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\textit{Acta Mathematica Universitatis Ostraviensis}, Vol. 17 (2009), No. 1, 51--66

Persistent URL: \url{http://dml.cz/dmlcz/137527}

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∂-Closed Sets in Biclosure Spaces

Chawalit Boonpok

Abstract. In the present paper, we introduce and study the concept of ∂-closed sets in biclosure spaces and investigate its behavior. We also introduce and study the concept of ∂-continuous maps.

1 Introduction
A bitopological space $(X, \mathcal{I}_1, \mathcal{I}_2)$ is a set $X$ together with two topologies $\mathcal{I}_1$ and $\mathcal{I}_2$ defined on $X$. The study of bitopological spaces was initiated by J. C. Kelly [6]. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. Closure spaces were studied in [1] (see also [2], [3], [9], [10]) as sets endowed with a grounded, extensive and monotone closure operator. In this paper, we introduce and study the concept of ∂-closed sets in biclosure spaces and characterize their properties. Moreover, we define the notions of ∂-continuity and ∂-irresoluteness by using ∂-closed sets and study some of their basic properties.

2 Preliminaries
A map $u: P(X) \to P(X)$ defined on the power set $P(X)$ of a set $X$ is called a closure operator on $X$ and the pair $(X, u)$ is called a closure space if the following axioms are satisfied:

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator $u$ on a set $X$ is called additive (respectively, idempotent) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is closed in the closure space $(X, u)$ if $uA = A$ and it is open if

2000 Mathematics Subject Classification: 54A05
Key Words and Phrases: closure operator, closure space, biclosure space, ∂-closed set, ∂-continuous map.
its complement is closed. The empty set and the whole space are both open and closed.

A closure space \((Y, v)\) is said to be a subspaces of \((X, u)\) if \(Y \subseteq X\) and \(vA = uA \cap Y\) for each subset \(A \subseteq Y\). If \(Y\) is closed in \((X, u)\), then the subspace \((Y, v)\) of \((X, u)\) is said to be closed too. A closure space \((X, u)\) is said to be a \(T_0\)-space if, for any pair of points \(x, y \in X\), from \(x \in u\{y\}\) and \(y \in u\{x\}\) it follows that \(x = y\), and it is called a \(T_1\)-space if each singleton subset of \(X\) is closed or open.

Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \(f : (X, u) \to (Y, v)\) is said to be continuous if \(f(uA) \subseteq v(f(A))\) for every subset \(A \subseteq X\).

One can see that a map \(f : (X, u) \to (Y, v)\) is continuous if and only if

\[
uf^{-1}(B) \subseteq f^{-1}(vB)
\]

for every subset \(B \subseteq Y\). Clearly, if \(f : (X, u) \to (Y, v)\) is continuous, then \(f^{-1}(F)\) is a closed subset of \((X, u)\) for every closed subset \(F\) of \((Y, v)\).

Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \(f : (X, u) \to (Y, v)\) is said to be closed (resp. open) if \(f(F)\) is a closed (resp. open) subset of \((Y, v)\) whenever \(F\) is a closed (resp. open) subset of \((X, u)\).

The product of a family \(\{(X_\alpha, u_\alpha) : \alpha \in I\}\) of closure spaces, denoted by

\[
\prod_{\alpha \in I} (X_\alpha, u_\alpha),
\]

is the closure space \(\left( \prod_{\alpha \in I} X_\alpha, u \right)\) where \(\prod_{\alpha \in I} X_\alpha\) denotes the cartesian product of sets \(X_\alpha, \alpha \in I,\) and \(u\) is the closure operator generated by the projections

\[
\pi_\alpha : \prod_{\alpha \in I} (X_\alpha, u) \to (X_\alpha, u),
\]

\(\alpha \in I, \) i.e., is defined by

\[
uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)
\]

for each \(A \subseteq \prod_{\alpha \in I} X_\alpha\).

Clearly, if \(\{(X_\alpha, u_\alpha) : \alpha \in I\}\) is a family of closure spaces, then the projection map \(\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta)\) is closed and continuous for every \(\beta \in I\).

**Proposition 1.** Let \(\{(X_\alpha, u_\alpha) : \alpha \in I\}\) be a family of closure spaces and let \(\beta \in I\). Then \(F\) is a closed subset of \((X_\beta, u_\beta)\) if and only if \(F \times \prod_{\alpha \in I, \alpha \neq \beta} X_\alpha\) is a closed subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\).

**Proof.** Let \(F\) be a closed subset of \((X_\beta, u_\beta)\). Since \(\pi_\beta\) is continuous, \(\pi_\beta^{-1}(F)\) is a closed subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\). But

\[
\pi_\beta^{-1}(F) = F \times \prod_{\alpha \neq \beta} X_\alpha,
\]

hence \(F \times \prod_{\alpha \neq \beta} X_\alpha\) is a closed subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\).
Conversely, let \( F \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \) be a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). Since \( \pi_\beta \) is closed,

\[
\pi_\beta \left( F \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \right) = F
\]
is a closed subset of \( (X_\beta, u_\beta) \). \( \square \)

The following statement is evident:

**Proposition 2.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I \} \) be a family of closure spaces and let \( \beta \in I \). Then \( G \) is an open subset of \( (X_\beta, u_\beta) \) if and only if \( G \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \) is an open subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

**Definition 1.** Let \( (X, u) \) be a closure space. A subset \( A \subseteq X \) is called a generalized closed set, briefly a \( g \)-closed set, if \( uA \subseteq \overline{A} \) whenever \( G \) is an open subset of \( (X, u) \) with \( A \subseteq G \). A subset \( A \subseteq X \) is called a generalized open set, briefly a \( g \)-open set, if its complement is \( g \)-closed.

**Proposition 3.** Let \( (X, u) \) be a closure space. A set \( A \subseteq X \) is \( g \)-open if and only if \( F \subseteq X - u(X - A) \) whenever \( F \) is a closed subset of \( (X, u) \) with \( F \subseteq A \).

**Proof.** Suppose that \( A \) is \( g \)-open and let \( F \subseteq A \) be a closed subset of \( (X, u) \). Then \( X - A \subseteq X - F \). But \( X - A \) is \( g \)-closed and \( X - F \) is open. It follows that \( u(X - A) \subseteq X - F \) and hence \( F \subseteq X - u(X - A) \).

Conversely, let \( X - A \subseteq G \) where \( G \) is open. Then \( X - G \subseteq A \). Since \( X - G \) is closed, \( X - G \subseteq X - u(X - A) \). Therefore, \( u(X - A) \subseteq G \). Hence, \( X - A \) is \( g \)-closed and so \( A \) is \( g \)-open. \( \square \)

**Proposition 4.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in I \} \) be a family of closure spaces and let \( \beta \in I \). Then \( G \) is a \( g \)-open subset of \( (X_\beta, u_\beta) \) if and only if \( G \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \) is a \( g \)-open subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

**Proof.** Let \( F \) be a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) such that \( F \subseteq G \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \). Then \( \pi_\beta (F) \subseteq G \). Since \( \pi_\beta (F) \) is closed and \( G \) is \( g \)-open in \( (X_\beta, u_\beta) \),

\[
\pi_\beta (F) \subseteq X_\beta - u_\beta (X_\beta - G).
\]

Therefore,

\[
F \subseteq \pi_\beta^{-1}(X_\beta - u_\beta (X_\beta - G)) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha \pi_\alpha \left( \prod_{\alpha \in I} X_\alpha - G \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \right).
\]

By Proposition 4, \( G \times \prod_{\alpha \neq \beta \atop \alpha \in I} X_\alpha \) is a \( g \)-open subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).
Conversely, let $F$ be a closed subset of $(X_\beta, u_\beta)$ such that $F \subseteq G$. Then

$$F \times \prod_{\alpha \neq \beta} X_\alpha \subseteq G \times \prod_{\alpha \neq \beta} X_\alpha.$$ 

Since $F \times \prod_{\alpha \neq \beta} X_\alpha$ is closed and $G \times \prod_{\alpha \neq \beta} X_\alpha$ is g-open in $\prod_{\alpha \neq \beta} (X_\alpha, u_\alpha)$,

$$F \times \prod_{\alpha \neq \beta} X_\alpha \subseteq \prod_{\alpha \neq \beta} X_\alpha - \prod_{\alpha \neq \beta} u_\alpha \pi_\alpha \left( \prod_{\alpha \neq \beta} X_\alpha - G \times \prod_{\alpha \neq \beta} X_\alpha \right)$$

by Proposition 4. Therefore,

$$\prod_{\alpha \neq \beta} u_\alpha \pi_\alpha \left( (X_\beta - G) \times \prod_{\alpha \neq \beta} X_\alpha \right) \subseteq \prod_{\alpha \neq \beta} X_\alpha - F \times \prod_{\alpha \neq \beta} X_\alpha = (X_\beta - F) \times \prod_{\alpha \neq \beta} X_\alpha.$$ 

Consequently, $u_\beta (X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - u_\beta (X_\beta - G)$. Hence, $G$ is a g-open subset of $(X_\beta, u_\beta)$. \hfill \Box

**Proposition 5.** Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then $F$ is a g-closed subset of $(X_\beta, u_\beta)$ if and only if $F \times \prod_{\alpha \neq \beta} X_\alpha$ is a g-closed subset of $\prod_{\alpha \neq \beta} (X_\alpha, u_\alpha)$.

**Proof.** Let $F$ be a g-closed subset of $(X_\beta, u_\beta)$. Then $X_\beta - F$ is a g-open subset of $(X_\beta, u_\beta)$. By Proposition 5,

$$(X_\beta - F) \times \prod_{\alpha \neq \beta} X_\alpha = \prod_{\alpha \neq \beta} X_\alpha - F \times \prod_{\alpha \neq \beta} X_\alpha$$

is a g-open subset of $\prod_{\alpha \neq \beta} (X_\alpha, u_\alpha)$. Hence, $F \times \prod_{\alpha \neq \beta} X_\alpha$ is a g-closed subset of $\prod_{\alpha \neq \beta} (X_\alpha, u_\alpha)$.

Conversely, let $G$ be an open subset of $(X_\beta, u_\beta)$ such that $F \subseteq G$. Then

$$F \times \prod_{\alpha \neq \beta} X_\alpha \subseteq G \times \prod_{\alpha \neq \beta} X_\alpha.$$ 

Since $F \times \prod_{\alpha \neq \beta} X_\alpha$ is g-closed and $G \times \prod_{\alpha \neq \beta} X_\alpha$ is open in $\prod_{\alpha \neq \beta} (X_\alpha, u_\alpha)$,

$$\prod_{\alpha \neq \beta} u_\alpha \pi_\alpha \left( F \times \prod_{\alpha \neq \beta} X_\alpha \right) \subseteq G \times \prod_{\alpha \neq \beta} X_\alpha.$$ 

Consequently, $u_\beta F \subseteq G$. Therefore, $F$ is a g-closed subset of $(X_\beta, u_\beta)$. \hfill \Box
Proposition 6. Let \( \{(X_\alpha, u_\alpha) : \alpha \in I\} \) be a family of closure spaces. For each \( \beta \in I \), let
\[
\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta)
\]
be the projection map. If \( F \) is a g-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \), then \( \pi_\beta(F) \) is a g-closed subset of \( (X_\beta, u_\beta) \).

Proof. Let \( F \) be a g-closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \) and let \( G \) be an open subset of \( (X_\beta, u_\beta) \) such that \( \pi_\beta(F) \subseteq G \). Then
\[
F \subseteq \pi_\beta^{-1}(G) = G \times \prod_{\alpha \in I, \alpha \neq \beta} X_\alpha.
\]
Since \( G \times \prod_{\alpha \in I, \alpha \neq \beta} X_\alpha \) is an open subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \),
\[
\prod_{\alpha \in I} u_\alpha \pi_\alpha(F) \subseteq G \times \prod_{\alpha \in I, \alpha \neq \beta} X_\alpha.
\]
Consequently, \( u_\beta \pi_\beta(F) \subseteq G \). Hence, \( \pi_\beta(F) \) is a g-closed subset of \( (X_\beta, u_\beta) \). \( \square \)

Definition 2. A biclosure space is a triple \( (X, u_1, u_2) \) where \( X \) is a set and \( u_1, u_2 \) are two closure operators on \( X \).

Definition 3. A subset \( A \) of a biclosure space \( (X, u_1, u_2) \) is called closed if \( u_1 u_2 A = A \) and it is open if its complement is closed.

Clearly, \( A \) is a closed subset of a biclosure space \( (X, u_1, u_2) \) if and only if \( A \) is both a closed subset of \( (X, u_1) \) and \( (X, u_2) \).

Let \( A \) be a closed subset of a biclosure space \( (X, u_1, u_2) \). The following conditions are equivalent

(i) \( u_2 u_1 A = A \),

(ii) \( u_1 A = A, u_2 A = A \).

Proposition 7. Let \( \{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\} \) be a family of biclosure spaces. Then \( F \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2) \) if and only if \( F \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha^1) \) and \( \prod_{\alpha \in I} (X_\alpha, u_\alpha^2) \).

Proof. Let \( F \) be a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2) \). Then
\[
F = \prod_{\alpha \in I} u_\alpha^1 \pi_\alpha \left( \prod_{\alpha \in I} u_\alpha^2 \pi_\alpha(F) \right).
\]
Since $F \subseteq \prod_{\alpha \in I} u^\alpha_{\pi\alpha}(F)$,
\[
\prod_{\alpha \in I} u^1_{\pi\alpha}(F) \subseteq \prod_{\alpha \in I} u^1_{\pi\alpha}\left(\prod_{\alpha \in I} u^2_{\pi\alpha}(F)\right) = F.
\]
Hence, $F$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$. Since $\prod_{\alpha \in I} u^2_{\pi\alpha}(F) \subseteq \prod_{\alpha \in I} u^2_{\pi\alpha}(F)$,
\[
\prod_{\alpha \in I} u^2_{\pi\alpha}(F) \subseteq \prod_{\alpha \in I} u^1_{\pi\alpha}\left(\prod_{\alpha \in I} u^2_{\pi\alpha}(F)\right) = F.
\]
Therefore, $F$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)$.

Conversely, let $F$ be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$ and $\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)$. Then $F = \prod_{\alpha \in I} u^1_{\pi\alpha}(F)$ and $F = \prod_{\alpha \in I} u^2_{\pi\alpha}(F)$. Consequently,
\[
F = \prod_{\alpha \in I} u^1_{\pi\alpha}\left(\prod_{\alpha \in I} u^2_{\pi\alpha}(F)\right).
\]
Hence, $F$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. $\square$

**Proposition 8.** Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then $F$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

**Proof.** Let $F$ be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F$ is a closed subset of $(X_\beta, u^1_\beta)$ and $(X_\beta, u^2_\beta)$, respectively. Therefore, $F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$ and $\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)$. Consequently, $F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F \times \prod_{\alpha \in I} X_\alpha$ be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$ and $\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)$, respectively. Hence, $F$ is a closed subset of $(X_\beta, u^1_\beta)$ and $(X_\beta, u^2_\beta)$. Consequently, $F$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. $\square$

**Definition 4.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be biclosure spaces and let $i \in \{1, 2\}$. A map
\[
f : (X, u_1, u_2) \to (Y, v_1, v_2)
\]
is called $i$-continuous if the map $f : (X, u_i) \to (Y, v_i)$ is continuous. A map $f$ is called continuous if $f$ is $i$-continuous for each $i \in \{1, 2\}$. 
Definition 5. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces and let \(i \in \{1, 2\}\). A map 
\[ f : (X, u_1, u_2) \to (Y, v_1, v_2) \]
is called \(i\)-closed (resp. \(i\)-open) if the map \(f : (X, u_i) \to (Y, v_i)\) is closed (resp. open). A map \(f\) is called closed (resp. open) if \(f\) is \(i\)-closed (resp. \(i\)-open) for each \(i \in \{1, 2\}\).

3 \(\partial\)-Closed Sets

In this section, we introduce a new class of \(\partial\)-closed sets in biclosure spaces and study some of its properties.

Definition 6. A subset \(A\) of a biclosure space \((X, u_1, u_2)\) is called \(\partial\)-closed if \(u_2 A \subseteq G\) whenever \(G\) is a g-open subset of \((X, u_1)\) with \(A \subseteq G\). The complement of a \(\partial\)-closed set is called \(\partial\)-open.

Remark 1. For a subset \(A\) of a biclosure space \((X, u_1, u_2)\), the following implications hold:

\[ A \text{ is closed} \Rightarrow A \text{ is } \partial\text{-closed} \]

The implication is not reversible as shown by the following example.

Example 1. Let \(X = \{a, b\}\) and define a closure operator \(u_1\) on \(X\) by 
\[ u_1 \emptyset = \emptyset, \quad u_1 \{a\} = u_1 \{b\} = u_1 X = X. \]
Define a closure operator \(u_2\) on \(X\) by 
\[ u_2 \emptyset = \emptyset, \quad u_2 \{a\} = \{a\}, \quad u_2 \{b\} = u_2 X = X. \]
Then \(\{a\}\) is \(\partial\)-closed but it is not closed.

Proposition 9. Let \((X, u_1, u_2)\) be a biclosure space and let \(u_2\) be additive. If \(A\) and \(B\) are \(\partial\)-closed subsets of \((X, u_1, u_2)\), then \(A \cup B\) is \(\partial\)-closed.

Proof. Let \(U\) be a g-open subset of \((X, u_1)\) such that \(A \cup B \subseteq U\). Then \(A \subseteq U\) and \(B \subseteq U\). Since \(A\) and \(B\) are \(\partial\)-closed, \(u_2 A \subseteq U\) and \(u_2 B \subseteq U\). Since \(u_2\) is additive,
\[ u_2 (A \cup B) = u_2 A \cup u_2 B \subseteq U. \]
Hence, \(A \cup B\) is \(\partial\)-closed. \(\square\)

Proposition 10. Let \((X, u_1, u_2)\) be a biclosure space and let \(u_2\) be idempotent. If \(A\) is a \(\partial\)-closed subset and \(A \subseteq B \subseteq u_2 A\), then \(B\) is \(\partial\)-closed.

Proof. Let \(G\) be a g-open subset of \((X, u_1)\) such that \(B \subseteq G\). Then \(A \subseteq G\). Since \(A\) is \(\partial\)-closed, \(u_2 A \subseteq G\). Since \(u_2\) is idempotent,
\[ u_2 B \subseteq u_2 u_2 A = u_2 A \subseteq G. \]
Hence, \(B\) is \(\partial\)-closed. \(\square\)
Proposition 11. Let \((X, u_1, u_2)\) be a biclosure space and let \(A \subseteq X\). If \(A\) is \(\partial\)-closed, then \(u_2 A - A\) has no nonempty g-closed subset of \((X, u_1)\).

Proof. Let \(F\) be a g-closed subset of \((X, u_1)\) such that \(F \subseteq u_2 A - A\). Then \(A \subseteq X - F\). Since \(A\) is \(\partial\)-closed and \(X - F\) is a g-open subset of \((X, u_1)\),

\[
u_2 A \subseteq X - F.
\]

Hence, \(F \subseteq X - u_2 A\). Consequently,

\[
F \subseteq (X - u_2 A) \cap u_2 A = \emptyset.
\]

Therefore, \(F = \emptyset\). \(\square\)

Proposition 12. Let \((X, u_1, u_2)\) be a biclosure space and let \(A \subseteq X\). Then \(A\) is \(\partial\)-open if and only if

\[
F \subseteq X - u_2(X - A)
\]

for every \(F\) is a g-closed subset of \((X, u_1)\) with \(F \subseteq A\).

Proof. Assume that \(A\) is \(\partial\)-open and let \(F\) be a g-closed subset of \((X, u_1)\). Then \(X - A \subseteq X - F\). Since \(X - A\) is \(\partial\)-closed and \(X - F\) is g-open subset of \((X, u_1)\),

\[
u_2(X - A) \subseteq X - F.
\]

Hence \(F \subseteq X - u_2(X - A)\).

Conversely, let \(U\) be a g-open subset of \((X, u_1)\) such that \(X - A \subseteq U\). Then \(X - U \subseteq A\). Since \(X - U\) is g-closed subset of \((X, u_1)\),

\[
X - U \subseteq X - u_2(X - A).
\]

Consequently, \(u_2(X - A) \subseteq U\). Hence, \(X - A\) is \(\partial\)-closed and so \(A\) is \(\partial\)-open. \(\square\)

Proposition 13. Let \((X, u_1, u_2)\) be a biclosure space. If \(A \subseteq X\) is \(\partial\)-closed, then \(u_2 A - A\) is \(\partial\)-open.

Proof. Suppose that \(A\) is \(\partial\)-closed and let \(F\) be a g-closed subset of \((X, u_1)\) such that \(F \subseteq u_2 A - A\). By Proposition 11, \(F = \emptyset\) and hence

\[
F \subseteq X - u_2(X - (u_2 A - A)).
\]

By Proposition 12, \(u_2 A - A\) is \(\partial\)-open. \(\square\)

Proposition 14. Let \(\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}\) be a family of biclosure spaces and let \(\beta \in I\). Then \(G\) is a \(\partial\)-open subset of \((X_\beta, u_\beta^1, u_\beta^2)\) if and only if \(G \times \prod_{\alpha \neq \beta, \alpha \in I} X_\alpha\) is a \(\partial\)-open subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)\).
Proof. Let $F$ be a $g$-closed subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^1)$ such that $F \subseteq G \times \prod_{\alpha \in I} X_{\alpha}$. Then $\pi_{\beta}(F) \subseteq G$. Since $\pi_{\beta}(F)$ is $g$-closed in $(X_{\beta}, u_{\beta}^1)$,

$$\pi_{\beta}(F) \subseteq X_{\beta} - u_{\beta}^2(X_{\beta} - G).$$

Therefore,

$$F \subseteq \pi_{\beta}^{-1}(X_{\beta} - u_{\beta}^2(X_{\beta} - G)) = \prod_{\alpha \in I} X_{\alpha} - \prod_{\alpha \in I} u_{\alpha}^2 \pi_{\alpha}\left(\prod_{\alpha \in I} X_{\alpha} - G \times \prod_{\alpha \in I} X_{\alpha}\right).$$

By Proposition 12, $G \times \prod_{\alpha \in I} X_{\alpha}$ is a $\partial$-open subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2)$.

Conversely, let $F$ be a $g$-closed subset of $(X_{\beta}, u_{\beta}^1)$ such that $F \subseteq G$. Then

$$F \times \prod_{\alpha \in I} X_{\alpha} \subseteq G \times \prod_{\alpha \in I} X_{\alpha}.$$ 

Since $F \times \prod_{\alpha \in I} X_{\alpha}$ is $g$-closed and $G \times \prod_{\alpha \in I} X_{\alpha}$ is $\partial$-open in $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^1)$,

$$F \times \prod_{\alpha \in I} X_{\alpha} \subseteq \prod_{\alpha \in I} X_{\alpha} - \prod_{\alpha \in I} u_{\alpha}^2 \pi_{\alpha}\left(\prod_{\alpha \in I} X_{\alpha} - G \times \prod_{\alpha \in I} X_{\alpha}\right)$$

by Proposition 12. Therefore,

$$\prod_{\alpha \in I} u_{\alpha}^2 \pi_{\alpha}\left((X_{\beta} - G) \times \prod_{\alpha \in I} X_{\alpha}\right) \subseteq \prod_{\alpha \in I} X_{\alpha} - F \times \prod_{\alpha \in I} X_{\alpha} = (X_{\beta} - F) \times \prod_{\alpha \in I} X_{\alpha}.$$

Consequently,

$$u_{\beta}^2(X_{\beta} - G) \subseteq X_{\beta} - F$$

implies

$$F \subseteq X_{\beta} - u_{\beta}^2(X_{\beta} - G).$$

Hence, $G$ is a $\partial$-open subset of $(X_{\beta}, u_{\beta}^1, u_{\beta}^2)$. 

**Proposition 15.** Let $\{(X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then $F$ is a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^1, u_{\beta}^2)$ if and only if $F \times \prod_{\alpha \in I} X_{\alpha}$ is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2)$.

Proof. Let $F$ be a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^1, u_{\beta}^2)$. Then $X_{\beta} - F$ is a $\partial$-open subset of $(X_{\beta}, u_{\beta}^1, u_{\beta}^2)$. By Proposition 14,

$$(X_{\beta} - F) \times \prod_{\alpha \in I} X_{\alpha} = \prod_{\alpha \in I} X_{\alpha} - F \times \prod_{\alpha \in I} X_{\alpha}$$
is a $\partial$-open subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2})$. Hence, $F \times \prod_{\alpha \in I} X_{\alpha}$ is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2})$.

Conversely, let $G$ be a g-open subset of $(X_{\beta}, u_{\beta}^{1})$ such that $F \subseteq G$. Then

$$F \times \prod_{\alpha \in I \setminus \beta} X_{\alpha} \subseteq G \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}.$$ 

Since $F \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}$ is a $\partial$-closed and $G \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}$ is g-open in $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1})$, $\prod_{\alpha \in I} u_{\alpha}^{2} \pi_{\alpha} (F \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}) \subseteq G \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}$.

Consequently, $u_{\beta}^{2} F \subseteq G$. Therefore, $F$ is a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$. $\square$

**Proposition 16.** Let $\{ (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2}) : \alpha \in I \}$ be a family of biclosure spaces. For each $\beta \in I$, let

$$\pi_{\beta} : \prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2}) \to (X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$$

be the projection map. Then

(i) If $F$ is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2})$, then $\pi_{\beta}(F)$ is a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$.

(ii) If $F$ is a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$, then $\pi_{\beta}^{-1}(F)$ is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2})$.

**Proof.** (i) Let $F$ be a $\partial$-closed subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1}, u_{\alpha}^{2})$ and let $G$ be a g-open subset of $(X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$ such that $\pi_{\beta}(F) \subseteq G$. Then

$$F \subseteq \pi_{\beta}^{-1}(G) = G \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}.$$ 

Since $G \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}$ is a g-open subset of $\prod_{\alpha \in I} (X_{\alpha}, u_{\alpha}^{1})$,

$$\prod_{\alpha \in I} u_{\alpha}^{2} \pi_{\alpha}(F) \subseteq G \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}.$$ 

Consequently, $u_{\beta}^{2} \pi_{\beta}(F) \subseteq G$. Hence, $\pi_{\beta}(F)$ is a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$.

(ii) Let $F$ be a $\partial$-closed subset of $(X_{\beta}, u_{\beta}^{1}, u_{\beta}^{2})$. Then

$$\pi_{\beta}^{-1}(F) = F \times \prod_{\alpha \in I \setminus \beta} X_{\alpha}.$$
By Proposition 15, $F \times \prod_{\alpha \neq \beta} X_\alpha$ is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$. Hence, $
abla_{\beta}^{-1}(F)$ is a $\partial$-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

4 $\partial$-Continuous Maps

In this section, we introduce the concept of $\partial$-continuous maps by using $\partial$-closed sets. These maps are investigated and studied.

**Definition 7.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be biclosure spaces. A map $f: (X, u_1, u_2) \to (Y, v_1, v_2)$ is called $\partial$-closed (resp. $\partial$-open) if $f(F)$ is a $\partial$-closed (resp. $\partial$-open) subset of $(Y, v_1, v_2)$ for every closed (resp. open) subset of $(X, u_1, u_2)$.

**Proposition 17.** Let $(X, u_1, u_2)$, $(Y, v_1, v_2)$ and $(Z, w_1, w_2)$ be biclosure spaces. If $g \circ f: (X, u_1, u_2) \to (Z, w_1, w_2)$ is $\partial$-closed and $f: (X, u_1, u_2) \to (Y, v_1, v_2)$ is surjective and continuous, then $g: (Y, v_1, v_2) \to (Z, w_1, w_2)$ is $\partial$-closed.

**Proof.** Let $F$ be a closed subset of $(Y, v_1, v_2)$. Then $F$ is a closed subset of $(Y, v_1)$ and $(Y, v_2)$, respectively. Since $f$ is continuous, $f^{-1}(F)$ is a closed subset of $(X, u_1)$ and $(X, u_2)$, respectively. Consequently, $f^{-1}(F)$ is a closed subset of $(X, u_1, u_2)$. Since $g \circ f$ is $\partial$-closed and $f$ is surjective, $g \circ f(f^{-1}(F)) = g(F)$ is a $\partial$-closed subset of $(Z, w_1, w_2)$. Therefore, $g$ is $\partial$-closed.

**Definition 8.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be biclosure spaces. A map $f: (X, u_1, u_2) \to (Y, v_1, v_2)$ is called $\partial$-continuous if $f^{-1}(F)$ is a $\partial$-closed subset of $(X, u_1, u_2)$ for every closed subset $F$ of $(Y, v_1, v_2)$.

Clearly, it is easy to prove that $f: (X, u_1, u_2) \to (Y, v_1, v_2)$ is $\partial$-continuous if and only if $f^{-1}(G)$ is a $\partial$-open subset of $(X, u_1, u_2)$ for every open subset $G$ of $(Y, v_1, v_2)$.
Proposition 18. Let \((X, u_1, u_2), (Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. If
\[ g \circ f : (X, u_1, u_2) \rightarrow (Z, w_1, w_2) \]
is closed and
\[ g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2) \]
is injective and \(\partial\)-continuous, then
\[ f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \]
is \(\partial\)-closed.

Proof. Let \(F\) be a closed subset of \((X, u_1, u_2)\). Then \(F\) is a closed subset of \((X, u_1)\) and \((X, u_2)\), respectively. Since \(g \circ f\) is closed, \(g \circ f(F)\) is a closed subset of \((Z, w_1)\) and \((Z, w_2)\), respectively. Consequently, \(g \circ f(F)\) is a closed subset of \((Z, w_1, w_2)\). Since \(g\) is \(\partial\)-continuous and injective, 
\[ g^{-1}(g \circ f(F)) = f(F) \]
is a \(\partial\)-closed subset of \((Y, v_1, v_2)\). Therefore, \(f\) is \(\partial\)-closed. \(\square\)

Definition 9. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. A map
\[ f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \]
is called \(\partial\)-irresolute if \(f^{-1}(F)\) is a \(\partial\)-closed subset of \((X, u_1, u_2)\) for every \(\partial\)-closed subset \(F\) of \((Y, v_1, v_2)\).

Clearly, it is easy to prove that
\[ f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \]
is \(\partial\)-irresolute if and only if \(f^{-1}(G)\) is a \(\partial\)-open subset of \((X, u_1, u_2)\) for every \(\partial\)-open subset \(G\) of \((Y, v_1, v_2)\).

The following statement is obvious:

Proposition 19. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. If
\[ f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \]
is \(\partial\)-irresolute, then \(f\) is \(\partial\)-continuous.

The converse need not be true as can be seen from the following example.

Example 2. Let \(X = \{a, b\} = Y\) and define closure operators \(u_1\) and \(u_2\) on \(X\) by
\[
\begin{align*}
  u_1 \emptyset &= \emptyset, & u_1 \{a\} &= \{a\}, & u_1 \{b\} &= u_1 X = X, \\
  u_2 \emptyset &= \emptyset, & u_2 \{a\} &= u_2 \{b\} &= u_2 X = X.
\end{align*}
\]
Define closure operators \(v_1\) and \(v_2\) on \(Y\) by
\[
\begin{align*}
  v_1 \emptyset &= \emptyset, & v_1 \{b\} &= \{b\}, & v_1 \{a\} &= v_1 Y = Y, \\
  v_2 \emptyset &= \emptyset, & v_2 \{a\} &= v_2 \{b\} &= v_2 Y = Y.
\end{align*}
\]
Let

\[ f : (X, u_1, u_2) \to (Y, v_1, v_2) \]

be the identity map. Then \( f \) is \( \partial \)-continuous but it is not \( \partial \)-irresolute because \( \{b\} \) is a \( \partial \)-closed subset of \((Y, v_1, v_2)\) but \( f^{-1}(\{b\}) = \{b\} \) is not \( \partial \)-closed subset of \((X, u_1, u_2)\).

**Definition 10.** A biclosure space \((X, u_1, u_2)\) is called a \( T^*_2 \)-biclosure space if every \( \partial \)-closed subset of \((X, u_1, u_2)\) is a closed subset of \((X, u_2)\).

**Proposition 20.** Let \((X, u_1, u_2)\) be a biclosure space. Then \((X, u_1, u_2)\) is a \( T^*_2 \)-biclosure space if and only if every singleton subset of \( X \) is either a \( g \)-closed subset of \((X, u_1)\) or an open subset of \((X, u_2)\).

**Proof.** Let \( x \in X \) and suppose that \( \{x\} \) is not a \( g \)-closed subset of \((X, u_1)\). Then \( X - \{x\} \) is not a \( g \)-open subset of \((X, u_1)\). The only \( g \)-open subset of \((X, u_1)\) containing \( X - \{x\} \) is \( X \), hence \( X - \{x\} \) is a \( \partial \)-closed subset of \((X, u_1, u_2)\). Since \((X, u_1, u_2)\) is a \( T^*_2 \)-biclosure space, \( X - \{x\} \) is a closed subset of \((X, u_2)\). Consequently, \( \{x\} \) is an open subset of \((X, u_2)\).

Conversely, let \( A \) be a \( \partial \)-closed subset of \((X, u_1, u_2)\). Suppose that \( x \notin A \). Then \( \{x\} \subseteq X - A \) and we have \( A \subseteq X - \{x\} \). If \( \{x\} \) is an open subset of \((X, u_2)\), then \( X - \{x\} \) is a closed subset of \((X, u_2)\). Consequently,

\[ u_2 A \subseteq u_2(X - \{x\}) = X - \{x\}, \]

thus \( x \notin u_2 A \). If \( \{x\} \) is a \( g \)-closed subset of \((X, u_1)\), then \( X - \{x\} \) is a \( g \)-open subset of \((X, u_1)\). Since \( A \) is a \( \partial \)-closed, \( u_2 A \subseteq X - \{x\} \). Therefore, \( x \notin u_2 A \). So, we always have \( u_2 A \subseteq A \). Thus \( u_2 A = A \) or, equivalently, \( A \) is a closed subset of \((X, u_2)\). Therefore, \((X, u_1, u_2)\) is a \( T^*_2 \)-biclosure space. \( \square \)

**Proposition 21.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. Let

\[ f : (X, u_1, u_2) \to (Y, v_1, v_2) \]

be surjective, \( 2 \)-closed and \( \partial \)-irresolute. If \((X, u_1, u_2)\) is a \( T^*_2 \)-biclosure space, then \((Y, v_1, v_2)\) is a \( T^*_2 \)-biclosure space.

**Proof.** Let \( F \) be a \( \partial \)-closed subset of \((Y, v_1, v_2)\). Since \( f \) is \( \partial \)-irresolute, \( f^{-1}(F) \) is a \( \partial \)-closed subset of \((X, u_1, u_2)\). Since \((X, u_1, u_2)\) is a \( T^*_2 \)-biclosure space, \( f^{-1}(F) \) is a closed subset of \((X, u_2)\). Since \( f \) is \( 2 \)-closed and surjective, \( F \) is a closed subset of \((Y, v_2)\). Hence, \((Y, v_1, v_2)\) is a \( T^*_2 \)-biclosure space. \( \square \)

**Proposition 22.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. Let

\[ f : (X, u_1, u_2) \to (Y, v_1, v_2) \]

and

\[ g : (Y, v_1, v_2) \to (Z, w_1, w_2) \]

be maps. Then
(i) $g \circ f$ is $\partial$-continuous if $g$ is continuous and $f$ is $\partial$-continuous.

(ii) $g \circ f$ is $\partial$-irresolute if $f$ and $g$ are $\partial$-irresolute.

(iii) $g \circ f$ is $\partial$-continuous if $g$ is $\partial$-continuous and $f$ is $\partial$-irresolute.

Proof. (i) Let $F$ be a closed subset of $(Z, w_1, w_2)$. Then $F$ is a closed subset of $(Z, w_1)$ and $(Z, w_2)$, respectively. Since $g$ is continuous, $g^{-1}(F)$ is a closed subset of $(Y, v_1)$ and $(Y, v_2)$, respectively. Consequently, $g^{-1}(F)$ is closed subset of $(Y, v_1, v_2)$. Since $f$ is $\partial$-continuous, $f^{-1}(g^{-1}(F))$ is a $\partial$-closed subset of $(X, u_1, u_2)$. Therefore, $(g \circ f)^{-1}(F)$ is a $\partial$-closed subset of $(X, u_1, u_2)$. Hence, $g \circ f$ is $\partial$-continuous.

The proofs of (ii)–(iii) are similar. □

**Proposition 23.** Let $\{(X_\alpha, u_1^\alpha, u_2^\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map

$$\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_1^\alpha, u_2^\alpha) \to (X_\beta, u_1^\beta, u_2^\beta)$$

is continuous.

Proof. Let $A \subseteq \prod_{\alpha \in I} X_\alpha$. Then

$$\pi_\beta \left( \prod_{\alpha \in I} u_1^\alpha \pi_\alpha(A) \right) = u_1^\beta \pi_\beta(A).$$

Hence,

$$\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_1^\alpha) \to (X_\beta, u_1^\beta)$$

is continuous. Similarly, since

$$\pi_\beta \left( \prod_{\alpha \in I} u_2^\alpha \pi_\alpha(A) \right) = u_2^\beta \pi_\beta(A).$$

Therefore,

$$\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_2^\alpha) \to (X_\beta, u_2^\beta)$$

is continuous. Consequently,

$$\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_1^\alpha, u_2^\alpha) \to (X_\beta, u_1^\beta, u_2^\beta)$$

is continuous. □

**Proposition 24.** Let $(X, u_1, u_2)$ be a biclosure space and let $\{(Y_\alpha, v_1^\alpha, v_2^\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Let $f : X \to \prod_{\alpha \in I} Y_\alpha$ be a map. If

$$f : (X, u_1, u_2) \to \prod_{\alpha \in I} (Y_\alpha, v_1^\alpha, v_2^\alpha)$$
is $\partial$-continuous, then
\[ \pi_\alpha \circ f : (X, u_1, u_2) \to (Y_\alpha, v_1^\alpha, v_2^\alpha) \]
is $\partial$-continuous for each $\alpha \in I$.

**Proof.** Let $f$ be $\partial$-continuous. Since $\pi_\alpha$ is continuous for each $\alpha \in I$, also $\pi_\alpha \circ f$ is $\partial$-continuous for each $\alpha \in I$. $\square$

**Proposition 25.** Let \( \{(X_\alpha, u_1^\alpha, u_2^\alpha) : \alpha \in I\} \) and \( \{(Y_\alpha, v_1^\alpha, v_2^\alpha) : \alpha \in I\} \) be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha : (X_\alpha, u_1^\alpha, u_2^\alpha) \to (Y_\alpha, v_1^\alpha, v_2^\alpha)$ be a map and
\[
f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha
\]
be the map defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. If
\[
f : \prod_{\alpha \in I}(X_\alpha, u_1^\alpha, u_2^\alpha) \to \prod_{\alpha \in I}(Y_\alpha, v_1^\alpha, v_2^\alpha)
\]
is $\partial$-continuous, then
\[
f_\alpha : (X_\alpha, u_1^\alpha, u_2^\alpha) \to (Y_\alpha, v_1^\alpha, v_2^\alpha)
\]
is $\partial$-continuous for each $\alpha \in I$.

**Proof.** Let $F$ be a closed subset of $(Y_\beta, v_1^\beta, v_2^\beta)$. Then $F \times \prod_{\alpha \neq \beta} Y_\alpha$ is a closed subset of $\prod_{\alpha \in I}(Y_\alpha, v_1^\alpha, v_2^\alpha)$. Since $f$ is $\partial$-continuous,
\[
f^{-1}\left( F \times \prod_{\alpha \neq \beta} Y_\alpha \right) = f_\beta^{-1}(F) \times \prod_{\alpha \neq \beta} X_\alpha
\]
is a $\partial$-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_1^\alpha, u_2^\alpha)$. By Proposition 15, $f_\beta^{-1}(F)$ is a $\partial$-closed subset of $(X_\beta, u_1^\beta, u_2^\beta)$. Hence, $f_\beta$ is $\partial$-continuous. $\square$

**References**


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