

Víctor Ayala; José Ayala-Hoffmann; Ivan de Azevedo Tribuzy  
Controllability of invariant control systems at uniform time

*Kybernetika*, Vol. 45 (2009), No. 3, 405--416

Persistent URL: <http://dml.cz/dmlcz/140016>

## Terms of use:

© Institute of Information Theory and Automation AS CR, 2009

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

## CONTROLLABILITY OF INVARIANT CONTROL SYSTEMS AT UNIFORM TIME

VÍCTOR AYALA, JOSÉ AYALA-HOFFMANN AND IVAN DE AZEVEDO TRIBUZY

Let  $G$  be a compact and connected semisimple Lie group and  $\Sigma$  an invariant control systems on  $G$ . Our aim in this work is to give a new proof of Theorem 1 proved by Jurdjevic and Sussmann in [6]. Precisely, to find a positive time  $s_\Sigma$  such that the system turns out controllable at uniform time  $s_\Sigma$ . Our proof is different, elementary and the main argument comes directly from the definition of semisimple Lie group. The uniform time is not arbitrary. Finally, if  $A = \bigcap_{t>0} A(t, e)$  denotes the reachable set from arbitrary uniform time, we conjecture that it is possible to determine  $A$  as the intersection of the isotropy groups of orbits of  $G$ -representations which contains  $\exp(\mathfrak{z})$ , where  $\mathfrak{z}$  is the Lie algebra determined by the control vectors.

*Keywords:* uniform-time, compact, semisimple, reverse-system

*AMS Subject Classification:* 93B0512

### 1. INTRODUCTION

Let  $G$  be a compact and connected Lie group with Lie algebra  $\mathfrak{g}$ . An invariant control systems  $\Sigma = (G, \mathcal{D})$  as in Definition 2.2 is said to be controllable at uniform time, if there exists a positive time  $s_\Sigma$  such that for every couple of points  $x$  and  $y$  in  $G$ , there exists a control  $u = u(x, y)$  transferring  $x$  to  $y$  at exact time  $s_\Sigma$ .

In [6] the authors proved

**Theorem 1.1.** Let  $G$  be a connected, compact and semisimple Lie group and  $\Sigma$  an invariant control system on  $G$ . If  $\Sigma$  satisfy the Lie algebra rank condition, then  $\Sigma$  is controllable at uniform time.

The proof is based on the following topological results, see [5]:

**Theorem 1.2.** Let  $M$  be a manifold whose universal covering space is compact. Then, every system having the accessibility property has the strong accessibility property.

In this paper we give an alternative proof of Theorem 1.1. In fact, we show the strong accessibility property of  $\Sigma$  just by using the fact that any semisimple Lie algebra does not contains ideals of co-dimension 1, see [9] and also [12].

It turns out that to prove Theorem 1.1 is enough to show the existence of a time  $s_+$  such that, the accessibility set from the identity element of  $G$  at exact time  $s_+$  coincides with  $G$ .

Let  $H$  be the normal Lie subgroup of  $G$  with Lie algebra given by the ideal in  $\mathfrak{g}$

$$\mathfrak{h} = \text{ideal}_{\mathfrak{g}} \{Y^1, Y^2, \dots, Y^m\}$$

generated by the control vectors of the system. Since the algebra  $\mathfrak{g}$  does not contains ideals of co-dimension one, it follows that for every positive time  $t$ , the left translation of  $H$  by  $\exp(tX)$  coincides with  $G$ . Next, we show the existence of a positive time  $t_+$ , such that the identity element  $e$  belongs to the open set  $\text{int } A(t_+, e)$ , as defined in Section 3. From this fact, we find a time  $s_+$  such that any point of the manifold can be reached by the identity element in exact  $s_+$  units of time, i. e.,  $A(s_+, e) = G$ .

**Remark 1.3.** As Example 5.2 shows, Proposition 3.5 does not imply that you can reach any point of  $G$  from  $e$  in arbitrary time. But, in some particular cases the uniform time could be arbitrary. For instance, it happens in the homogeneous case, i. e., when the system does not has drift vector field, i. e.,  $X = 0$ . For a more general results in this direction see [7].

The uniform time  $s_{\Sigma}$  depends on the *order of generation* of  $G$ , which is defined as the minimum positive integer  $k$  such that every element in  $G$  can be expressed as  $k$  product of exponential. This notion is closed related to the concept of *uniformly completely controllable* set of left invariant vector fields on Lie groups, analyzed in [10], see also [11].

Finally, we apply the main results in Theorem 3.6 to a special class of homogeneous systems: the affine systems. As usual, we start with a bilinear control systems in  $\mathbb{R}^n$  in such a way that its associated Lie algebra generate a connected, compact and semisimple Lie group  $G$ . It follows that there exists a positive time  $s_{\Sigma}$  such that the affine system determined by the bilinear one is controllable at uniform time  $s_{\Sigma}$  on the orbit  $G(x_0) \subset \mathbb{R}^n$ , for any initial condition  $x_0 \in \mathbb{R}^n$ .

This article is organized as follows. Section 2 contains the definition of invariant control system  $\Sigma$  and the uniform time controllable notion. In Section 3 we recall some basic results of  $\Sigma$  and we prove the existence of a uniform time for semisimple Lie groups. In Section 4 we apply the main results to a special class of affine systems and Section 5 contains a number of examples.

## 2. PRELIMINARIES

Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ .

**Definition 2.1.** An invariant control system  $\Sigma = (G, \mathcal{D})$  is determined by a family  $\mathcal{D}$  of differential equations given by

$$\mathcal{D} = \left\{ X^u = X + \sum_{j=1}^m u_j Y^j : u \in \mathcal{U} \right\}.$$

The drift vector field  $X$  and the control vectors  $Y^j, j = 1, 2, \dots, m$ , are elements of  $\mathfrak{g}$  considered as left invariant vector fields. The admissible control functions are elements in the class

$$\mathcal{U}_K = \{ u : \mathbb{R} \rightarrow K \subset \mathbb{R}^m \mid u(t) \text{ is a piecewise constant function} \}.$$

Here,  $K = \mathbb{R}^m$ : the unrestricted controls, or the cube  $[-1, 1]^m$ : the bounded controls, or the boundary set  $\partial[-1, 1]^m$ : the bang-bang controls.

Except for explicitly mention, any results will be independent of the particular type of the admissible class of control.

For each  $u \in \mathcal{U}$  and for any initial condition  $x \in G$ , the ordinary differential equation determined by  $X^u$  has an unique global solution  $X_t^u(x), t \in \mathbb{R}$ , with  $X_0^u(x) = x$ , where  $(X_t^u)_{t \in \mathbb{R}}$  is the 1-parameter group of  $X^u$ .

We assume that  $\Sigma$  satisfies the Lie algebra rank condition (*LARC*), i. e.,

$$\text{Span}_{\mathcal{L}\mathcal{A}}(\mathcal{D})(x) = T_x G, \text{ for any } x \in G.$$

Here,  $[\cdot, \cdot]$  denotes the usual Lie brackets between vector fields and  $\text{Span}_{\mathcal{L}\mathcal{A}}(\mathcal{D})$  is the Lie algebra generated by the vector fields in  $\mathcal{D}$ , i. e., the real vector space generated by  $X, Y^j, j = 1, \dots, m$ , and closed by the Lie brackets. The *LARC* property imposed just on the identity element  $e$  of  $G$ , means that the finite increasing sequence of subspaces  $\mathcal{D} + [\mathcal{D}, \mathcal{D}] + [\mathcal{D}, [\mathcal{D}, \mathcal{D}]] + \dots$  coincides with  $\mathfrak{g}$ , which implies the *LARC* condition.

Furthermore, if  $u$  is a constant control the invariance property of  $X^u$  implies that  $X_t^u(x) = x \cdot X_t^u(e)$ . The invariant control system  $-\Sigma$  is defined as the system  $\Sigma$  but replacing the drift vector  $X$  by  $-X$ . Actually, in any case the set  $K$  in Definition 2.1 is symmetric.

**Definition 2.2.** A control system  $\Sigma = (G, \mathcal{D})$  is said to be

- a) *Transitive* if for any two arbitrary points  $x, y \in G$  there exists an admissible control  $u = u(x, y) \in \mathcal{U}$  and a time  $t = t(x, y) \in \mathbb{R}$  such that  $X_t^u(x) = y$
- b) *Controllable* if for any two arbitrary points  $x, y \in G$  there exists an admissible control  $u \in \mathcal{U}$  and a time  $t = t(x, y) \geq 0$  such that  $X_t^u(x) = y$
- c) *Controllable at uniform time*  $t > 0$ , if for any two arbitrary points  $x, y \in G$  there exists an admissible control  $u = u(x, y) \in \mathcal{U}$  such that  $X_t^u(x) = y$ .

Our aim in this work is to give a new proof of Theorem 1, i. e., to find a positive time  $s_\Sigma$  such that the system turns out controllable at uniform time  $s_\Sigma$ .

3. THE EXISTENCE OF AN UNIFORM TIME FOR  $\Sigma$

In the sequel  $G$  will denote a compact and connected Lie group. It is well known that under the LARC condition, the analytic control system  $\Sigma = (G, \mathcal{D})$  is transitive. Since the system is invariant for the  $G$ -group action, it turns out that  $\Sigma$  is also controllable [6], which of course is a necessary condition for our study. We denote by

$$\mathfrak{h} = \text{ideal}_{\mathfrak{g}} \{Y^1, Y^2, \dots, Y^m\}$$

the ideal in  $\mathfrak{g}$  generated by the control vectors and by  $H$  the connected Lie subgroup of  $G$  associated to the Lie algebra  $\mathfrak{h}$ . Since  $\mathfrak{h}$  is an ideal, it follows that  $H$  is a normal subgroup of  $G$ , [4].

Let us consider a positive time  $t$ . The  $\Sigma$  accessibility set from the identity element  $e$  of  $G$  at exact time  $t$  is denoted by

$$A(t, e) = \{X_t^u(x) : u \in U\}.$$

In the sequel, we follow reference [6]

**Proposition 3.1.** Let  $\Sigma$  be a left invariant control system on a compact and connected Lie group  $G$ . Therefore, for each  $t > 0$

i) With respect to the  $H$ -topology,

$$\text{cl}(\text{int}_{\exp(tX)H} A(t, e)) = \text{cl}(A(t, e))$$

ii)  $A(t, e) \subset \exp(tX)H$ .

In other words,  $A(t, e)$  is contained in a  $G$ -submanifold of co-dimension 0 or 1, depending on the fact  $X \in \mathfrak{h}$  or not. In particular,  $A(t, e)$  has non empty interior in this submanifold. Just observe that the relation  $\mathfrak{h} = \mathfrak{g}$  could be possible.

**Example 3.2.** Let us consider the Torus  $T^2 = S^1 \times S^1$  and the invariant control system  $\Sigma = (T^2, \mathcal{D})$  with

$$\mathcal{D} = \{X^u = X + uY : u \in \mathcal{U}\}.$$

Here,  $X = (\frac{\partial}{\partial x}, 0)$  and  $Y = (0, \frac{\partial}{\partial y})$ . The non null component of  $X$  and  $Y$  are invariant vector fields on the sphere  $S^1$ . Since  $T$  is commutative,  $\mathfrak{h}$  has dimension 1. So, the subgroup  $H$  has co-dimension 1 and for each positive time  $t$ , the accessibly set  $A(t, e)$  at exact time  $t$  is contained in the one dimensional submanifold  $\exp(tX)H$  of  $T^2$ . Therefore, in this situation, we can not expect uniform controllability. From the general theory of Lie groups we know that any Abelian Lie group  $G$  has the form

$$G = T^n \times \mathbb{R}^m$$

for some non negative integers numbers  $n, m$ . Since we assume  $G$  compact, Example 3.2 shows that we can not expect the uniform time property for the Abelian case.

**Example 3.3.** On the rotational group  $G = \text{SO}(3)$  consider the system

$$\mathcal{D} = \{X^u = X + uY : u \in \mathcal{U}\}.$$

where  $X$  and  $Y$  are any two arbitrary linear independent skewsymmetric matrices. Then  $\mathfrak{h} = \mathfrak{g}$ .

**Remark 3.4.** According to Proposition 3.1 and the previous examples, it is natural to concentrate our study on invariant control systems  $\Sigma$ , when the drift vector field  $X$  belongs to the ideal  $\mathfrak{h}$ . We focus the problem on semisimple Lie groups.

Let  $\mathfrak{C}$  be any classical simple Lie algebra over  $\mathbb{C}$  and consider a compact real form  $\mathfrak{g}$  of  $\mathfrak{C}$ , that is,  $\mathfrak{g}$  is a real Lie algebra with  $\mathfrak{C} = \mathfrak{g} + i\mathfrak{g}$ . Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . According to the classification of Lie groups and Lie algebras, we have: up to isomorphism,  $G$  must be the Special Orthogonal group

$$\text{SO}(n) = \{A \in M(n, \mathbb{R}) : A^T A = \text{Id and } \det A = 1\}$$

or the Unitary group

$$U(n) = \left\{ A \in M(n, \mathbb{C}) : \bar{A}^T A = \text{Id} \right\}$$

or the Spinors group, which is defined by the short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1$$

as the double cover of the special orthogonal group.

By definition, a semisimple Lie algebra is a direct sum of simple Lie algebras. Therefore, a semisimple Lie group is a finite product of simple Lie groups. Since the semisimple Lie algebra  $\mathfrak{g}$  does not contains ideals of co-dimension 1, for every positive time  $t$  we get

$$\exp(tX)H = G.$$

Under this hypothesis, we have the following

**Proposition 3.5.** Let  $G$  be a compact and connected semisimple Lie group. Consider an invariant control system  $\Sigma = (G, \mathcal{D})$  which satisfies LARC. Then, there exists a positive time  $s_+$  such that,

$$G = A(s_+, e).$$

*Proof.* We consider the invariant control systems  $\Sigma$  and  $-\Sigma$ . Since the Lie group  $G$  is semisimple, Proposition 3.1 implies that for each positive time  $t$ ,

$$\text{cl}(\text{int}_G A(t, e)) = \text{cl}(A(t, e)) \quad \text{and} \quad \text{cl}(\text{int}_G A_-(t, e)) = \text{cl}(A_-(t, e)).$$

where  $A_-(t, e)$  denotes the accessibility set of  $-\Sigma$  at the exact time  $t$ . On the other hand, since both systems are controllable we get

$$\bigcup_{t \geq 0} A(t, e) = G = \bigcup_{s \geq 0} A_-(s, e).$$

It turns out that there are times  $t, s > 0$  and a point  $x \in G$  with

$$x \in \text{int}_G A(t, e) \cap \text{int}_G A_-(s, e).$$

We claim:

$$e \in \text{int} A(t + s, e).$$

In fact, consider the control  $u = u(e, x)$  steering  $e$  to the state  $x$  at time  $t$  by a trajectory of  $\Sigma$  and the control  $v = v(e, x)$  steering the identity element to  $x$  at time  $s$  by a trajectory of  $-\Sigma$ . The concatenation  $w = v * u$  steering  $e$  to  $e$  by a  $\Sigma$ -trajectory at time  $t_+ = t + s$ , i.e.,  $e \in A(t_+, e)$ . Since  $x \in \text{int}_G A(t, e)$  the  $\Sigma$ -diffeomorphism  $X_s^v$  induced by the control  $v$  at time  $s$  shows the claim.

Next, take any neighborhood  $V$  of  $e$  such that  $V \subset A(t_+, e)$ . Since  $G$  is connected and  $A(e) = \bigcup_{t \geq 0} A(t, e)$  is a semigroup, [6], there exists a positive integer number  $k$  such that

$$G = V^k \subset A(t_+, e)^k \subset A(kt_+, e).$$

The proof is complete by taking  $s_+ = kt_+$ . □

We are able to prove the main results of the paper

**Theorem 3.6.** Let  $G$  be a compact and connected semisimple Lie group and  $\Sigma = (G, \mathcal{D})$  an invariant control system on  $G$  which satisfies LARC. Then, there exists a positive time  $s_\Sigma$  such that  $\Sigma$  is controllable at uniform time  $s_\Sigma$ .

*Proof.* In Proposition 3.5 we already proved the existence of a time  $s_+$  for  $\Sigma$  such that starting from  $e$  it is possible to reach any point of  $G$  at exact time  $s_+$ . Now, we consider the invariant transitive control system  $-\Sigma$  on  $G$ . Again from Proposition 3.5, there exists a positive time  $s_-$  such that through  $-\Sigma$  it is possible to reach  $e$  from any point of the Lie group  $G$  at the exact time  $s_-$ . Take

$$s_\Sigma = s_+ + s_-$$

Therefore, any two arbitrary points of  $G$  can be connected in exactly time  $s_\Sigma$  and  $\Sigma$  is controllable at uniform time. □

**Remark 3.7.** The proof of Proposition 3.5 depends on the existence of a state  $x$  in the interior of an accessible set of  $\Sigma$  and  $-\Sigma$  simultaneously. In particular, the proof doesn't show that you can reach any point of  $G$  from  $e$  in arbitrary time. Furthermore, the uniform time  $s_\Sigma$  depends on the integer number  $k$ , which depends on a neighborhood  $V$  inside of  $A(t_+, e)$ . But, the uniform time could be arbitrary.

For instance, this is the situation when the Lie algebra generated by the control vectors

$$\mathfrak{z} = \text{Span}_{\mathcal{L}\mathcal{A}} \{Y^1, Y^2, \dots, Y^m\}$$

coincides with the Lie algebra of  $G$ , i. e.,  $X \in \mathfrak{z} = \mathfrak{g}$ . In fact, in this special case and when the system consider the class of unrestricted control  $\mathcal{U}_{\mathbb{R}^m}$ , we have

$$A(t, e) = G, \text{ for every } t > 0$$

see [7]. In a more general set up, assume that the set

$$A = \bigcap_{t>0} A(t, e)$$

is not empty. Then, in [3], the author proves that  $A$  is a closed and connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{a}$  such that  $\mathfrak{z} \subset \mathfrak{a} \subset \mathfrak{g}$ . Of course,

$$A = G \Leftrightarrow \text{the time } s_\Sigma \text{ given by Theorem 3.6 is arbitrary.}$$

As indicated by an anonymous referee, the property of uniform time at arbitrary positive  $t$  is strictly related to the condition

$$\text{int}_G A(t, e) \cap \text{int}_G A_-(t, e) \neq \emptyset, \text{ for each } t > 0.$$

Unfortunately, we don't know any algebraic or geometric property on  $G$  or on the system, in order to obtain this condition. However, by following Example 5.2 we conjecture later that  $A$  is the intersection of the isotropy groups of orbits of  $G$ -representations which contains  $\exp(\mathfrak{z})$ .

**Remark 3.8.** Consider the number

$$s_* = \inf_s \{s > 0 : \exists V \subset A(s, e), V \text{ a neighborhood of } e \in \text{int}A(s, e)\}.$$

Except the case  $A = G$ , the time  $s_*$  is positive and this number is related to the notion of *order of generation* of  $G$ , which is defined as the minimum positive integer  $k$  such that every element in  $G$  can be expressed as  $k$  product of exponential and also to the concept of *uniformly completely controllable*, see [10], and [11].

#### 4. AFFINE SYSTEMS

In this section we apply Theorem 3.6 to a special class of homogeneous systems: the affine systems. Definition 2.1 consider the notion of invariant control systems by using left invariant vector fields. Obviously, everything remains true if we consider the same class of systems but with right invariant vector fields.

Let  $M$  be a differential manifold and let  $\Sigma = (G, \mathcal{D})$  a right invariant control system on the connected Lie group  $G$ , which acts transitively on  $M$  by

$$p : G \times M \rightarrow M.$$

The dynamic set  $\mathcal{D} \subset \mathfrak{g}$  induces the non linear homogeneous system

$$p_*(\Sigma) = (M, p_*(\mathcal{D}))$$

where  $p_*$  denotes the differential of  $p$ . We have, [8]

**Theorem 4.1.** If the system  $\Sigma$  is controllable on  $G$ , then  $p_*(\Sigma)$  is controllable on  $M$ . Furthermore, for every positive time  $s$

$$A_\Sigma(s, x) = G \Rightarrow A_{p_*(\Sigma)}(s, p(x)) = M$$

Therefore, we get

**Theorem 4.2.** Let  $G$  be a compact and connected semisimple Lie group acting transitively on  $M$ . Consider the right-invariant control system  $\Sigma = (G, \mathcal{D})$ . Hence, there exists a positive time  $s_{p_*(\Sigma)}$  such that the affine system

$$p_*(\Sigma) = (M, p_*(\mathcal{D}))$$

is controllable at uniform time  $s_{p_*(\Sigma)}$ .

*Proof.* By Theorem 3.6,  $\Sigma$  is controllable at uniform time  $s_\Sigma$ . The proof follows from Theorem 4.1 and the fact  $s_\Sigma = s_{p_*(\Sigma)}$ . □

As a generic example, consider the Bilinear Control System  $\Sigma_B$  on  $\mathbb{R}^n$

$$\dot{x}(t) = \left[ A_0 + \sum_{i=1}^m u_i(t)A_i \right] x(t)$$

determined by the matrices  $A_i \in \mathfrak{gl}(n, \mathbb{R})$ ,  $i = 0, 1, \dots, m$ , and with  $u \in \mathcal{U}$  as in Definition 2.1. Let  $G$  be the connected Lie group with Lie subalgebra

$$\mathfrak{g} = \text{Span}_{\mathcal{L}\mathcal{A}} \{A_0, A_1, \dots, A_m\} \subset \mathfrak{gl}(n, \mathbb{R}).$$

Therefore, the control system

$$\dot{X}(t) = \left[ A_0 + \sum_{i=1}^m u_i(t)A_i \right] X(t), X \in G$$

induced by  $\Sigma_B$  is right invariant on  $G$ . Assume that  $G$  is compact and semisimple. Thus, the affine system

$$\dot{x}(t) = \left[ A_0 + \sum_{i=1}^m u_i(t)A_i \right] x(t), x(0) = x_0$$

defined on the orbit

$$G(x_0) = \{x \in \mathbb{R}^n : x = gx_0 \text{ with } g \in G\}$$

is controllable at uniform time on  $G(x_0)$ .

### 5. EXAMPLES

**Example 5.1.** Let us consider the bilinear control system

$$\dot{x}(t) = [A_0 + u(t)A_1]x(t)$$

where  $A_0$  and  $A_1$  generate the Lie algebra

$$\mathfrak{g} = \text{Span}_{\mathcal{L}\mathcal{A}} \{A_0, A_1\} = \mathfrak{so}(n, \mathbb{R})$$

of the skewsymmetric matrices of order  $n$ . The associated compact Lie group is  $G = \text{SO}(n)$ . Recall that for each natural number  $n$  the group  $\text{SO}(n)$  is simple, except the semisimple case  $\text{SO}(4)$ . It turns out that the affine system is controllable at uniform time on the manifold  $M = S^{n-1}$ .

For a concrete general example, consider  $A_0 = \sum_{i=1}^{n-2} E_{i,i+1} - E_{i+1,i}$  and  $A_1 = E_{n-1,n} - E_{n,n-1}$ . Here the matrices  $E_{i,j}$   $1 \leq i, j \leq n$  are the canonical basis elements of  $\mathfrak{gl}(n, \mathbb{R})$ . Then,  $A_0$  and  $A_1$  generates  $\mathfrak{so}(n, \mathbb{R})$ .

The example below shows that the uniform time given in Theorem 3.6 is not arbitrary.

**Example 5.2.** On  $\mathbb{R}^3$  consider the bilinear control system  $\Sigma_B$  :

$$\dot{x}(t) = [X + u(t)Y]x(t),$$

with  $X = -E_{1,3} + E_{3,1}$  and  $Y = E_{1,2} - E_{2,1}$  and  $u \in \mathcal{U}$ .

As Theorem 3.6 shows, the lifting invariant control system  $\Sigma$  determined by  $\Sigma_B = p_*(\Sigma)$ , is controllable at uniform time on  $\text{SO}(3)$ . So, Theorem 4.1 implies that the bilinear control system is controllable at uniform time on  $S^2$  considered, for instance, as the orbit of  $G$  on  $e_2 = (0, 1, 0)$ .

As before we denote  $A = \bigcap_{t>0} A(t, e)$ . Consider the class of unrestricted admissible controls  $\mathcal{U}_{\mathbb{R}}$ . We claim:

*the subset  $A(e_2) \subset S^2$  where any two points can be reached by any arbitrary uniform positive time, is reduced to the Equator circle.*

Consider the function

$$h : S^2 \rightarrow \mathbb{R}, \text{ given by } h(x) = \langle e_3, x \rangle$$

on the sphere. Here  $\langle \cdot, \cdot \rangle$  denotes the usual inner product of  $\mathbb{R}^3$  and  $\beta(t) = h(x(t))$ , where  $x(t)$  is a solution of  $\Sigma_B$  with control  $u$ . We have,

$$\dot{\beta}(t) = (dh)_{x(t)}(Xx(t) + u(t)Yx(t))$$

Since  $h$  is constant on any trajectory determined by the invariant vector field  $Y$ , we get  $(dh)_{x(t)}(Yx(t)) = 0$ . Let  $x(t)$  be the solution connecting an arbitrary point  $x_0$  from the Equator to the North Pole at positive time  $s_0$ . Thus, we obtain

$$\beta(0) = 0, \beta(s_0) = 1 \text{ and } \beta(s) = \int_0^s \dot{\beta}(\zeta) d\zeta.$$

In particular,

$$1 \leq \int_0^{s_0} \left| \dot{\beta}(s) \right| ds < Ms_0$$

where  $M$  is an appropriate bound for the integral. In fact,  $(dh)_{(\cdot)}$  is bounded on the sphere. So, we get  $\frac{1}{M} < s_\Sigma$ . Therefore, the North Pole doesn't belong to the subset of the sphere obtained by the action of  $A$  on  $e_2$ . This fact shows that in general the uniform time can not be arbitrary. Furthermore, since the Lie group  $SO(3)$  is simple, the only subalgebra  $\mathfrak{a}$  with  $\mathfrak{z} \subset \mathfrak{a} \subset \mathfrak{g}$  is  $\mathfrak{g}$ . Then,  $\mathfrak{z} = \mathfrak{a}$ . In other words,  $A = \exp(\mathbb{R}Y)$  and for any arbitrary positive time  $t$

$$\{\text{Equator} = \exp(tuY)e_2 : u \in \mathcal{U}_\mathbb{R}\}.$$

Finally, a simple inspection shows that any trajectory of  $p_*(\Sigma)$  connecting the poles needs at least  $\pi$  units of time. Then,  $\pi \leq s_\Sigma$ . On the other hand, if one of any two arbitrary points of the sphere is not a pole, then by sending them to the Equator we see that they can reach each other at  $\pi$  units of time. In fact, we can waste the time at the Equator. Therefore,

$$s_\Sigma = \pi$$

is a uniform time for the system.

**Example 5.3.** On  $\mathfrak{g} = \mathfrak{su}(2)$  consider the bilinear control system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + u(t) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} x(t).$$

In this case,  $G = SU(2)$ , and the affine system is controllable at uniform time on  $M = S^3$ .

**Remark 5.4.** We hope to extend the main results of this paper to the class of Linear Control Systems on Lie Groups, introduced in [1]. In this case, the drift vector field  $X$  belongs to the normalizer  $\mathfrak{n}_{X^\infty(G)}$  of  $\mathfrak{g}$  in the Lie algebra  $X^\infty(G)$  of all  $C^\infty$  vector fields defined on  $G$ . A results in [1] shows that the normalizer is the semidirect product of the Lie algebra  $\mathfrak{g}$  of  $G$  with the Lie algebra  $\mathfrak{aut}(G)$  of the Lie group  $\text{Aut}(G)$ , i. e.,

$$X \in \mathfrak{n}_{X^\infty(G)} = \mathfrak{g} \otimes_s \mathfrak{aut}(G).$$

When  $G$  is compact and semisimple there exists a controllable uniform time to the first component of  $X$ . On the other hand, if  $X \in \mathfrak{aut}(G)$  it follows that  $X_t(e) = e$  for every positive time  $t$ , exactly as in the linear case on  $\mathbb{R}^n$ . Unfortunately,  $S_\Sigma(e)$  is not a semigroup as in the invariant case, see [1]. Here

$$S_\Sigma = \{X_{t_1}^{u_1} \circ X_{t_2}^{u_2} \circ \dots \circ X_{t_k}^{u_k} : u_i \in \mathcal{U}, t_i \geq 0, k \in \mathbb{N}\}$$

is the set of all possible flow compositions coming from the system with non negative times. Furthermore, in [2] the authors give the solution of the differential equation determined by any element  $X$  in  $\mathfrak{n}_{X^\infty(G)}$ .

Finally, by analyzing Example 5.2 we conjecture

**Conjecture 5.5.** Let  $G$  be a connected Lie group and  $\Sigma = (G, \mathcal{D})$  an invariant control system. Assume that  $A = \bigcap_{t>0} A(t, e)$  is not empty. Then, the closed Lie subgroup  $A$  is the intersection of the isotropy groups of orbits of  $G$ -representations which contains  $\exp(\mathfrak{z})$ . Here,  $\mathfrak{z}$  is the Lie algebra determined by the control vectors.

**Acknowledgement 5.6.** The authors are grateful to an anonymous referee which pointed out that the main results in this paper was stated and proved by V. Jurdjevic and H. Sussmann in [6]. On the other hand, he observed that the proof of Proposition 10 was incorrect. Certainly, his suggestions and remarks improved the paper.

#### ACKNOWLEDGEMENT

The first author was partially supported by Proyectos FONDECYT n° 1060981, n° 1060977 and Programa AMAZONAS SENIOR, Projeto n° 507/2007, FAPEAM.

(Received March 10, 2008.)

#### REFERENCES

---

- [1] V. Ayala and L. San Martin: Controllability properties of a class of control systems on Lie groups. *Lectures Notes in Control and Inform. Sci.* 1 (2001), 258, 83–92.
- [2] V. Ayala and J. Tirao: Linear control systems on Lie groups and controllability. *Amer. Math. Soc. Symposia in Pure Mathematics* 64 (1999), 47–64.
- [3] Domenico D’Alessandro: Small time controllability of systems on compact Lie groups and spin angular momentum. *J. Math. Phys.* 42 (2001) 9, 4488–4496.
- [4] S. Helgason: *Differential Geometry, Lie groups and Symmetric Spaces*. Academic Press, New York 1978.
- [5] V. Jurdjevic and H. J. Sussmann: Controllability of nonlinear systems. *J. Differential Equations* 12 (1972), 95–116.
- [6] V. Jurdjevic and H. J. Sussmann: Control systems on Lie groups. *J. Differential Equations* 12 (1972), 313–329.
- [7] H. Kunita: Support of diffusion processes and controllability problems. In: *Proc. Internat. Symposium on Stochastic Differential Equations* (K. Ito, ed.), Wiley, New York 1978, pp. 163–185.
- [8] Y. Sachkov: *Control Theory on Lie Groups*. Lecture Notes SISSA, 2006.
- [9] L. San Martin: *Algebras de Lie*. Editorial UNICAMP, Campinas, SP, 1999.
- [10] F. Silva Leite: Uniform controllable sets of left-invariant vector fields on compact Lie groups. *Systems Control Lett.* 7 (1986), 213–216.
- [11] F. Silva Leite: Uniform controllable sets of left-invariant vector fields on non compact Lie groups. *Systems Control Letters* 6 (1986), 329–335.
- [12] F. W. Warner: *Foundations of Differential Manifolds and Lie Groups*. Scott Foreman, Glenview 1971.

*VictorAyala Universidad Catolica del Norte Avda. Angamos 0610. Antofagasta. Chile.  
e-mail: vayala@ucn.cl*

*JoseAyala-Hoffmann Universidad Arturo Prat. Iquique. Chile.  
e-mail: jayalhoff@gmail.com*

*Ivande Azevedo Tribuzy Universidade Federal do Amazonas. Manaus. Brazil.  
e-mail: ivan@argo.com.br*