clearly, there exists an increasing sequence \( a = (a_k)_{k=1}^\infty \) of positive integers such that greatest common divisor \( (a_i, a_j) = 1 \) for every \( i \neq j \) and \( \varphi(a_k)/a_k \to 0 \), such a sequence \( (a_k)_{k=1}^\infty \) can be construct by multiplication of sufficiently long interval of consecutive primes. To this sequence \( (a_k)_{k=1}^\infty \) it can be construct an increasing sequence \( (b_k)_{k=1}^\infty \) of positive integers such that \( b_{k+1} > b_2 k \) and \( a_{k+1}|b_k + 1, a_{k+2}|b_k + 2, \ldots, a_{k+k}|b_k + k \). This follows from Chinese remainder theorem. Put \( A = \bigcup_{k=1}^\infty ([b_k + 1, b_k + k] \cap \mathbb{N} \) and let \( x = (x_n)_{n=1}^\infty \) be the characteristic function of \( A \) thus \( x_n = 1 \) for \( n \in A \) and \( x_n = 0 \) otherwise. Then \( x = (x_n)_{n=1}^\infty \) is \( \varphi \)-convergent to 0 but it is not \( \mathcal{I}_u \)-convergent since \( u(A) = 0, \overline{m}(A) = 1 \).
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Both \( \varphi \)-convergence and almost convergence are not compatible, as we can see Example 5 and Example 3 simultaneously show that \( c_\varphi \setminus F \neq \emptyset \) and \( F \setminus c_\varphi \neq \emptyset \), respectively.

It is well known that if \( E_0 \) is a closed linear subspace of a linear normed space \( E \) and \( E_0 \neq E \), then \( E_0 \) is a nowhere dense set in \( E \) (see [G], [K, p.37, Ex.4]). This fact evokes the question about the porosity of \( E_0 \). The solution is given by the following theorem (see [KMSS, Th.2.5]):

**Theorem 1** Suppose that \( E \) is a linear normed space and \( E_0 \) is its closed linear subspace, \( E_0 \neq E \). Then \( E_0 \) is a very porous set in \( E \), in more detail

a) If \( x \in E \setminus E_0 \), then \( p(x, E_0) = 1 \),

b) If \( x \in E_0 \), then \( p(x, E_0) = 1/2 \).

In [S] is proved that \( m_0 \) is a closed linear subspace of the space \( m, m_0 \neq m \). That is why \( m_0 \) is a very porous set in the space \( m \). According to (1) we have the following Lemma.

**Lemma 1** Each of sets \( m_1, uw_p \) and \( w_p \) \((0 < p < \infty)\) is a very porous set in \( m \).

In [L] is proved that \( F \) is a closed linear subspace of the space \( m, F \neq m \). On that account \( F \) is a very porous set in the space \( m \). Also from this fact we get that each of sets \( m_1 \) and \( uw_p \) \((0 < p < \infty)\) is a very porous set in \( m \).

For the proof of the next Theorem 2 we first prove the following lemma.

**Lemma 2** The set \( m_1 \) is closed in \( m \).

**Proof** Let \( x^{(k)} = (x_j^{(k)})_{j=1}^\infty \) \((k = 1, 2, \ldots)\) belong to \( m_1 \), \( x^{(k)} \to x, x = (x_j)_{j=1}^\infty \) in \( m \) i.e. \( \|x^{(k)} - x\| \to 0 \) by \( k \to \infty \). Since \( m_1 \subset F \) and \( F \) is a closed set in \( m \) we get that \( x \in F \). Hence \( x^{(k)} \in m_1 \subset F \) we have \( x^{(k)} \) is almost convergent to some number \( L_k \) for all \( k = 1, 2, \ldots \). We shall prove that the sequence \( (L_k)_{k=1}^\infty \) is convergent to some number \( L \) and the sequence \( x = (x_j)_{j=1}^\infty \) is \( L \)-convergent to \( L \). A simple estimation gives

\[
|L_k - L_r| \leq \left| \frac{x_{n+1}^{(k)} + x_{n+2}^{(k)} + \cdots + x_{n+p}^{(k)}}{p} - L_k \right| + \\
+ \left| \frac{x_{n+1}^{(k)} + x_{n+2}^{(k)} + \cdots + x_{n+p}^{(k)}}{p} - \frac{x_{n+1}^{(r)} + x_{n+2}^{(r)} + \cdots + x_{n+p}^{(r)}}{p} \right| + \\
+ \left| \frac{x_{n+1}^{(r)} + x_{n+2}^{(r)} + \cdots + x_{n+p}^{(r)}}{p} - L_r \right|. 
\]

(2)

Let \( \varepsilon > 0 \). Since \( x^{(k)} \to x \) in \( m \), there exists an \( n_0 \) such that for \( k, r > n_0 \) we have \( \|x^{(k)} - x^{(r)}\| \leq \frac{\varepsilon}{3} \). Let us choose fixed \( k, r \) such that \( k, r > n_0 \). Since \( x^{(k)} = (x_j^{(k)})_{j=1}^\infty \) is almost convergent to \( L_k \) and \( x^{(r)} = (x_j^{(r)})_{j=1}^\infty \) is almost convergent to \( L_r \), there exists an \( p_0 \) such that the first and third summand in (2) is less than \( \frac{\varepsilon}{3} \).
for $p > p_0$, $n = 1, 2, \ldots$, respectively. The second summand is also less than $\frac{\varepsilon}{3}$, what can be shown as follows:

$$\left| \frac{x_{n+1}^{(k)} + x_{n+2}^{(k)} + \cdots + x_{n+p}^{(k)}}{p} - \frac{x_{n+1}^{(r)} + x_{n+2}^{(r)} + \cdots + x_{n+p}^{(r)}}{p} \right| \leq \frac{1}{p} \sum_{j=1}^{p} \left| x_{n+j}^{(k)} - x_{n+j}^{(r)} \right| \leq \frac{1}{p} \sum_{j=1}^{p} \| x^{(k)} - x^{(r)} \| = \| x^{(k)} - x^{(r)} \| < \frac{\varepsilon}{3}.$$ 

Consequently for $k, r > n_0$ we obtain $|L_k - L_r| < \varepsilon$. Since $(L_k)_{k=1}^{\infty}$ is a Cauchy sequence, then there exists an $L$ such that $L = \lim_{k \to \infty} L_k$. Further, let $\eta > 0$. Put $A_\eta = \{ k \in \mathbb{N} ; |x_k - L| \geq \eta \}$. Since $x^{(k)} \to x$ in $m$, there exists an $r \in \mathbb{N}$ such that $\| x^{(r)} - x \| < \frac{\eta}{3}$ and $\| L_r - L \| < \frac{\eta}{3}$ simultaneously. Put $B_\eta = \{ k \in \mathbb{N} ; |x_k^{(r)} - L_r| \geq \frac{\eta}{3} \}$. For an arbitrary $k \in \mathbb{N}$ we have

$$|x_k - L| \leq |x_k - x_k^{(r)}| + |x_k^{(r)} - L_r| + |L_r - L| < \frac{2\eta}{3} + |x_k^{(r)} - L_r|.$$ (3)

Hence if $k \in A_\eta$, according to (3) we have $k \in B_\eta$, thus $A_\eta \subset B_\eta$. Using the fact that $u(B_\eta) = 0$ we get $u(A_\eta) = 0$ and therefore $I_u - \lim x_k = L$ thus $x = (x_k)_{k=1}^{\infty} \in m_1$.

**Theorem 2** The set $m_1$ is a perfect, very porous and not separable set in $m_0$.

**Proof** The facts that $m_0$ is a closed set in $m$ and Lemma 2 imply, that $m_1$ is a closed set in $m_0$. Since $m_1$ is a linear space and $m_1 \neq m_0$ (see Example 2) by Theorem 1 we obtain that $m_1$ is a very porous set in $m_0$. Further, if $x = (x_j)_{j=1}^{\infty} \in m_1$ then for every $\eta > 0$ a sequence $x_1 + \eta, x_2, x_3, \ldots, x_j, \ldots$ also belongs to $m_1$. So we get that $m_1$ is dense in itself. To prove that $m_1$ is not separable it is sufficient to construct uncountable many sequences belong to $m_1$ having the distance 1 from each other. Let $B \subset \mathbb{N}$ be an infinite set such that $u(B) = 0$ (see Example 1). Put $M$ the set of those sequences $x = (x_j)_{j=1}^{\infty}, x \in m_1$ for which $x_j$ is equal 0 or 1 if $j \in B$ and $x_j = 0$ otherwise. Evidently $card M = c$ - power of continuum and for all $x, y \in M$ such that $x \neq y$ we have $|x - y| = 1$.

**Remark** Since $m_1 = uw_p, (0 < p < \infty)$ we have that $uw_p$ is a perfect, very porous and not separable set in $m_0$.

It is easily to verify that $c_1$ is a closed linear subspace of the space $m$ and $m_0 \neq c_1$ (see Example 3). So we get the following proposition.

**Theorem 3** The set $m_0$ is a perfect, very porous and not separable set in $c_1$.

**Proof** The proof is analogous as the proof of the previous theorem. Non separability of $m_0$ immediately is implied by the non separability of $m_1$.

**Remark** Again since $m_0 = w_p, (0 < p < \infty)$ we have that $w_p$ is a perfect, very porous and not separable set in $c_1$.

As we mentioned (see [L]), $F$ is a closed and non separable set in $m$, moreover $F \subset c_1, F \neq c_1$. So we have a similar proposition as Theorem 3.

**Theorem 4** The set $F$ is a perfect, very porous and not separable set in $c_1$.
The following theorem investigates topological properties of subspace $c_\varphi$ in $m_0$.

**Theorem 5** The set $c_\varphi$ is a perfect, very porous and not separable set in $m_0$.

**Proof** First of all we shall prove that the set $c_\varphi$ is a closed set in $m$. Let $x^{(k)} = (x_j^{(k)})_{j=1}^{\infty} \ (k=1,2,\ldots)$ belong to $c_\varphi$, $x^{(k)} \to x$, $x = (x_j)_{j=1}^{\infty}$ in $m$. Since $x^{(k)} \in c_\varphi$, we have $x^{(k)}$ is $\varphi$-convergent to some number $L_k$, for all $k = 1, 2, \ldots$ To prove that $x \in c_\varphi$ it is sufficient to show that the sequence $(L_k)_{k=1}^{\infty}$ is convergent to some number $L$ and the sequence $x = (x_j)_{j=1}^{\infty}$ is $\varphi$-convergent to $L$. For $n, k, j \in N$ we put

$$
S_n(x^{(k)}, L_k) = \left| \frac{\sum_{d\mid n} \varphi(d)x_d^{(k)}}{n} - L_k \right|,
$$

$$
S_n(x^{(k)}, x^{(j)}) = \left| \frac{\sum_{d\mid n} \varphi(d)x_d^{(k)}}{n} - \frac{\sum_{d\mid n} \varphi(d)x_d^{(j)}}{n} \right|.
$$

For $n = 1, 2, \ldots$ a simple estimation gives

$$
S_n(x^{(k)}, x^{(j)}) \leq \frac{\sum_{d\mid n} \varphi(d)|x_d^{(k)} - x_d^{(j)}|}{n} \leq \|x^{(k)} - x^{(j)}\| |\sum_{d\mid n} \varphi(d)|n = \|x^{(k)} - x^{(j)}\| (4)
$$

Since $x^{(k)} = (x_j^{(k)})_{j=1}^{\infty}$ is Cauchy sequence then for $\varepsilon > 0$ there exists a $k_0$ such that for arbitrary $k, j > k_0$ we have $S_n(x^{(k)}, x^{(j)}) \leq \frac{\varepsilon}{3}$ for $n = 1, 2, \ldots$ Let us choose fixed $k, j$ such that $k, j > k_0$. Since $x^{(k)} = (x_j^{(k)})_{j=1}^{\infty}$ is $\varphi$-convergent to $L_k$ and $x^{(j)} = (x_i^{(j)})_{i=1}^{\infty}$ is $\varphi$-convergent to $L_j$, there exists an $n_0$ such that for each $n > n_0$ we have $S_n(x^{(k)}, L_k) < \frac{\varepsilon}{3}$, $S_n(x^{(j)}, L_j) < \frac{\varepsilon}{3}$ and the simple estimation yields

$$
|L_k - L_j| \leq S_n(x^{(k)}, L_k) + S_n(x^{(k)}, x^{(j)}) + S_n(x^{(j)}, L_j) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
$$

The fact that $(L_k)_{k=1}^{\infty}$ is a Cauchy sequence implies the existence of a number $L$ such that $L = \lim_{k\to\infty} L_k$. Further, let $\eta > 0$. Since $x^{(k)} \to x$ in $m$, there exists an $r \in N$ such that $\|x^{(r)} - x\| \leq \frac{\varepsilon}{3}$ and $|L_r - L| < \frac{\varepsilon}{3}$ simultaneously. Let $r$ be a fixed natural number. Because $x^{(r)} = (x_j^{(r)})_{j=1}^{\infty}$ is $\varphi$-convergent to $L_r$, so there exists an $n_0$ such that for each $n > n_0$ we have $S_n(x^{(r)}, L_r) < \frac{\varepsilon}{3}$. From this and applying (4) we have

$$
S_n(x, L) \leq S_n(x, x^{(r)}) + S_n(x^{(r)}, L_r) + |L_r - L| < \eta
$$

for $n > n_0$ and thus the sequence $x = (x_j)_{j=1}^{\infty}$ is $\varphi$-convergent to $L$. The fact that $c_\varphi$ and $m_0$ are closed sets in $m$ implies that $c_\varphi$ is a closed set in $m_0$. Since $c_\varphi$ is a linear space and $c_\varphi \neq m_0$ (see Example 1) by Theorem 1 we obtain that $c_\varphi$ is a very porous set in $m_0$.

Further, if $x = (x_j)_{j=1}^{\infty} \in c_\varphi$ then for every $t > 0$ a sequence $x_1 + t, x_2, x_3, \ldots, x_j, \ldots$ also belongs to $c_\varphi$. So we get that $c_\varphi$ is dense in itself.

Again to prove that $c_\varphi$ is not separable it is sufficient to construct uncountable many sequences belong to $c_\varphi$ having the distance 1 from each other. Let $A = \{p_1, p_1 p_2, \ldots, p_1 p_2 \ldots p_k, \ldots\}$ be a set defined in Example 4. Then the sequence $x = (x_n)_{n=1}^{\infty}$ defined as $x_n = 1$ for $n \in A$ and $x_n = 0$ otherwise is $\varphi$-convergent to 0. Consider $K$ the set of those sequences $y = (y_k)_{k=1}^{\infty}$, for which $y_k = 0$ or 1 if $k \in A$ and $y_k = 0$ otherwise. Evidently card $K = c$. For each $y \in K$ we have
\[\sum_{d|n} \varphi(d)y_d \leq \sum_{d|n} \varphi(d)x_d \text{ for } n = 1, 2, \ldots.\] Thus \(y = (y_k)_{k=1}^\infty\) is \(\varphi\)-convergent to 0 and so \(y \in c_\varphi\). Moreover if \(z, y \in K\) such that \(x \neq y\) we have \(||z - y|| = 1\). \(\square\)

4. Concluding remarks

This section contains a brief addition of results and problems concerning the uniform density and \(\varphi\)-convergence.

- The notion of uniform density \(u(A)\) defined in Section 2(ii), can be found in different parts of number theory. It is also known as Banach density, since \(u(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n c_A(n + i)\), uniformly in \(n\), see Section 2(v). Here \(c_A(x)\) is the characteristic function of \(A\).

In [KN, p.40, Def. 5.1] was studied the concept of well-distributed sequence as follows: A sequence \(x = (x_n)_{n=1}^\infty\) in \([0, 1)\), is said to be \textit{well-distributed sequence} if for every interval \(I \subset [0, 1)\) we have \(\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N c_I(x_{n+i}) = |I|\) uniformly in \(n\). Here \(|I|\) is the length of \(I\). In [KN, p. 42, Ex. 5.2] is shown that the sequence \(x = (\lfloor n\theta \rfloor)_{n=1}^\infty\) is (i.e. \(x_{n+1}/x_n \geq c > 1\)) are Hartman sequences, where \(\lfloor y \rfloor\) means the integer part of \(y\). Of course the uniform density of such sequences is 0. \textit{Beatty sequence} \(x = ([n\beta + \gamma])_{n=1}^\infty\), where \(\beta > 1\) is irrational and \(\gamma\) is appropriate is also Hartman.

- As we mentioned in Section 2(vii) \(\varphi\)-convergence was introduced by I.J. Schoenberg (1959) [5]. Denote the \(\varphi\)-transformation of \(x = (x_n)_{n=1}^\infty\) as \(y = (y_n)_{n=1}^\infty\) where \(y_n = \frac{1}{n} \sum_{d|n} \varphi(d)x_d\) and \(x_n \to L\) denotes the classical limit. Schoenberg’s main results are:
  
  (i) If \(y_n \to L\) then \(x_{n_k} \to L\) for every subsequence \((x_{n_k})_{k=1}^\infty\), for which \(\varphi(n_k)/n_k \geq \delta > 0\).
  
  (ii) If \(y_n \to L\) then \(x_n\) is statistical convergent to \(L\).
  
  (iii) \(x_n = \frac{1}{\varphi(n)} \sum_{d|n} \mu(d)\varphi(d)y_d\), that is a result of Möbius’ inversion formula. \(^2\)

These results lead to the following open problems:
- Test some number-theoretic statistical convergent sequences whether they are also \(\varphi\)-convergent, e.g. the sequence \(\frac{\omega(n)}{\log \log n}\) which is statistical convergent to 1, see also [SP, p. 2–35, 2.3.23].
- Input any convergent sequence \(y_n \to L\) into the inversion formula (iii), then the result is a sequence \(x_n\) which is statistical convergent and \(\varphi\)-convergent to \(L\), simultaneously. Find \(y_n \to L\) such that \(x_n\) is no \(I_u\)-converges to \(L\) (different from Example 5).

\(^2\)The Möbius function is defined by \(\mu(n) = (-1)^{\omega(n)}\) for square-free \(n\) and \(\mu(n) = 0\) others. Here \(\omega(n)\) is the number of different primes dividing \(n\).
5. Obituary notice
Professor Tibor Šalát died on 2005 May 14th.

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