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The Generalized Criterion of Dieudonné for Valuated $p$-Groups

Peter Danchev

Abstract. We prove that if $G$ is an abelian $p$-group with a nice subgroup $A$ so that $G/A$ is a $\Sigma$-group, then $G$ is a $\Sigma$-group if and only if $A$ is a $\Sigma$-subgroup in $G$ provided that $A$ is equipped with a valuation induced by the restricted height function on $G$. In particular, if in addition $A$ is pure in $G$, $G$ is a $\Sigma$-group precisely when $A$ is a $\Sigma$-group.


1. Introduction

Let all groups under discussion be $p$-primary abelian groups, that are commutative groups each element of which has a finite order equal to a power of $p$, written additively as is customary when regarding such groups.

In 1952, Jean Dieudonné [8] proves his remarkable generalization of the classical Kulikov’s criterion [10] for direct sums of torsion cyclic groups (e.g., cf. [9] too). His generalized version possesses a rather convincing for applications form, which shows that the structure of a group to be a direct sum of cycles depends on this how its subgroup is situated inside the whole group such that the quotient of the group modulo its subgroup is a direct sum of cycles.

The Dieudonné’s affirmation was strengthened by us in a subsequent series of papers ([3]-[7]) for various exotic classes of primary groups. More especially, in [4] and [5] respectively, we have proved the valuated versions of the Generalized Dieudonné Criterion for $\sigma$-summable groups as well as for summable groups and totally projective groups both with countable lengths, respectively.

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Our purpose here is to give an enlarged formulation, in terms of valuated subgroups, of the corresponding result in [7] for Σ-groups. Also, the approach in constructing the generating subgroups is quite different than that of [7]. All our other unexplained specially notions and notation will be the same as those in [9].

2. The main result

Before formulating the chief theorem, we need a few conventions. First of all, we recollect a necessary and sufficient condition for a group to be a Σ-group.

**Criterion ([11]).** The group $H$ is a Σ-group $\iff H[p] = \cup_{n<\omega} H_n$, $H_n \subseteq H_{n+1} \leq H[p] : H_n \cap p^n H \subseteq p^n H = \cap_{i<\omega} p^i H$, $\forall n \geq 1$.

Assume that $K < H$. Define $K(\alpha) = K \cap p^\alpha H$ for any ordinal number $\alpha$. We shall say that $K$ is a proper valuated subgroup of $H$ by using the valuation inherited from the height function on $G$. Thereby, $K$ is a Σ-group with respect to this valuation (more precisely $K$ is a Σ-subgroup in $H$), provided that $K[p] = \cup_{n<\omega} K_n, K_n \subseteq K_{n+1} \leq K[p] : K_n \cap p^n H \subseteq p^n H$, i.e. $K_n \cap K(n) \subseteq K(\omega)$. In particular, if $K$ is pure in $H$, $K$ being a Σ-subgroup in $H$ reduces to $K$ is a Σ-group.

And so, we have done much of the groundwork necessary to proceed by proving the following assertion (compare with the corresponding result from [7]).

**Theorem.** Suppose $G$ is a group with a nice valuated subgroup $A$ endowed with the valuation produced by the restricted height valuation of $G$. If $G/A$ is a Σ-group, then $G$ is a Σ-group if and only if $A$ is a Σ-subgroup in $G$. In particular, under these circumstances, $G$ is a Σ-group only when $A$ is a Σ-group, provided $A$ is pure in $G$.

**Proof.** Foremost, we treat the necessity. According to the foregoing criterion, we write $G[p] = \cup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G[p]$ and $G_n \cap p^n G \subseteq p^n G$ for each $n \geq 1$. Therefore $A[p] = \cup_{n<\omega} A_n$, where we put $A_n = G_n \cap A$. Thus $A_n \subseteq A_{n+1} \leq A[p]$ and $A_n \cap A(n) = A_n \cap p^n G = G_n \cap p^n G \cap A \subseteq p^n G \cap A = A(\omega)$. By virtue of the comments alluded to above, we are finished.

As for the sufficiency, we write down via the preceding criterion that $A[p] = \cup_{n<\omega} A_n$, where, for every natural number $n$, $A_n \subseteq A_{n+1} \leq A[p]$ and $A_n \cap p^n G \subseteq p^n G$. Hypothesis also implies that $(G/A)[p] = \cup_{n<\omega} [G_n/A]$, where, for all positive integers $n$, $(G_n/A)\cap p^n G = (G_n/A)\cap [(p^n G + A)/A] \subseteq p^n (G/A) = (p^n G + A)/A$; equivalently, by using the modular law from [9], we have $G_n \cap p^n G \subseteq p^n G + A$. Furthermore, since $(G[p] + A)/A \subseteq (G/A)[p]$, it is a routine technical exercise to obtain that $G[p] = \cup_{n<\omega} G_n[p]$.

Next, we select a family of groups $(C_n)_{n<\omega}$ so that $C_n \subseteq C_{n+1} \leq G[p]$, so that $C_n \cap A = 0$ and so that $(C_n \oplus A)/A = (G_n/A) \cap [(G[p] + A)/A]$. Utilizing the classical modular law from [8], the last equality is equivalent to $C_n \oplus A = G_n[p] + A$ where $C_n \leq G_n[p]$.

We claim that $G[p] = \cup_{n<\omega} (C_n \oplus A_n)$. In order to verify that, letting $g \in G[p]$, hence $g + A \in (G_k/A) \cap [(G[p] + A)/A]$ for some $k \in \mathbb{N}$. It is obvious then that $g + A \subseteq C_m \oplus A$, whence $g \in C_m \oplus A$. Finally, $g \in C_t \oplus A_t$ for some $t \geq 1$, so the claim sustained.
Now, we choose an ascending chain \((P_n)_{n<\omega}\) of subgroups of \(G[p]\) such that, for every \(n \geq 1\), \(P_n \subseteq C_n\) with \(\cup_{n<\omega} P_n = \cup_{n<\omega} C_n\) and such that \((P_n \oplus A_n) \cap p^nG \subseteq p^\omega G + A_n\). The choice is possible because of the inclusions \((C_n \oplus A_n) \cap p^nG \subseteq G_n \cap p^nG \subseteq p^\omega G\).

It is straightforward that \(G[p] = \cup_{n<\omega} (P_n \oplus A_n)\) where \((P_n \oplus A_n)_{n<\omega}\) forms an increasing tower of subgroups. What suffices to argue is that \((P_n \oplus A_n) \cap p^nG \subseteq p^\omega G\). In fact, \((P_n \oplus A_n) \cap p^nG \subseteq p^\omega G + A_n\); hence, with the modular law from [9] at hand, we derive \((P_n \oplus A_n) \cap p^nG \subseteq (p^\omega G + A_n) \cap p^nG = p^\omega G + (p^n G \cap A_n) = p^\omega G\).

As a final step, we apply our criterion to complete the proof. QED

As a direct consequence, we yield the following affirmation which was proved in ([2], p. 267, Theorem) by the usage of another technique.

**Corollary** ([2]). Let \(G\) be a group with a balanced (nice and isotype) subgroup \(A\) such that \(G/A\) is a \(\Sigma\)-group. Then \(G\) is a \(\Sigma\)-group if and only if \(A\) is a \(\Sigma\)-group.

**References**


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