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Between Closed Sets and Generalized Closed Sets in Closure Spaces

Chawalit Boonpok and Jeeranunt Khampakdee

Abstract. The purpose of the present paper is to define and study $\partial$-closed sets in closure spaces obtained as generalization of the usual closed sets. We introduce the concepts of $\partial$-continuous and $\partial$-closed maps by using $\partial$-closed sets and investigate some of their properties.

1 Introduction

Generalized closed sets, briefly g-closed sets, in a topological space were introduced by N. Levine [10] in order to extend some important properties of closed sets to a larger family of sets. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by g-closed subsets. K. Balachandran, P. Sundaram and H. Maki [2] introduced the notion of generalized continuous maps, briefly g-continuous maps, by using g-closed sets and studied some of their properties.

Closure spaces were introduced by E. Čech in [4] and then studied by many mathematicians, see e.g. [5], [6], [14] and [15]. The concepts of generalized closed sets and generalized continuous maps of topological spaces were extended to closure spaces in [3]. In this paper, we introduce and study a new class of closed sets in closure spaces lying, as for generality, between the class of closed sets and the class of generalized closed sets. Using the concept of $\partial$-closed sets, we define two new kinds of spaces, namely $T'_1$-spaces and $T''_1$-spaces, and introduce $\partial$-continuous and $\partial$-closed maps. The two kinds of spaces and the two kinds of maps are investigated.

2 Preliminaries

A map $u: P(X) \to P(X)$ defined on the power set $P(X)$ of a set $X$ is called a closure operator on $X$ and the pair $(X, u)$ is called a closure space if the following axioms are satisfied:

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(N1) \( u\emptyset = \emptyset \),
(N2) \( A \subseteq uA \) for every \( A \subseteq X \),
(N3) \( A \subseteq B \Rightarrow uA \subseteq uB \) for all \( A, B \subseteq X \).

A closure operator \( u \) on a set \( X \) is called additive (respectively, idempotent) if \( A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB \) (respectively, \( A \subseteq X \Rightarrow uuA = uA \)). A subset \( A \subseteq X \) is closed in the closure space \((X, u)\) if \( uA = A \) and it is open if its complement is closed. The empty set and the whole space are both open and closed. Let \((X, u_1)\) and \((X, u_2)\) be closure spaces. The closure \( u_1 \) is said to be finer than the closure \( u_2 \), or \( u_2 \) is said to be coarser than \( u_1 \), by symbols \( u_1 \leq u_2 \), if \( u_2 A \supseteq u_1 A \) for every \( A \subseteq X \). The relation \( \leq \) is a partial order on the set of all closure operators on \( X \).

A closure space \((Y, v)\) is said to be a subspace of \((X, u)\) if \( Y \subseteq X \) and \( vA = uA \cap Y \) for each subset \( A \subseteq Y \). If \( Y \) is closed in \((X, u)\), then the subspace \((Y, v)\) of \((X, u)\) is said to be closed too. A closure space \((X, u)\) is said to be a \( T_0 \)-space if, for any pair of points \( x, y \in X \), from \( x \in u\{y\} \) and \( y \in u\{x\} \) it follows that \( x = y \), and it is called a \( T_1 \)-space if each singleton subset of \( X \) is closed or open.

Let \((Y, v)\) be a closed subspace of \((X, u)\). If \( F \) is a closed subset of \((Y, v)\), then \( F \) is a closed subset of \((X, u)\).

Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \( f : (X, u) \to (Y, v) \) is said to be continuous if \( f(uA) \subseteq v(f(A)) \) for every subset \( A \subseteq X \).

One can see that a map \( f : (X, u) \to (Y, v) \) is continuous if and only if \( uf^{-1}(B) \subseteq f^{-1}(vB) \) for every subset \( B \subseteq Y \). Clearly, if \( f : (X, u) \to (Y, v) \) is continuous, then \( f^{-1}(F) \) is a closed subset of \((X, u)\) for every closed subset \( F \) of \((Y, v)\).

Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \( f : (X, u) \to (Y, v) \) is said to be closed (resp. open) if \( f(F) \) is a closed (resp. open) subset of \((Y, v)\) whenever \( F \) is a closed (resp. open) subset of \((X, u)\).

The product of a family \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) of closure spaces, denoted by \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \), is the closure space \( (\prod_{\alpha \in I} X_\alpha, u) \) where \( uA = \prod_{\alpha \in I} u_\alpha(A) \) for each \( A \subseteq \prod_{\alpha \in I} X_\alpha \).

Clearly, if \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) is a family of closure spaces, then the projection map \( \pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta) \) is closed and continuous for every \( \beta \in I \).

**Proposition 1.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces, let \( \beta \in I \) and let \( F \subseteq X_\beta \). Then \( F \) is a closed subset of \((X_\beta, u_\beta)\) if and only if \( F \times \prod_{\alpha \neq \beta} X_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

**Proof.** Let \( F \) be a closed subset of \((X_\beta, u_\beta)\). Since \( \pi_\beta \) is continuous, \( \pi_\beta^{-1}(F) \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). But \( \pi_\beta^{-1}(F) = F \times \prod_{\alpha \in I, \alpha \neq \beta} X_\alpha \), hence \( F \times \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).
Conversely, let $F \times \prod_{\alpha \in I} X_{\alpha}$ be a closed subset of $\prod_{\alpha \neq \beta} (X_\alpha, u_\alpha)$. Since $\pi_\beta$ is closed, 
\[ \pi_\beta \left( F \times \prod_{\alpha \neq \beta} X_\alpha \right) = F \] is a closed subset of $(X_\beta, u_\beta)$. \hfill \square

**Proposition 2.** Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then $G$ is an open subset of $(X_\beta, u_\beta)$ if and only if $G \times \prod_{\alpha \neq \beta} X_\alpha$ is an open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

### 3 Generalized closed sets

**Definition 1.** Let $(X, u)$ be a closure space. A subset $A \subseteq X$ is called a generalized closed set, briefly a $g$-closed set, if $uA \subseteq G$ whenever $G$ is an open subset of $(X, u)$ with $A \subseteq G$. A subset $A \subseteq X$ is called a generalized open set, briefly a $g$-open set, if its complement is $g$-closed.

The following statement is evident:

**Proposition 3.** Let $(X, u)$ be a closure space and let $(Y, v)$ be a closed subspace of $(X, u)$. If $F$ is a $g$-closed subset of $(Y, v)$, then $F$ is a $g$-closed subset of $(X, u)$.

**Theorem 1.** Let $(X, u)$ be a closure space. Then $(X, u)$ is a $T_{\frac{3}{2}}$-space if and only if every $g$-closed subset of $(X, u)$ is closed.

**Proof.** Let $(X, u)$ be a $T_{\frac{3}{2}}$-space and let $M$ be a $g$-closed subset of $(X, u)$. Suppose that $x \notin M$. Then $\{x\} \subseteq X - M$ and hence $M \subseteq X - \{x\}$. Since $M$ is $g$-closed and $X - \{x\}$ is open, $uM \subseteq X - \{x\}$ or, equivalently, $\{x\} \subseteq X - uM$. Therefore, $x \notin uM$ and thus $uM \subseteq M$. Hence, $M$ is a closed subset of $(X, u)$.

Conversely, suppose that $\{x\}$ is not closed. Then $X - \{x\}$ is not open. This implies that $X$ is the only open set containing $X - \{x\}$. Therefore, $X - \{x\}$ is a $g$-closed subset of $(X, u)$. Consequently, $X - \{x\}$ is closed. Hence, $\{x\}$ is an open subset of $(X, u)$. Therefore, $(X, u)$ is a $T_{\frac{3}{2}}$-space. \hfill \square

**Proposition 4.** Let $(X, u)$ be a closure space and let $(Y, v)$ be a closed subspace of $(X, u)$. If $(X, u)$ is a $T_{\frac{3}{2}}$-space, then $(Y, v)$ is a $T_{\frac{3}{2}}$-space too.

**Proof.** Let $F$ be a $g$-closed subset of $(Y, v)$. Then $F$ is a $g$-closed subset of $(X, u)$. Since $(X, u)$ is a $T_{\frac{3}{2}}$-space, $F$ is a closed subset of $(X, u)$. This implies that $F$ is a closed subset of $(Y, v)$. Therefore, $(Y, v)$ is a $T_{\frac{3}{2}}$-space. \hfill \square

**Proposition 5.** Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $F \subseteq X_\beta$. Then $F$ is a $g$-closed subset of $(X_\beta, u_\beta)$ if and only if $F \times \prod_{\alpha \neq \beta} X_\alpha$ is a $g$-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

**Proposition 6.** Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then $G$ is a $g$-open subset of $(X_\beta, u_\beta)$ if and only if $G \times \prod_{\alpha \neq \beta} X_\alpha$ is a $g$-open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.
Proposition 7. Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces. For each \( \beta \in I \), let \( \pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta \) be the projection map. Then

(i) If \( F \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \), then \( \pi_\beta(F) \) is a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \).

(ii) If \( F \) is a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \), then \( \pi_\beta^{-1}(F) \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

Definition 2. Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \( f : (X, u) \to (Y, v) \) is called generalized continuous, briefly \( g \)-continuous, if \( f^{-1}(F) \) is a \( \partial \)-closed subset of \((X, u)\) for every closed subset \( F \) of \((Y, v)\).

Clearly, a map \( f : (X, u) \to (Y, v) \) is \( g \)-continuous if and only if \( f^{-1}(G) \) is a \( g \)-open subset of \((X, u)\) for every open subset \( G \) of \((Y, v)\).

4 \( \partial \)-Closed Sets in Closure Spaces

In this section, we introduce and study a new class of closed sets lying, as for generality, between the class of closed sets and the class of generalized closed sets.

Definition 3. A subset \( A \) of closure space \((X, u)\) is called a \( \partial \)-closed set if \( uA \subseteq G \) whenever \( G \) is a \( g \)-open subset of \((X, u)\) with \( A \subseteq G \). A subset \( A \) of \( X \) is called a \( \partial \)-open set if its complement is a \( \partial \)-closed subset of \((X, u)\).

Remark 1. For a subset \( A \) of a closure space \((X, u)\), the following implications hold:

\[ A \text{ is closed} \Rightarrow A \text{ is } \partial \text{-closed} \Rightarrow A \text{ is } g \text{-closed}. \]

None of these implications is reversible as shown by the following examples.

Example 1. Let \( X = \{1, 2, 3, 4\} \) and define a closure operator \( u \) on \( X \) by \( u \emptyset = \emptyset \), \( u\{1\} = \{1, 3\}, u\{2\} = \{2, 3\}, u\{3\} = u\{4\} = u\{3, 4\} = \{3, 4\} \) and \( u\{1, 2\} = u\{1, 3\} = u\{1, 4\} = u\{2, 3\} = u\{2, 4\} = u\{1, 2, 3\} = u\{1, 2, 4\} = u\{2, 3, 4\} = u\{1, 3, 4\} = uX = X \). Then \( \{1, 2, 3\} \) is \( \partial \)-closed set but it is not closed.

Example 2. Let \( X = \{1, 2\} \) and define a closure operator \( u \) on \( X \) by \( u \emptyset = \emptyset \) and \( u\{1\} = u\{2\} = uX = X \). Then \( \{1\} \) is \( g \)-closed but it is not \( \partial \)-closed.

The following statement is evident:

Proposition 8. Let \((X, u)\) be a closure space. If a subset \( A \) of \((X, u)\) is both \( g \)-open and \( \partial \)-closed, then \( A \) is closed.

Proposition 9. Let \((X, u)\) be a closure space and let \( u \) be idempotent. If \( A \) is a \( \partial \)-closed subset of \((X, u)\) such that \( A \subseteq B \subseteq uA \), then \( B \) is a \( \partial \)-closed subset of \((X, u)\).
Proof. Let $G$ be a g-open subset of $(X, u)$ such that $B \subseteq G$. Then $A \subseteq G$. Since $A$ is $\partial$-closed, $uA \subseteq G$. As $u$ is idempotent, $uB \subseteq uuA = uA \subseteq G$. Hence, $B$ is $\partial$-closed.

Proposition 10. Let $(X, u)$ be a closure space. If $A$ is $\partial$-closed, then $uA - A$ has no nonempty g-closed subset.

Proof. Suppose that $A$ is $\partial$-closed. Let $F$ be a g-closed subset of $uA - A$. Then $F \subseteq uA \cap (X - A)$ and so $A \subseteq X - F$. Consequently, $F \subseteq X - uA$. Since $F \subseteq uA$, $F \subseteq (X - uA) \cap uA = \emptyset$, thus $F = \emptyset$. Therefore, $uA - A$ contains no nonempty closed set.

Theorem 2. Let $(X, u)$ be a closure space. A set $A \subseteq X$ is $\partial$-open if and only if $F \subseteq X - u(X - A)$ whenever $F$ is a g-closed subset of $(X, u)$ with $F \subseteq A$.

Proof. Suppose that $A$ is $\partial$-open and let $F \subseteq A$ be a g-closed subset of $(X, u)$. Then $X - A \subseteq X - F$. But $X - A$ is $\partial$-closed and $X - F$ is g-open. It follows that $u(X - A) \subseteq X - F$ and hence $F \subseteq X - u(X - A)$.

Conversely, let $X - A \subseteq G$ where $G$ is g-open. Then $X - G \subseteq A$. Since $X - G$ is g-closed, $X - G \subseteq X - u(X - A)$. Therefore, $u(X - A) \subseteq G$. Hence, $X - A$ is $\partial$-closed and so $A$ is $\partial$-open.

Proposition 11. Let $\{ (X_\alpha, u_\alpha) : \alpha \in I \}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then $G$ is a $\partial$-open subset of $(X_\beta, u_\beta)$ if and only if $G \times \prod_{\alpha \neq \beta} X_\alpha$ is a $\partial$-open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Proof. Let $F$ be a $\partial$-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$ such that $F \subseteq G \times \prod_{\alpha \neq \beta} X_\alpha$.

Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is g-closed and $G$ is $\partial$-open in $(X_\beta, u_\beta)$, $\pi_\beta(F) \subseteq X_\beta - u_\beta(X_\beta - G)$. Therefore,

$$F \subseteq \pi_\beta^{-1}(X_\beta - u_\beta(X_\beta - G)) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha \pi_\alpha \left( \prod_{\alpha \in I} X_\alpha - G \times \prod_{\alpha \neq \beta} X_\alpha \right).$$

By Theorem 2, $G \times \prod_{\alpha \neq \beta} X_\alpha$ is a $\partial$-open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Conversely, let $F$ be a g-closed subset of $(X_\beta, u_\beta)$ such that $F \subseteq G$. Then $F \times \prod_{\alpha \neq \beta} X_\alpha \subseteq G \times \prod_{\alpha \in I} X_\alpha$. Since $F \times \prod_{\alpha \neq \beta} X_\alpha$ is g-closed and $G \times \prod_{\alpha \neq \beta} X_\alpha$ is $\partial$-open in $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$,

$$F \times \prod_{\alpha \in I} X_\alpha \subseteq \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha \pi_\alpha \left( \prod_{\alpha \in I} X_\alpha - G \times \prod_{\alpha \neq \beta} X_\alpha \right).$$
by Theorem 2. Therefore,
\[ \prod_{\alpha \in I} u_\alpha \pi_\alpha \left( (X_\beta - G) \times \prod_{\alpha \neq \beta} X_\alpha \right) \subseteq \prod_{\alpha \in I} X_\alpha - F \times \prod_{\alpha \neq \beta} X_\alpha = (X_\beta - F) \times \prod_{\alpha \neq \beta} X_\alpha . \]

Consequently, \( u_\beta (X_\beta - G) \subseteq X_\beta - F \) implies \( F \subseteq X_\beta - u_\beta (X_\beta - G) \). Hence, \( G \) is a \( \partial \)-open subset of \( (X_\beta, u_\beta) \).

**Proposition 12.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces, let \( \beta \in I \) and let \( F \subseteq X_\beta \). Then \( F \) is a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \) if and only if \( F \times \prod_{\alpha \neq \beta} X_\alpha \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

**Proof.** Let \( F \) be a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \). Then \( X_\beta - F \) is a \( \partial \)-open subset of \( (X_\beta, u_\beta) \). By Proposition 11,
\[ (X_\beta - F) \times \prod_{\alpha \in I \setminus \{\beta\}} X_\alpha = \prod_{\alpha \neq \beta} X_\alpha - F \times \prod_{\alpha \neq \beta} X_\alpha \]
is a \( \partial \)-open subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). Hence, \( F \times \prod_{\alpha \neq \beta} X_\alpha \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

Conversely, let \( G \) be a \( g \)-open subset of \( (X_\beta, u_\beta) \) such that \( F \subseteq G \). Then \( F \times \prod_{\alpha \neq \beta} X_\alpha \subseteq G \times \prod_{\alpha \neq \beta} X_\alpha \). Since \( F \times \prod_{\alpha \neq \beta} X_\alpha \) is \( \partial \)-closed and \( G \times \prod_{\alpha \neq \beta} X_\alpha \) is \( g \)-open in \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \),
\[ \prod_{\alpha \in I \setminus \{\beta\}} u_\alpha \pi_\alpha \left( F \times \prod_{\alpha \neq \beta} X_\alpha \right) \subseteq G \times \prod_{\alpha \in I \setminus \{\beta\}} X_\alpha . \]
Consequently, \( u_\beta F \subseteq G \). Therefore, \( F \) is a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \). □

**Proposition 13.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces. For each \( \beta \in I \), let \( \pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta \) be the projection map. Then

(i) If \( F \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \), then \( \pi_\beta (F) \) is a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \).

(ii) If \( F \) is a \( \partial \)-closed subset of \( (X_\beta, u_\beta) \), then \( \pi_\beta^{-1} (F) \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

**Proof.** (i) Let \( F \) be a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) and let \( G \) be a \( g \)-open subset of \( (X_\beta, u_\beta) \) such that \( \pi_\beta (F) \subseteq G \). Then \( F \subseteq \pi_\beta^{-1} (G) = G \times \prod_{\alpha \neq \beta} X_\alpha \). Since \( F \) is
\( \partial \)-closed and \( G \times \prod_{\alpha \neq \beta} X_{\alpha} \) is g-open, \( \prod_{\alpha \in I} u_{\alpha} \pi_{\alpha}(F) \subseteq G \times \prod_{\alpha \in I} X_{\alpha} \). Consequently, \( u_{\beta} \pi_{\beta}(F) \subseteq G \). Hence, \( \pi_{\beta}(F) \) is a \( \partial \)-closed subset of \((X_{\beta}, u_{\beta})\).

(ii) Let \( F \) be a \( \partial \)-closed subset of \((X_{\beta}, u_{\beta})\). Then \( \pi_{\beta}^{-1}(F) = F \times \prod_{\alpha \in I} X_{\alpha} \). By Proposition 12, \( F \times \prod_{\alpha \in I} X_{\alpha} \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}) \). Therefore, \( \pi_{\beta}^{-1}(F) \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}) \). \( \square \)

5 \( T'_{1/2} \)-spaces and \( T''_{1/2} \)-spaces

As applications of \( \partial \)-closed sets, two new kinds of spaces, namely \( T'_{1/2} \)-spaces and \( T''_{1/2} \)-spaces, are introduce.

**Definition 4.** A closure space \((X, u)\) is said to be a \( T'_{1/2} \)-space if every \( \partial \)-closed subset of \((X, u)\) is closed.

**Definition 5.** A closure space \((X, u)\) is said to be a \( T''_{1/2} \)-space if every g-closed subset of \((X, u)\) is \( \partial \)-closed.

We note that the concepts of a \( T'_{1/2} \)-space and a \( T''_{1/2} \)-space are independent as shown in the following examples.

**Example 3.** Let \( X = \{a, b, c, d\} \) and define a closure operator \( u \) on \( X \) by \( u\emptyset = \emptyset \), \( u\{a\} = \{a, c\} \), \( u\{b\} = \{b, c\} \), \( u\{c\} = u\{d\} = u\{c, d\} = \{c, d\} \) and \( u\{a, b\} = u\{a, c\} = u\{a, d\} = u\{b, c\} = u\{b, d\} = u\{a, b, c\} = u\{a, b, d\} = u\{b, c, d\} = uX = X \). Then \((X, u)\) is a \( T''_{1/2} \)-space. But \((X, u)\) is not a \( T'_{1/2} \)-space since \( \{a, c, d\} \) is \( \partial \)-closed but it is not a closed subset of \((X, u)\).

**Example 4.** Let \( X = \{a, b, c\} \) and define a closure operator \( u \) on \( X \) by \( u\emptyset = \emptyset \), \( u\{a\} = \{a\} \), \( u\{b\} = \{b\} \), \( u\{c\} = \{a, c\} \) and \( u\{a, b\} = u\{a, c\} = u\{b, c\} = uX = X \). Then \((X, u)\) is not a \( T''_{1/2} \)-space since \( \{c\} \) is g-closed but it is not a \( \partial \)-closed subset of \((X, u)\). However, \((X, u)\) is a \( T'_{1/2} \)-space.

**Example 5.** Let \( X = \{p, q\} \) and define a closure operator \( u \) on \( X \) by \( u\emptyset = \emptyset \), \( u\{p\} = u\{q\} = uX = X \). Then \((X, u)\) is both a \( T'_{1/2} \)-space and a \( T''_{1/2} \)-space.

**Proposition 14.** Let \((X, u)\) be a closure space. Then

(i) If \((X, u)\) is a \( T'_{1/2} \)-space, then every singleton subset of \( X \) is either g-closed or open.

(ii) If every singleton subset of \( X \) is a g-closed subset of \((X, u)\), then \((X, u)\) is a \( T'_{1/2} \)-space.
Proof. (i) Suppose that \((X, u)\) is a \(T'_2\)-space. Let \(x \in X\) and assume that \(\{x\}\) is not g-closed. Then \(X - \{x\}\) is not g-open. This implies \(X - \{x\}\) is \(\partial\)-closed since \(X\) is the only g-open set which contains \(X - \{x\}\). Since \((X, u)\) is a \(T'_2\)-space, \(X - \{x\}\) is closed or equivalently, \(\{x\}\) is open.

(ii) Let \(A\) be a \(\partial\)-closed subset of \((X, u)\). Suppose that \(x \notin A\). Then \(\{x\} \subseteq X - A\) and we have \(A \subseteq X - \{x\}\). Since \(A\) is \(\partial\)-closed and \(X - \{x\}\) is g-open, \(uA \subseteq X - \{x\}\), i.e., \(\{x\} \subseteq X - uA\). Hence, \(x \notin uA\) and thus \(uA \subseteq A\). Therefore, \(A\) is a closed subset of \((X, u)\). Hence, \((X, u)\) is a \(T'_2\)-space. \(\square\)

**Proposition 15.** Let \((X, u)\) be a closure space. If \((X, u)\) is a \(T''_2\)-space, then every singleton subset of \(X\) is either \(\partial\)-open or closed.

Proof. It follows from Proposition 14 (i). \(\square\)

Clearly, if \((X, u)\) is a \(T'_2\)-space, then \((X, u)\) is a \(T''_2\)-space. The converse need not be true as can be seen from the following example.

**Example 6.** In example 3, \((X, u)\) is not a \(T'_2\)-space since \(\{a, c, d\}\) is g-closed but it is not closed in \((X, u)\). However, \((X, u)\) is a \(T''_2\)-space.

Clearly, if \((X, u)\) is a \(T'_2\)-space, then \((X, u)\) is a \(T'_2\)-space. The converse need not be true as can be seen from the following example.

**Example 7.** Let \(X = \{p, q\}\) and define a closure operator \(u\) on \(X\) by \(u\emptyset = \emptyset\), \(u\{p\} = u\{q\} = uX = X\). Then \((X, u)\) is not a \(T'_2\)-space since \(\{p\}\) is g-closed but it is not closed in \((X, u)\). However, \((X, u)\) is a \(T'_2\)-space.

The following statement is evident:

**Proposition 16.** Let \((X, u)\) be a closure space. Then \((X, u)\) is a \(T'_2\)-space if and only if \((X, u)\) is both a \(T'_2\)-space and a \(T''_2\)-space.

**Proposition 17.** Let \(\{(X_\alpha, u_\alpha) : \alpha \in I\}\) be a family of closure spaces. Then \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\) is a \(T'_2\)-space if and only if \((X_\alpha, u_\alpha)\) is a \(T'_2\)-space for each \(\alpha \in I\).

Proof. Suppose that \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\) is a \(T'_2\)-space. Let \(\beta \in I\) and let \(F\) be a \(\partial\)-closed subset of \((X_\beta, u_\beta)\). Then \(F \times \prod_{\alpha \neq \beta} X_\alpha\) is a \(\partial\)-closed subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\). Since \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\) is a \(T'_2\)-space, \(F \times \prod_{\alpha \neq \beta} X_\alpha\) is a closed subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\). Consequently, \(F\) is a closed subset of \((X_\beta, u_\beta)\). Hence, \((X_\beta, u_\beta)\) is a \(T'_2\)-space.

Conversely, suppose that \((X_\alpha, u_\alpha)\) is a \(T'_2\)-space for each \(\alpha \in I\). Let \(F\) be a \(\partial\)-closed subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\) and let \((x_\alpha)_{\alpha \in I} \notin F\). Then there exists \(\beta \in I\) such
that \( x_\beta \notin \pi_\beta(F) \). Since \( \pi_\beta(F) \) is \( \partial \)-closed and \((X_\beta, u_\beta)\) is a T\(_1\)\(_\beta\)-space, \( \pi_\beta(F) \) is a closed subset of \((X_\beta, u_\beta)\). Thus, \( x_\beta \notin u_\beta \pi_\beta(F) \) implies \((x_\alpha)_{\alpha \in I} \notin \prod_{\alpha \in I} u_\alpha \pi_\alpha(F)\). Therefore, \( F \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). Hence, \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) is a T\(_1\)\(_\beta\)-space.

**Proposition 18.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces. Then \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) is a T\(_1\)\(_\beta\)-space if and only if \((X_\alpha, u_\alpha)\) is a T\(_1\)\(_\beta\)-space for each \( \alpha \in I \).

Proof. It follows from Proposition 17.

**Proposition 19.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces. If \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) is a T\(_\beta\)\(_\alpha\)-space, then \((X_\alpha, u_\alpha)\) is a T\(_\beta\)\(_\alpha\)-space for each \( \alpha \in I \).

Proof. Suppose that \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) is a T\(_\beta\)\(_\alpha\)-space. Let \( \beta \in I \) and let \( F \) be a g-closed subset of \((X_\beta, u_\beta)\). Then \( F \times \prod_{\alpha \neq \beta \in I} X_\alpha \) is a g-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). Since \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) is a T\(_\beta\)\(_\alpha\)-space, \( F \times \prod_{\alpha \neq \beta \in I} X_\alpha \) is a \( \partial \)-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). Then \( F \) is a \( \partial \)-closed subset of \((X_\beta, u_\beta)\). Hence, \((X_\beta, u_\beta)\) is a T\(_\beta\)\(_\alpha\)-space.

### 6 \( \partial \)-Continuous Maps

In this section, we investigate a new class of maps called \( \partial \)-continuous maps. These maps are defined by the help of g-closed sets and they lie, as for generality, properly between the class of continuous maps and the class of generalized continuous maps. We also introduce the notion of \( \partial \)-closed maps and study some of its properties.

**Definition 6.** Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \( f : (X, u) \to (Y, v) \) is said to be \( \partial \)-continuous if \( f^{-1}(F) \) is a \( \partial \)-closed subset of \((X, u)\) for every closed subset \( F \) of \((Y, v)\).

Clearly, it is easy to prove that a map \( f : (X, u) \to (Y, v) \) is \( \partial \)-continuous if and only if \( f^{-1}(G) \) is a \( \partial \)-open subset of \((X, u)\) for every open subset \( G \) of \((Y, v)\).

**Remark 2.** The following implications hold for any map \( f : (X, u) \to (Y, v) \):

\[
\text{f is continuous} \Rightarrow \text{f is \( \partial \)-continuous} \Rightarrow \text{f is g-continuous}.
\]

None of these implications is reversible as shown by the following examples.

**Example 8.** Let \( X = \{1, 2\} = Y \) and define a closure operator \( u \) on \( X \) by \( u\emptyset = \emptyset \), \( u\{1\} = \{1\} \) and \( u\{2\} = uX = X \). Define a closure operator \( v \) on \( Y \) by \( v\emptyset = \emptyset \), \( v\{1\} = \{1\} \), \( v\{2\} = \{2\} \) and \( vY = Y \). Let \( \varphi : (X, u) \to (Y, v) \) be defined by \( \varphi(1) = \varphi(2) = 1 \). Then \( \varphi \) is \( \partial \)-continuous but \( \varphi \) is not continuous because \( \varphi(u\{2\}) \notin v\varphi(\{2\}) \).
Example 9. Let $X = \{1, 2\} = Y$ and define a closure operator $u$ on $X$ by $u\emptyset = \emptyset$, $u\{1\} = u\{2\} = uX = X$. Define a closure operator $v$ on $Y$ by $v\emptyset = \emptyset$, $v\{1\} = \{1\}$, $v\{2\} = \{2\}$ and $vY = Y$. Let $\varphi: (X, u) \to (Y, v)$ be the identity map. Then $\varphi$ is $g$-continuous but $\varphi$ is not $\partial$-continuous because $\{1\}$ is a closed subset of $(Y, v)$ but $\varphi^{-1}(\{1\}) = \{1\}$ is not a $\partial$-closed subset of $(X, u)$.

Proposition 20. Let $(X, u)$ be a $T_2''$-space and let $(Y, v)$ be a closure space. If $f: (X, u) \to (Y, v)$ is $g$-continuous, then $f$ is $\partial$-continuous.

Proof. Let $F$ be a closed subset of $(Y, v)$, since $f$ is $g$-continuous, $f^{-1}(F)$ is a $g$-closed subset of $(X, u)$. Since $(X, u)$ is a $T_2''$-space, $f^{-1}(F)$ is a $\partial$-closed subset of $(X, u)$. Hence, $f$ is $\partial$-continuous.

The following statement is obvious:

Proposition 21. Let $(X, u)$, $(Y, v)$ and $(Z, w)$ be closure spaces. If $f: (X, u) \to (Y, v)$ is $\partial$-continuous and $g: (Y, v) \to (Z, w)$ is continuous, then $g \circ f: (X, u) \to (Z, w)$ is $\partial$-continuous.

Proposition 22. Let $(X, u)$ and $(Z, w)$ be closure spaces and let $(Y, v)$ be a $T_2$-space. If $f: (X, u) \to (Y, v)$ is $g$-continuous and $g: (Y, v) \to (Z, w)$ is $\partial$-continuous, then $g \circ f: (X, u) \to (Z, w)$ is $\partial$-continuous.

Proof. Let $F$ be a closed subset of $(Z, w)$. Since $g$ is $g$-continuous, $g^{-1}(F)$ is a $g$-closed subset of $(Y, v)$. Since $(Y, v)$ is a $T_2$-space, $g^{-1}(F)$ is a closed subset of $(Y, v)$. Since $f$ is $\partial$-continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is a $\partial$-closed subset of $(X, u)$. Therefore, $g \circ f$ is $\partial$-continuous.

Proposition 23. Let $(X, u)$ and $(Z, w)$ be closure spaces and let $(Y, v)$ be a $T_1'$-space. If $f: (X, u) \to (Y, v)$ and $g: (Y, v) \to (Z, w)$ are $\partial$-continuous, then $g \circ f: (X, u) \to (Z, w)$ is $\partial$-continuous too.

Proof. Let $F$ be a closed subset of $(Z, w)$. Since $g$ is $\partial$-continuous, $g^{-1}(F)$ is a $\partial$-closed subset of $(Y, v)$. Since $(Y, v)$ is a $T_1'$-space, $g^{-1}(F)$ is a closed subset of $(Y, v)$ which implies that $(g \circ f)^{-1}(F)$ is a $\partial$-closed subset of $(X, u)$. Hence, $g \circ f$ is $\partial$-continuous.

Proposition 24. Let $\{\{X_\alpha, u_\alpha\} : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be families of closure spaces. For each $\alpha \in I$, let $f_\alpha: X_\alpha \to Y_\alpha$ be a map and $f: \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be the map defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. If $f: \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to \prod_{\alpha \in I} (Y_\alpha, v_\alpha)$ is $\partial$-continuous, then $f_\alpha: (X_\alpha, u_\alpha) \to (Y_\alpha, v_\alpha)$ is $\partial$-continuous for each $\alpha \in I$.

Proof. Let $\beta \in I$ and let $F$ be a closed subset of $(Y_\beta, v_\beta)$. Then $F \times \prod_{\alpha \in I \setminus \{\beta\}} Y_\alpha$ is a closed subset of $\prod_{\alpha \in I} (Y_\alpha, v_\alpha)$. Since $f$ is $\partial$-continuous, $f^{-1}(F) = f_\beta^{-1}(F) \times \prod_{\alpha \in I \setminus \{\beta\}} X_\alpha$.
is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. By Proposition 12, $f_\beta^{-1}(F)$ is a $\partial$-closed subset of $(X_\beta, u_\beta)$. Hence, $f_\beta$ is $\partial$-continuous. \hfill \square

**Definition 7.** Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f : (X, u) \to (Y, v)$ is called $\partial$-closed if $f(F)$ is a $\partial$-closed subset of $(Y, v)$ for every closed subset $F$ of $(X, u)$.

Every closed map is $\partial$-closed but the converse is not true as may be seen from the following example.

**Example 10.** Let $X = \{1, 2, 3, 4\} = Y$ and define a closure operator $u$ on $X$ by $u\emptyset = \emptyset$, $u\{1, 3, 4\} = \{1, 3, 4\}$ and $u\{1\} = u\{2\} = u\{3\} = u\{4\} = u\{1, 2\} = u\{1, 3\} = u\{1, 4\} = u\{2, 3\} = u\{2, 4\} = u\{3, 4\} = u\{1, 2, 3\} = u\{1, 2, 4\} = u\{1, 3, 4\} = u\{2, 3, 4\} = uX = X$. Define a closure operator $v$ on $Y$ by $v\emptyset = \emptyset$, $v\{1\} = \{1, 3\}$, $v\{2\} = \{2, 3\}$, $v\{3\} = v\{4\} = v\{3, 4\} = \{3, 4\}$ and $v\{1, 2\} = v\{1, 3\} = v\{1, 4\} = v\{2, 3\} = v\{2, 4\} = v\{1, 2, 3\} = v\{1, 2, 4\} = v\{1, 3, 4\} = v\{2, 3, 4\} = vY = Y$. Let $f : (X, u) \to (Y, v)$ be the identity map. Then $f$ is $\partial$-closed but it is not closed because $\{1, 3, 4\}$ is a closed subset of $(X, u)$ but $f(\{1, 3, 4\}) = \{1, 3, 4\}$ is not a closed subset of $(Y, v)$.

The following statement is evident:

**Proposition 25.** Let $(X, u)$, $(Y, v)$ and $(Z, w)$ be closure spaces, let $f : (X, u) \to (Y, v)$ and $g : (Y, u) \to (Z, w)$ be maps. Then

(i) If $f$ is $\partial$-closed and $g$ is closed, then $g \circ f$ is $\partial$-closed.

(ii) If $g \circ f$ is $\partial$-closed and $f$ is continuous and surjective, then $g$ is $\partial$-closed.

(iii) If $g \circ f$ is closed and $g$ is $\partial$-continuous and injective, then $f$ is $\partial$-closed.

**Proposition 26.** Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f : (X, u) \to (Y, v)$ is $\partial$-closed if and only if, for each subset $B$ of $Y$ and each open subset $G$ with $f^{-1}(B) \subseteq G$, there is a $\partial$-open subset $V$ of $(Y, v)$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq G$.

**Proof.** Suppose that $f$ is $\partial$-closed. Let $B$ be a subset of $(Y, v)$ and $G$ be an open subset of $(X, u)$ such that $f^{-1}(B) \subseteq G$. Then $f(X - G)$ is a $\partial$-closed subset of $(Y, v)$. Let $V = Y - f(X - G)$. Then $V$ is $\partial$-open and

$$f^{-1}(V) = f^{-1}(Y - f(X - G)) = X - f^{-1}(f(X - G)) \subseteq X - (X - G) = G.$$ 

Therefore, $V$ is $\partial$-open, $B \subseteq V$ and $f^{-1}(V) \subseteq G$.

Conversely, suppose that $F$ is a closed subset of $(X, u)$. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a $\partial$-open subset $V$ of $(Y, v)$ such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, $F \subseteq X - f^{-1}(V)$. Hence,

$$Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$$

implies that $f(F) = Y - V$. Thus $f(F)$ is $\partial$-closed. Therefore, $f$ is $\partial$-closed. \hfill \square
Proposition 27. Let \((X, u)\) be a closure space and let \(\{(Y_\alpha, v_\alpha) : \alpha \in I\}\) be a family of closure spaces. Let \(f : X \to \prod_{\alpha \in I} Y_\alpha\) be a map. If \(f : (X, u) \to \prod_{\alpha \in I} (Y_\alpha, v_\alpha)\) is \(\partial\)-closed, then \(\pi_\alpha \circ f : (X, u) \to (Y_\alpha, v_\alpha)\) is \(\partial\)-closed for each \(\alpha \in I\).

Proof. Let \(f\) be \(\partial\)-closed. Since \(\pi_\alpha\) is closed for each \(\alpha \in I\), also \(\pi_\alpha \circ f\) is \(\partial\)-closed for each \(\alpha \in I\). \(\square\)

References


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