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A survey of results on density modulo 1 of double sequences containing algebraic numbers

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Abstract. In this survey article we start from the famous Furstenberg theorem on non-lacunary semigroups of integers, and next we present its generalizations and some related results.

1 Introduction

1.1 Furstenberg's theorem

Let S be a multiplicative semigroup of integers. The semigroup S is said to be *lacunary* if the members $\{s \in S : s > 0\}$ are of the form s_0^k, $k \in \mathbb{N}$, $s_0 \in \mathbb{N}^*$. Otherwise, S is *non-lacunary*. In 1967 Furstenberg proved the following result

Theorem 1 ([9, Theorem IV.1]). *If S is a non-lacunary semigroup of integers and ξ is an irrational number, then the set $\{s\xi : s \in S\}$ is dense modulo 1.*

Let $0 \neq p, q \in \mathbb{Z}$. We say that p and q are *multiplicatively independent* if they are not both integer powers of the same integer, or equivalently the ratio $\log p / \log q \notin \mathbb{Q}$. For example, p, q relatively prime are multiplicatively independent.

Since the multiplicative semigroup $S = \langle p, q \rangle$ generated by multiplicatively independent p and q is clearly non-lacunary, we get the following

Corollary 1. *If $p, q > 1$ are multiplicatively independent integers then for every irrational ξ the set*

$$\{p^n q^m \xi : n, m \in \mathbb{N}\} \tag{1}$$

is dense modulo 1.

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Remark 1. It is clear that for $\xi \in \mathbb{Q}$ the set (1) is finite.

Remark 2. As noted by Furstenberg [9] there exist irrationals ξ such that the set (1) is not uniformly distributed modulo 1. Let the sequence $\{2^n 3^m : n, m \in \mathbb{N}\}$ be arranged in increasing order as $\{s_n : n \in \mathbb{N}\}$. Then $s_n\xi$ is not uniformly distributed modulo 1 for $\xi = \sum_k 6^{-n_k}$ with n_k of sufficiently rapid growth. See [1] for some quantitative results in this direction.

Remark 3. Effective versions of Furstenberg's theorem are given in a recent work by Bourgain *et al.* [7]. In particular, an estimate on the rate in

$$\overline{\{p^n q^m \xi \mod 1 : n, m \in \mathbb{N}\}} = [0, 1]$$

in terms of the Diophantine properties of ξ is given.

1.2 Proofs of Furstenberg's theorem

Furstenberg's proof is based on disjointness of dynamical systems – the important notion introduced by Furstenberg in the same paper [9]. Furstenberg's original proof is also outlined in [25]. In 1994 Boshernitzan in [6] gave an elementary proof of Furstenberg's theorem based on topological dynamics methods. Another possible approach is through the renewal theorem and is sketched in [10].

2 Generalizations of Furstenberg's theorem

2.1 ID-semigrous of endomorphisms

In this subsection we focus on the *higher dimensional* analogues of Theorem 1.

2.1.1 Commutative semigroups

Notice that, in terms of dynamical systems, Furstenberg's theorem says that the orbits of the (commutative) semigroup of endomorphisms generated by the multiplicatively independent integers p and q and acting on the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by

$$p.\xi = p\xi \mod 1 \text{ and } q.\xi = q\xi \mod 1$$

are finite or dense, or equivalently, see [6], [12] for details, the only infinite closed p- and q-invariant subset of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is $\mathbb{T}$ itself, or that the infinite invariant subset of $\mathbb{T}$ is dense. Clearly, there are many closed infinite p-invariant (or q-invariant) proper subsets of $\mathbb{T}$. For example, consider $\mathbb{T}$ with multiplication by 2 and 3. Then the Cantor set

$$\mathcal{C} = \left\{ x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \in [0, 1] : x_i \in \{0, 2\} \right\}$$

is 3-invariant. Hence, Furstenberg's theorem gives a remarkable rigidity property of the joint p- and q-action on the 1-dimensional torus[1].

[1] We note that beside this topological rigidity result given by Theorem 1 there is Furstenberg's conjecture about measure rigidity.

Conjecture 1 (Furstenberg 1967). *Let p, q be two multiplicative independent positive integers. Any Borel measure μ on $\mathbb{T}$ ergodic under the action of the semigroup $S = \langle p, q \rangle$ generated by*

The above reformulation of Furstenberg's theorem suggest a possible generalization. Namely, to consider the semigroups of endomorphisms of the higher dimensional torus or other compact groups.

Following [2], [3] we have the following

Definition 1. We say that the semigroup Σ of endomorphisms of a compact group G has the *ID-property*, or that Σ is an *ID-semigroup*, if the only infinite closed Σ-invariant[2] subset of G is G itself.[3]

According to Definition 1, Furstenberg's theorem says that the semigroup $S = \langle p, q \rangle \subset \mathbb{N}$ generated by two multiplicatively independent integers p, q and acting on $\mathbb{T}$ is an ID-semigroup.

Generalization of Furstenberg's theorem to $\mathbb{T}^d$ was given by Berend in 1983. He gave necessary and sufficient conditions in arithmetical terms for a commutative semigroup of $\mathbb{T}^d$ to have the ID-property in the following theorem.

Theorem 2 (Berend, [2, Theorem 2.1]). *A commutative semigroup Σ of endomorphisms of $\mathbb{T}^d$ has the ID-property if and only if the following hold:*

(i) There exists an endomorphism $\sigma \in \Sigma$ such that the characteristic polynomial f_{σ^n} of σ^n is irreducible over $\mathbb{Z}$ for every positive integer n.

(ii) For every common eigenvector v of Σ there exists an endomorphism $\sigma_v \in \Sigma$ whose eigenvalue in the direction of v is of norm greater than 1.

(iii) The semigroup Σ contains a pair of multiplicatively independent endomorphisms[4].

In [3] Berend extended the above result to other compact Abelian groups. In particular, he gave necessary and sufficient conditions that guarantee that a given semigroup of endomorphisms of an a-adic solenoid Ω_a^d is an ID-semigroup [3, Theorem II.1].

2.1.2 Non-commutative semigroups

Recently some generalizations for non-commutative semigroup of endomorphisms acting on $\mathbb{T}^d$ have been obtained in [18], [19], [11], [12], [10]. For example Muchnik proved in [18] that if the semigroup Σ of $\mathrm{SL}(d, \mathbb{Z})$ is Zariski dense in $\mathrm{SL}(d, \mathbb{R})$, then Σ acting on $\mathbb{T}^d$ has the ID-property. In the next paper [19] from 2005 Muchnik generalized the results of Berend [2] proving the following

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p and q is either Lebesgue measure or an atomic measure supported on finitely many rational points.

More about this and related topics can be found in a survey article [15]

[2]Recall that that a subset $A \subset G$ is said to be *Σ-invariant* if $\Sigma A \subset A$.

[3]ID stands for *infinite invariant is dense*.

[4]We say, as we do in the case of integer numbers, that two endomorphisms σ and τ are *multiplicatively dependent* if there exist integers m and n, not simultaneously equal to 0, such that $\sigma^m = \tau^n$. Otherwise, we say that σ and τ are *multiplicatively independent*. We note that Berend used a slightly different terminology. Namely, *rationally independent* instead of multiplicatively independent.

Theorem 3 (Muchnik, [19, Theorem 1.1]). *Let Σ be a semigroup of invertible $d \times d$ matrices acting naturally on $\mathbb{T}^d$ and $\mathbb{Q}^d$. Let G be a group generated by Σ. The semigroup Σ is an ID-semigroup if and only if the following conditions are satisfied:*

(1) Σ acts strongly irreducibly on $\mathbb{Q}^d$, i.e., every subgroup H of G of finite index acts irreducibly on $\mathbb{Q}^d$.

(2) The group G is not cyclic-by-finite.

(3) For every G-invariant subspace $V \subset \mathbb{C}^d$, $S|_V$ is an unbounded semigroup in $\mathrm{End}(V)$.

Muchnik's proof extends the arguments of Berend to the non-commutative setting. Guivarc'h and Starkov [11] proved an important part of Muchnik's result using different methods (matricial analogues of the renewal theorem). For other results in this direction see also [12], [10].

2.2 Non-integer p and q

There is a natural question what happens if we replace integers p, q in Furstenberg's theorem by, say, algebraic numbers. This problem was considered by Berend in [5]. To state his result we need to introduce some notation.

Let K be a real algebraic number field (i.e., finite extension of $\mathbb{Q}$) and S a subsemigroup of its multiplicative group K^*. According to [5], the semigroup S is said to be DM_1 if $S\xi$ is dense modulo 1 for every $\xi \neq 0$, and *almost* DM_1 if $S\xi$ is dense modulo 1 for every $\xi \notin K$. We say that two numbers λ and μ are *multiplicatively dependent* if there are integers m and n, not both of which are 0, such that $\lambda^m = \mu^n$, and *multiplicatively independent* otherwise. The semigroup S is said to be *one-parameter* if all its elements are integer powers of a single number; *weakly one-parameter* if any two of its elements are rationally dependent, and *multi-parameter* otherwise. If $[K : \mathbb{Q}] = m$, we denote by $\mathrm{PS}(K)$ the semigroup consisting of all Pisot or Salem numbers[5] of degree m. For a subset $A \subset K$, we denote by $\mathbb{Q}(A)$ the subfield of K obtained by adjoining A to $\mathbb{Q}$. Then we have the following

Theorem 4 (Berend, [3, Theorem 2.1]). *Let K be a real algebraic number field and S a multi-parameter subsemigroup of $K^* \cap [-1, 1]^c$ with $\mathbb{Q}(S) = K$. Then S is almost* DM_1. *If, moreover, $S \not\subset$* $\mathrm{PS}(K)$, *then S is* DM_1.

The proof of the above theorem is based on the construction of an appropriate dynamical system of some a-adic solenoid. Then the necessary and sufficient conditions for the semigroup of endomorphisms of the solenoid to be ID-semigroup, obtained by Berend in [3], are used.

[5] A *Pisot number* is a real algebraic integer θ greater than 1 whose other conjugates $\theta_2, \ldots, \theta_n$ satisfy the inequalities $|\theta_2| < 1, \ldots, |\theta_n| < 1$.

A *Salem number* is a real algebraic integer θ greater than 1 whose other conjugates $\theta_2, \ldots, \theta_n$ satisfy the inequalities $|\theta_j| \leq 1$, $2 \leq j \leq n$, with equality for at least one j.

2.3 Sums of the expressions of the form (1)

Furstenberg also conjectured, that the set of the form $\{(p^m + q^n)\xi : m, n \in \mathbb{N}\}$ is dense modulo 1. As far as we know, this conjecture is still open, however there are some results concerning related questions. For example, Kra in her paper [14] was dealing with the expressions of the form

$$\sum_{i=1}^{k} p_i^n q_i^m \xi_i,$$

where p_i, q_i are multiplicatively independent integers. The main result of [14] is the following

Theorem 5 (Kra, [14, Theorem 1.2 and Corollary 2.2]). *Let $p_i, q_i \in \mathbb{N}$ be multiplicatively independent with $1 < p_i < q_i$ for $i = 1, \ldots, k$, $k \in \mathbb{N}$, $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j$, and $p_1 \leq p_2 \leq \ldots \leq p_k$. Then for distinct $\xi_1, \ldots, \xi_k \in [0, 1]$ with at least one $\xi_i \notin \mathbb{Q}$ the set*

$$\{\sum_{i=1}^{k} p_i^n q_i^m \xi_i : n, m \in \mathbb{N}\}$$

is dense modulo 1.

Furthermore, let r_m be any sequence of real numbers and $\xi \notin \mathbb{Q}$. Then, the set

$$\{p_1^m q_1^n \xi + r_m : m, n \in \mathbb{N}\} \tag{2}$$

is dense modulo 1.

The proof given in [14] is based on topological dynamics methods and Furstenberg's theorem plays a crucial role there. For an alternate proof of the first part of Kra's result via measure theoretic methods see [16].

Sets of the form (2) with integers replaced by algebraic numbers are considered in Sect. 3.4.

3 Expressions containing algebraic numbers

Inspired by Kra's Theorem 5 and Berend's result – Theorem 4, we stated in [20] the following conjecture.

Conjecture 2 ([20]). *If the pairs λ_i, μ_i are multiplicatively independent real algebraic numbers with absolute values greater than 1 and the pairs λ_i, μ_i and λ_j, μ_j are different for $i \neq j$, then for any real numbers $\xi_1, \ldots, \xi_k$ with at least one $\xi_j \notin \mathbb{Q}(\bigcup_{i=1}^{k}\{\lambda_i, \mu_i\})$ the set*

$$\{\sum_{i=1}^{k} \lambda_i^n \mu_i^m \xi_i : n, m \in \mathbb{N}\}$$

is dense modulo 1.

In the next subsections we are going to present some results around this conjecture and obtained by author in a series of papers [20], [22], [23].

3.1 Algebraic integer case

In [20] we proved a results which can be considered as the first step toward the proof of Conjecture 2. Namely, using some topological dynamics methods in the spirit of Berend [5] and Kra [14], we proved the following.[6]

Theorem 6 ([20, Theorem 1.6]). *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent real algebraic integers of degree 2. Assume that*

(i) $|\lambda_i|, |\mu_1| > 1$, $i = 1, 2$, *and the absolute values of their conjugates,* $\tilde{\lambda}_i, \tilde{\mu}_i$ *are also greater than 1.*

(ii) $\mu_i = g_i(\lambda_i)$, *for some* $g_i \in \mathbb{Z}[x]$, $i = 1, 2$.

(iii) In each pair (λ_i, μ_i) *there is at least one element with the property that for every* $n \in \mathbb{N}$, *its* n*-th power is irrational.*

(iv) There exist $k, l, k', l' \in \mathbb{N}$ *such that*

$$\min\{|\lambda_2|^k|\mu_2|^l, |\tilde{\lambda}_2|^k|\tilde{\mu}_2|^l\} > \max\{|\lambda_1|^k|\mu_1|^l, |\tilde{\lambda}_1|^k|\tilde{\mu}_1|^l\} \tag{3}$$

and

$$\min\{|\lambda_1|^{k'}|\mu_1|^{l'}, |\tilde{\lambda}_1|^{k'}|\tilde{\mu}_1|^{l'}\} > \max\{|\lambda_2|^{k'}|\mu_2|^{l'}, |\tilde{\lambda}_2|^{k'}|\tilde{\mu}_2|^{l'}\}. \tag{4}$$

Then for any real numbers ξ_1, ξ_2 *with at least one* $\xi_i \neq 0$, *there exists a natural number* κ *such that the set*

$$\{\lambda_1^n\mu_1^m\kappa\xi_1 + \lambda_2^n\mu_2^m\kappa\xi_2 : n, m \in \mathbb{N}\}$$

is dense modulo 1.

Remark 4. Equivalently, Theorem 6 says that for every ξ_1, ξ_2, with at least one ξ_i non-zero, there exists a natural number κ such that the set

$$\{\lambda_1^n\mu_1^m\xi_1 + \lambda_2^n\mu_2^m\xi_2 : n, m \in \mathbb{N}\} \tag{5}$$

is dense modulo $1/\kappa$.

It is not difficult to check that the following inequalities

$$|\lambda_2| > |\tilde{\lambda}_2| > |\lambda_1| > |\tilde{\lambda}_1| > 1 \text{ and } |\mu_1| > |\tilde{\mu}_1| > |\mu_2| > |\tilde{\mu}_2| > 1$$

imply (3) and (4). Hence, it is not difficult to construct examples:

$$(\sqrt{23}+1)^n(\sqrt{23}+2)^m\xi_1 + (\sqrt{61}+1)^n(\sqrt{61}-6)^m\xi_2.$$

In [24] it is shown that the condition $\mu_i \in \mathbb{Z}(\lambda_i)$ can be removed by imposing appropriate conditions on the norms of conjugates of λ_i, μ_i and the degree of the algebraic numbers $\lambda_i^n\mu_i^m$. In particular, the following theorem is proved.

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

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[6] In the proof of [20, Theorem 1.6] there is an incorrect statement. First of all, $\widetilde{X_\alpha^{\mathrm{ac}}}$ on p. 227 should be defined as $\mathcal{J}_{\kappa,\kappa}X_\alpha^{\mathrm{ac}}$, i.e., with $\kappa = \iota$. Then it is claimed that the set $\pi_1(S)$ is Σ_1-invariant. However, it is not, and so we can not apply the ergodic argument to conclude that $\pi_1(S) = \mathbb{T}$. What we only know is that for every pair (ξ_1, ξ_2) there exists a natural number κ such that $\kappa\pi_1(S) = \mathbb{T}$. See Sect. 3.2 for details.

Theorem 7 ([24]). *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent algebraic integers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates, $\tilde{\lambda}_1, \tilde{\mu}_1, \tilde{\lambda}_2, \tilde{\mu}_2$, are also greater than 1. Assume that the following conditions are satisfied:*

(i) for every $n, m \in \mathbb{N}$, $\deg_{\mathbb{Q}} \lambda_i^n \mu_i^m = 4$,

(ii) $|\lambda_i| > |\tilde{\lambda}_i| > 1$ and $|\mu_i| > |\tilde{\mu}_i| > 1$, $i = 1, 2$,

(iii) there exist $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, and two positive integers k, l, $k \neq l$ such that

$$|\tilde{\lambda}_2 \tilde{\mu}_2|^{\alpha} |\tilde{\lambda}_2^k \tilde{\mu}_2^l|^{\beta} > |\lambda_1 \mu_1|^{\alpha} |\lambda_1^k \mu_1^l|^{\beta}$$

and

$$|\tilde{\lambda}_1 \tilde{\mu}_1|^{\alpha'} |\tilde{\lambda}_1^k \tilde{\mu}_1^l|^{\beta'} > |\lambda_2 \mu_2|^{\alpha'} |\lambda_2^k \mu_2^l|^{\beta'}.$$

Then for any pair of real numbers ξ_1, ξ_2, with at least one ξ_i non-zero, the set (7) *is dense modulo* 1.

Remark 5. In order to check that given λ_i, μ_i satisfy condition (i) one can use Dubickas' result [8] which gives necessary and sufficient conditions (and also a simple sufficient condition) under which two algebraic numbers α and β over a field k satisfy $\deg_k(\alpha\beta) = \deg_k \alpha \deg_k \beta$.

As an example illustrating Theorem 7 we can take

$$(\sqrt{7}+1)^n (3\sqrt{3}+1)^m \xi_1 + (100\sqrt{5}+3)^n (2\sqrt{2}+1)^m \xi_2.$$

The case of algebraic numbers (not necessarily algebraic integers) is also studied in [24]. See Theorem 9 in Sect. 3.3 below.

3.2 Sketch of the proof of Theorem 6

The idea of the proof is to construct, using the companion matrices associated with λ_i's, an appropriate semigroup M of endomorphisms of the d-dimensional torus, for some $d > 1$. Then we have to chose a special point α in $\mathbb{T}^d$ such that looking at the coordinates of the orbit $M\alpha$ we can recognize the expression we are interested in. The next step is to prove that the orbit is large in some sense and that this implies density modulo 1 of our expression.

3.2.1 Companion matrices

Let $\nu > 1$ be a real algebraic integer of degree 2 with minimal (monic) polynomial $P_\nu \in \mathbb{Z}[x]$, $P_\nu(x) = x^2 + c_1 x + c_0$. A *companion matrix of P_ν or ν* is the matrix of the form

$$\sigma_\nu = \begin{pmatrix} 0 & 1 \\ -c_0 & -c_1 \end{pmatrix}. \tag{6}$$

3.2.2 Construction of the semigroups Σ_1, Σ_2 and M

We associate with λ_i, the companion matrices $\sigma_i = \sigma_{\lambda_i}$ and with μ_i we associate matrices $\tau_i = g_i(\sigma_i)$. For $i = 1, 2$, we denote by $\Sigma_i = \langle \sigma_i, \tau_i \rangle$ the semigroups generated by σ_i and τ_i. Using Theorem 4 we see that assumptions (i)–(iii) imply, that the semigroups Σ_1 and Σ_2 are commutative ID-semigroups of endomorphisms of $\mathbb{T}^2$. (Commutativity follows from condition (ii).) We put $M_\sigma = \left(\begin{smallmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{smallmatrix}\right)$ and $M_\tau = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$. The semigroup M of endomorphisms of $\mathbb{T}^2 \times \mathbb{T}^2$ is the semigroup $\langle M_\sigma, M_\tau \rangle$ generated by matrices M_σ and M_τ. Clearly, M is not an ID-semigroup due to reducibility.

3.2.3 The orbit $M\alpha$

Let $X_\alpha = M\alpha$ be the orbit of the point $\alpha = (\alpha_1, \alpha_2)$ under the action of M. Taking as α_1 and α_2 the common eigenvectors of the semigroups Σ_1 and Σ_2, respectively, $\alpha_1 = \xi_1(1, \lambda_1)$ and $\alpha_2 = \xi_2(1, \lambda_2)$, we get

$$X_\alpha = \{(\lambda_1^n \mu_1^m \xi_1, \lambda_1^{n+1} \mu_1^m \xi_1, \lambda_2^n \mu_2^m \xi_2, \lambda_2^{n+1} \mu_2^m \xi_2) : n, m \in \mathbb{N}\}.$$

Let X_α^{ac} denote the set of accumulation points of X_α. Clearly, X_α and X_α^{ac} are M-invariant. Furthermore, X_α^{ac} is closed and one can prove that that X_α^{ac} is infinite. Moreover, [20, Proposition 5.7] with the same assumptions as in Theorem 6, if $(0, 0) \in X_\alpha^{\mathrm{ac}}$ then one of the following holds:

(1) The point $(0, 0)$ is isolated in X_α^{ac}.

(2) The set X_α^{ac} contains the whole $\mathbb{T}^2 \times \{0\}$ or $\{0\} \times \mathbb{T}^2$.

In the prove of the above statement assumptions (3) and (4) are used. The next step is to show that there is a non-isolated rational point p/κ in X_α^{ac}. The fact that Σ_1 and Σ_2 are ID-semigroups plays the crucial role at this step. Hence, by the above dichotomy applied to $\kappa X_\alpha^{\mathrm{ac}}$ (what is justified since only invariance is important in the proof) we have that (2) holds and we conclude that the set $S := \{x + y : (x, y) \in \kappa X_\alpha^{\mathrm{ac}}\}$ is equal to the whole $\mathbb{T}^2$. Therefore, comparing this with the first and third coordinate of κX_α in $\mathbb{T}^2 \times \mathbb{T}^2$ we obtain the result taking projection of $S = \mathbb{T}^2$ onto the first coordinate.

Remark 6. We believe that under the assumptions of Theorem 6 the set (5) is in fact dense modulo 1 (i.e., $\kappa = 1$). However, in order to prove such a statement a much better understanding of the closed subsets of $\mathbb{T}^2$, invariant under the action of the semigroup M, is required. Specifically, if we knew that under the above assumptions the closure of the orbit X_α, contains $(0, 0)$ then we would have that for every ξ, with at least one ξ_i irrational, $\kappa = 1$. However, this seems to be a difficult problem as very little is known about reducible actions of linear semigroups on tori. In particular, description of the closed invariant sets and orbit closures is not known even in the "simplest" case of the semigroup of endomorphisms of $\mathbb{T}^2$ generated by the following two matrices $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Only some partial results are available in literature, [14], [17].

3.3 Extension to algebraic numbers

In [22] we removed the assumption that λ_i and μ_i are algebraic integers.

Theorem 8 ([22, Theorem 1.5]). *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent algebraic numbers of degree 2. Assume that*

(i) $|\lambda_i|, |\mu_1| > 1$, $i = 1, 2$, and the absolute values of their conjugates, $\tilde{\lambda}_i, \tilde{\mu}_i$ are also greater than 1.

(ii) $\mu_i = g_i(\lambda_i)$, for some $g_i \in \mathbb{Q}[x]$, $i = 1, 2$.

(iii) At least one element in each pair λ_i, μ_i has all non-negative powers irrational.

Let $S = \{\infty, p_1, p_2, \dots, p_s\}$, where for $k = 1, \dots, s$, $p_k \geq 2$ are the primes appearing in the denominators of coefficients of $g_1, g_2 \in \mathbb{Q}[x]$, and the minimal polynomials $P_{\lambda_1}, P_{\lambda_2} \in \mathbb{Q}[x]$ of λ_1, and λ_2, respectively.

Assume further that

(iv) there exist $k, l, k', l' \in \mathbb{N}$ such that

$$\min_{p\in S}(\min\{|\lambda_2|_p^k|\mu_2|_p^l, |\tilde{\lambda}_2|_p^k|\tilde{\mu}_2|_p^l\}) > \max_{p\in S}(\max\{|\lambda_1|_p^k|\mu_1|_p^l, |\tilde{\lambda}_1|_p^k|\tilde{\mu}_1|_p^l\})$$

and

$$\min_{p\in S}(\min\{|\lambda_1|_p^{k'}|\mu_1|_p^{l'}, |\tilde{\lambda}_1|_p^{k'}|\tilde{\mu}_1|_p^{l'}\}) > \max_{p\in S}(\max\{|\lambda_2|_p^{k'}|\mu_2|_p^{l'}, |\tilde{\lambda}_2|_p^{k'}|\tilde{\mu}_2|_p^{l'}\}),$$

where $|\cdot|_p$ is the p-adic norm, whereas $|\cdot|_\infty$ stands for the usual absolute value, and

$$\min\{|\lambda_i|_p, |\mu_i|_p, |\tilde{\lambda}_i|_p, |\tilde{\mu}_i|_p : i = 1, 2,\ p \in S\} > 1.$$

Then for any pair of real numbers ξ_1, ξ_2, with at least one ξ_i non-zero, there exists a natural number κ such that the set

$$\{\lambda_1^n \mu_1^m \kappa\xi_1 + \lambda_2^n \mu_2^m \kappa\xi_2 : n, m \in \mathbb{N}\}$$

is dense modulo 1.[7]

As an example, consider the following expression[8]

$$\left(\frac{17^{10}}{\sqrt{2}} + \frac{1}{3\cdot5\cdot7}\right)^n \left(\frac{11\cdot17^{10}}{\sqrt{2}} + \frac{11}{3\cdot5\cdot7} + \frac{17^{1000}}{7^3\cdot5^3\cdot2^6\cdot3^4}\right)^m + \\ + \left(\frac{17^{100}}{3^2\cdot2^3\cdot\sqrt{5}} + \frac{1}{7^2\cdot5^2}\right)^n \left(\frac{11\cdot17^{100}}{3^2\cdot2^3\cdot\sqrt{5}} + \frac{11}{7^2\cdot5^2} + 13\right)^m.$$

By Theorem 8 there is $\kappa \in \mathbb{N}$ such that the above set is dense modulo $1/\kappa$.

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[7]The statement in [22] is with $\kappa = 1$ although what is really proved is the statement given here. This is due to the same mistake as described in the footnote on p. 36.

[8]In [5, Proposition 4.2] it is proved that for λ and μ which are *effectively given* complex algebraic numbers (which is clearly the case here) it is possible effectively to decide whether or not they are multiplicatively independent.

Remark 7. An appropriate version of the above theorem in the case when not all of λ_i, μ_i are of degree 2, namely degree 1 of some numbers is allowed, is also given in [22].

Although the proof of Theorem 8 is more complicated than in the case of algebraic integers, the main steps remain the same. As before, we define $\sigma_i = \sigma_{\lambda_i}$ and $\tau_i = g(\sigma_i)$, where σ_{λ_i} are companion matrices. The main difference comes from the fact that the companion matrices σ_{λ_i} associated with λ_i, $i = 1, 2$ have rational coefficients. Therefore, we need to introduce some other compact group which will play the role of $\mathbb{T}^2 \times \mathbb{T}^2$. Let a be the product of all primes dividing the denominator of some entry of $\sigma_1, \sigma_2, \tau_1, \tau_2$. Similarly to [5], we consider the product of a-adic solenoids[9] $\Omega_a^2 \times \Omega_a^2$.

For $i = 1, 2$, we denote by $\Sigma_i = \langle \sigma_i, \tau_i \rangle$ the semigroups generated by σ_i and τ_i. Using [3, Theorem II.1], we check that Σ_1 and Σ_2 are commutative ID-semigroups of Ω_a^2. The semigroup M acting on $\Omega_a^2 \times \Omega_a^2$ is defined similarly. Now, the role of the rational points in $\mathbb{T}^2 \times \mathbb{T}^2$ play the torsion points in $\Omega_a^2 \times \Omega_a^2$.

In [24] the following theorem, which does not assume commutativity $\mu_i \in \mathbb{Q}(\lambda_i)$ is proved.

Theorem 9 ([24]). *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent real algebraic numbers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates, $\tilde{\lambda}_1, \tilde{\mu}_1, \tilde{\lambda}_2, \tilde{\mu}_2$, are also greater than 1. Assume that*

(i) for every $n, m \in \mathbb{N}$, $\deg_{\mathbb{Q}} \lambda_i^n \mu_i^m = 4$.

Let $p_1, p_2, \ldots, p_s \geq 2$ be the primes appearing in the denominators of coefficients of the minimal polynomials $P_{\lambda_i}, P_{\mu_i}, \in \mathbb{Q}[x]$ of λ_i and μ_i, $i = 1, 2$. We set $S = \{\infty, p_1, p_2, \ldots, p_s\}$. Assume further that the following conditions are satisfied:

(ii) $|\lambda_i|_\infty > |\tilde{\lambda}_i|_\infty > 1$ and $|\mu_i|_\infty > |\tilde{\mu}_i|_\infty > 1$, $i = 1, 2$,

(iii) there exist $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, and two positive integers k, l, $k \neq l$ such that

$$\min\left(\min_{p \in S\setminus\{\infty\}} |\lambda_2\mu_2|_p^\alpha |\lambda_2^k \mu_2^l|_p^\beta, |\tilde{\lambda}_2\tilde{\mu}_2|_\infty^\alpha |\tilde{\lambda}_2^k \tilde{\mu}_2^l|_\infty^\beta\right)$$
$$> \max\left(\max_{p \in S\setminus\{\infty\}} |\lambda_1\mu_1|_p^\alpha |\lambda_1^k \mu_1^l|_p^\beta, |\lambda_1\mu_1|_\infty^\alpha |\lambda_1^k \mu_1^l|_\infty^\beta\right)$$

[9] Let $a = p_1 p_2 \ldots p_s$, wher p_i are different primes. Consider $\mathbb{Z}[1/a]$ as a topological group with the discrete topology. The dual group $\widehat{\mathbb{Z}[1/a]}$ of $\mathbb{Z}[1/a]$ is called an *a-adic solenoid* and we denote it by Ω_a. The compact abelian group Ω_a^d may be considered as a quotient group of the additive group $\mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \ldots \times \mathbb{Q}_{p_s}^d$ by a discrete subgroup $B = \{(b, \underbrace{-b, \ldots, -b}_{s}) : b \in \mathbb{Z}[1/a]^d\}$. That is, $\Omega_a^d = \mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \ldots \times \mathbb{Q}_{p_s}^d / B$. (For more details on solenoids see [13].)

and

$$\min\left(\min_{p\in S\setminus\{\infty\}}|\lambda_1\mu_1|_p^{\alpha'}|\lambda_1^k\mu_1^l|_p^{\beta'},|\tilde{\lambda}_1\tilde{\mu}_1|_\infty^{\alpha'}|\tilde{\lambda}_1^k\tilde{\mu}_1^l|_\infty^{\beta'}\right)$$
$$>\max\left(\max_{p\in S\setminus\{\infty\}}|\lambda_2\mu_2|_p^{\alpha'}|\lambda_2^k\mu_2^l|_p^{\beta'},|\lambda_2\mu_2|_\infty^{\alpha'}|\lambda_2^k\mu_2^l|_\infty^{\beta'}\right),$$

(iv) $|\lambda_i|_p=|\tilde{\lambda}_i|_p>1$ *and* $|\mu_i|_p=|\tilde{\mu}_i|_p>1$, *for* $p\in S\setminus\{\infty\}$,

where $|\cdot|_p$ *is the* p*-adic norm, whereas* $|\cdot|_\infty$ *stands for the usual absolute value.*

Then for any pair of real numbers ξ_1,ξ_2, *with at least one* ξ_i *non-zero, there exists a natural number* κ *such that the set*

$$\{\lambda_1^n\mu_1^m\kappa\xi_1+\lambda_2^n\mu_2^m\kappa\xi_2 : n,m\in\mathbb{N}\} \tag{7}$$

is dense modulo 1.

As a result we are able to consider more general expressions containing algebraic numbers than those given by Theorem 8. For example, Theorem 9 implies that the following double-sequence[10]

$$\left(7\sqrt{2}+\frac{1}{2\cdot3\cdot5\cdot7}\right)^n\left(\frac{7^2}{\sqrt{5}}+\frac{1}{2^3\cdot3^2\cdot5^2\cdot7^2}\right)^m+$$
$$\left(7^5\sqrt{3}+\frac{1}{2^{11}\cdot3^{11}\cdot5^{11}\cdot7^{11}}\right)^n\left(\frac{7}{\sqrt{7}}+\frac{1}{2\cdot3\cdot5\cdot7}\right)^m,\quad n,m\in\mathbb{N}$$

is dense modulo $1/\kappa$ for some κ.

3.4 Other expressions

The following theorem is a generalization of the second part of Theorem 5.

Theorem 10 ([23, Theorem 1.3]). *Let* λ,μ *be a pair of multiplicatively independent real algebraic numbers, with their conjugates* $\lambda=\lambda_1,\lambda_2,\dots,\lambda_d$ *and* $\mu=\mu_1,\mu_2,\dots,\mu_r$, *such that* $\mu\in\mathbb{Q}(\lambda)$, *i.e.,* $\mu=g(\lambda)$ *for some* $g\in\mathbb{Q}[x]$.

Assume that λ *has the property that for every* $n\in\mathbb{N}$, λ^n *has the same degree over* $\mathbb{Q}$ *as* λ.

Let $S=\{\infty,p_1,p_2,\dots,p_s\}$, *where* $p_k\geq2$, $1\leq k\leq s$, *are the primes appearing in the denominators of the coefficients of* $g\in\mathbb{Q}[x]$ *and the minimal polynomial* $P_\lambda\in\mathbb{Q}[x]$ *of* λ.

Assume further that

$$\min_{p\in S}\min_{1\leq i\leq d}|\lambda_i|_p>1 \text{ and } \min_{p\in S}\min_{1\leq j\leq r}|\mu_j|_p>1.$$

[10]In order check that these pairs of algebraic numbers are multiplicatively independent we can use the following

Lemma 1 ([24]). *Let* $p,q>1$ *be the square-free numbers and* $a,b,c,d\in\mathbb{Q}$. *If* $p\neq q$ *then* $c\sqrt{p}+a$ *and* $d\sqrt{q}+b$ *are multiplicatively independent.*

See also footnote on p. 39.

Then, for any non-zero ξ and any sequence of real numbers r_m, the set

$$\{\mu^m \lambda^n \xi + r_m : m, n \in \mathbb{N}\}$$

is dense modulo 1.

Theorem 10 for algebraic integers was proved in [21].

Theorem 11. *[23, Theorem 1.4] Let λ, μ be a pair of multiplicatively independent real algebraic numbers satisfying conditions of Theorem 10. Then, for any non-zero ξ and any two real numbers r, β, the set*

$$\{\mu^m \lambda^n \xi + r^{m+n} \beta : m, n \in \mathbb{N}\} \tag{8}$$

are dense modulo 1.

The sets of the form (8) with λ and $\mu \in \mathbb{N}$ have been considered by Berend in [4]. In [23] we generalized his proof to the setting of algebraic numbers.

In the proofs of the above theorems topological dynamics methods (ergodicity and topological transitivity) together with an elementary algebraic numbers theory are used.

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