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The Generalized Criterion of Dieudonné for Primary Valuated Groups

Peter Danchev

Abstract. Let $G$ be an abelian reduced $p$-group with limit length and let $A$ be its valuated nice subgroup endowed with a valuation induced by the height valuation on $G$. If both $A$ and $G/A$ are either summable or totally projective groups of countable lengths, then $G$ is either summable or totally projective.


1. Introduction

Throughout this paper, suppose $G$ is an abelian $p$-group, where $p$ is an arbitrary but fixed prime, with a subgroup $A$. One of the most significant and applicable criteria in the abelian group theory is the so-termed Kulikov's criterion for direct sums of cycles ([Ku] and [Fu, v. I, p. 110, Theorem 18.1]). Later on, Dieudonné strengthens in [Di] this necessary and sufficient condition to a remarkable assertion, named as Dieudonné criterion, in which the structure of the whole group to be a direct sum of cyclic groups depends on a special subgroup and the factor-group modulo this subgroup. Specifically, assume that $G$ is an abelian $p$-group with a subgroup $A$ so that $G/A$ is a direct sum of cycles. Then $G$ is a direct sum of cyclic groups precisely when $A$ is contained in a basic subgroup of $G$.

Recently, we have enlarged in [D] this statement to the class of so-called $\sigma$-summable groups and also in [Da] for the valuated variant of $\sigma_\lambda$-summable groups ($\lambda$ is an ordinal cofinal with $\omega$). In this aspect, we improved in [Danc] and [Danch] the attainment of Dieudonné for other independent classes of torsion abelian groups.

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The aim of this brief note is to generalize some of the affirmations from [Dan], [Danc] and [Danch], especially for summable and totally projective groups, in the form of valuated subgroups having the standard height valuation.

For some basic concepts about valuated groups, used here, we refer for instance to [Da].

Before beginning, we need some preliminary technicalities.

**Criterion ([Ho]: [F, v. II, p. 123, Theorem 84.1]).** Let $H$ be a reduced abelian $p$-group of countable length. Then $H$ is summable if and only if $H[p] = \bigcup_{n<\omega} H_n$, $H_n \subseteq H_{n+1} \leq H$ and, for each non-negative integer $n$, $H_n$ is with a finite number of height values in $H$.

We recall that if $C \subseteq H$, then $C(\alpha) = C \cap p^\alpha H$ for some ordinal number $\alpha$.

2. Main Result

We are now prepared with the following (see the corresponding affirmation from [Dan] as well).

**Theorem.** Suppose that $G$ is a reduced abelian $p$-group (with limit length) and $A$ is its nice valuated subgroup endowed with a valuation produced by the restricted height valuation of $G$. If $A$ and $G/A$ are summable groups of countable lengths, then $G$ is summable.

**Proof.** We observe that there is a countable ordinal $\delta$ with the property that $0 = p^\delta(G/A) \supseteq (p^\delta G + A)/A$, hence $p^\delta G \subseteq A$. But $p^\nu A = 0$ for some countable ordinal $\nu$, whence $p^{\delta+\nu} G = 0$. Thus $\text{length}(G) \leq \delta + \nu$ and consequently $G$ possesses countable length.

In conjunction with the listed above Honda’s criterion, we write $(G/A)[p] = \bigcup_{n<\omega} G_n/A$, where $G_n \subseteq G_{n+1} \leq G$ and, for every $n \geq 0$, $(G_n/A) \setminus \{A\} \subseteq [p^{\alpha_1}(G/A) \setminus p^{\alpha_1+1}(G/A)] \cup \cdots \cup [p^{\alpha_n}(G/A) \setminus p^{\alpha_n+1}(G/A)]$ for some ordinals $\alpha_1, \ldots, \alpha_n$. Moreover, we write down $A[p] = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1} \leq A$ and, for any $n \geq 0$, $A_n \setminus \{0\} \subseteq [A(\beta_1) \setminus A(\beta_1 + 1)] \cup \cdots \cup [A(\beta_n) \setminus A(\beta_n + 1)]$ for some ordinals $\beta_1, \ldots, \beta_n$. Since $(G[p]+A)/A \subseteq (G/A)[p]$, we easily obtain that $G[p] = \bigcup_{n<\omega} G_n[p]$.

We now choose a family of groups $(C_n)_{n<\omega}$ such that $C_n \subseteq C_{n+1} \leq G[p]$, such that $C_n G \setminus A = 0$ and such that $(C_n + A)/A = (G_n/A) \cap [(G[p] + A)/A]$. By the utilization of the modular law, the last equality is equivalent to $C_n \cap A = G_n[p] + A$ where $C_n \leq G_n[p]$.

We claim that $G[p] = \bigcup_{n<\omega} (C_n \cap A_n)$. In order to check this, letting $g \in G[p]$, hence $g + A \in G_m/A$ for some $m \geq 1$. It is therefore obvious that $g + A \subseteq C_m \cap A$, whence $g \in C_m \cap A$. Finally, $g \in C_i \cap A_i$ for some $i \geq 1$ which substantiates our claim.

Furthermore, because of the niceness of $A$ in $G$, we have that $[(A \cap C_n)/A] \setminus \{A\} \subseteq [(p^{\alpha_1} G + A)/A] \cup \cdots \cup [(p^{\alpha_n} G + A)/A]$ for some ordinals $\alpha_1, \ldots, \alpha_n$. Consequently, $(A \cap C_n)/A \subseteq [(p^{\alpha_1} G + A)/A] \cup \cdots \cup [(p^{\alpha_n} G + A)/A]$. Since for each ordinal $\delta$ it is fulfilled that $(p^\delta G + A)/(p^\delta + A) \subseteq (p^{\delta+1} G + A)/(p^{\delta+1} G + A)$, we easily obtain that $(A \cap C_n)/A \subseteq [(p^{\alpha_1} G + A)/A] \cup \cdots \cup [(p^{\alpha_n} G + A)/A]$. Thus $C_n \setminus A = C_n \setminus \{0\} \subseteq [(p^{\alpha_1} G + A)/A] \cup \cdots \cup [(p^{\alpha_n} G + A)/A] + A$.

Now, we select an ascending tower of groups $(P_n)_{n<\omega}$ so that $P_n \subseteq C_n$ with $\bigcup_{n<\omega} P_n = \bigcup_{n<\omega} C_n$, so that $P_n \setminus \{0\} \subseteq [(p^{\alpha_1} G + A)/A] \cup \cdots \cup [(p^{\alpha_n} G + A)/A]$.
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Let \( G \) and \( A \) be primary valuated subgroups of \( G \) and let \( A \) be a primary valuated subgroup of \( G \). Then \( G \) is summable if and only if \( A \) is also summable.

**Proof.** It is clear that \( A \) is a nice valuated subgroup of \( G \) equipped with a valuation induced by the height function in \( G \). Henceforth, the Theorem works.

We thus conclude that \( (A \cap p^nG) \cap A[0,n] \subseteq (A \cap p^nG) \cap A[n] \subseteq (A \cap p^nG) \cap A[n+1] \), as desired. So, the proof is complete. \( QED \)

As immediate consequences, we yield the following.

**Corollary ([Danc]).** Let \( G \) be a reduced abelian \( p \)-group with countable (limit) length. If \( p^\gamma G \) and \( G/p^\gamma G \) are both summable, then \( G \) is summable.

**Proof.** It is clear that \( p^\gamma G \) is a nice valuated subgroup of \( G \) equipped with a valuation induced by the height function in \( G \). Henceforth, the Theorem works.

**Corollary ([Danchev]).** Let \( G \) be an abelian reduced \( p \)-group with countable (limit) length and \( A \) a balanced subgroup of \( G \). If \( A \) and \( G/A \) are summable groups, then \( A \) is also summable.

**Proof.** Since \( A \) is balanced in \( G \), it is by definition nice and isotype; so all heights may be computed in \( G \). Hereafter, we apply the Theorem. \( QED \)

By using the criterion for total projectivity from [HU, Theorem 2.1] and via the same reasoning appeared in the proof of the first theorem, we can record the following parallel statement (see also the corresponding assertion from [Danc] and [Danchev], respectively).

**Theorem ([Danchev]).** Suppose \( G \) is a reduced abelian \( p \)-group (of limit length) with a nice valuated subgroup \( A \) endowed with a restricted valuation induced by the height function on \( G \). If \( A \) and \( G/A \) are totally projective groups of countable length, then \( G \) is totally projective of countable length.

**References**


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