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# GLOBAL EXISTENCE OF STRONG SOLUTIONS TO THE ONE-DIMENSIONAL FULL MODEL FOR PHASE TRANSITIONS IN THERMOVISCOELASTIC MATERIALS

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*Dedicated to Jürgen Sprekels on the occasion of his 60th birthday*

*Abstract.* This paper is devoted to the analysis of a one-dimensional model for phase transition phenomena in thermoviscoelastic materials. The corresponding parabolic-hyperbolic PDE system features a *strongly nonlinear* internal energy balance equation, governing the evolution of the absolute temperature $\vartheta$, an evolution equation for the phase change parameter $\chi$, including constraints on the phase variable, and a hyperbolic stress-strain relation for the displacement variable $\mathbf{u}$. The main novelty of the model is that the equations for $\chi$ and $\mathbf{u}$ are coupled in such a way as to take into account the fact that the properties of the viscous and of the elastic parts influence the phase transition phenomenon in different ways. However, this brings about an elliptic degeneracy in the equation for $\mathbf{u}$ which needs to be carefully handled.

First, we prove a global well-posedness result for the related initial-boundary value problem. Secondly, we address the long-time behavior of the solutions in a simplified situation. We prove that the $\omega$-limit set of the solution trajectories is nonempty, connected and compact in a suitable topology, and that its elements solve the steady state system associated with the evolution problem.

*Keywords*: nonlinear and degenerating PDE system, global existence, uniqueness, long-time behavior of solutions, $\omega$-limit, phase transitions, thermoviscoelastic materials

*MSC 2010*: 76A10, 35L80, 35A05, 35B45, 35B50, 80A22

## 1. Introduction

In this paper we study the initial boundary-value problem for the following PDE system

$$\vartheta_t + \chi_t\vartheta - \Delta\vartheta = |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2 + g \quad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\chi_t - \Delta\chi + W'(\chi) = \vartheta - \vartheta_c + \tfrac{1}{2}|\varepsilon(\mathbf{u})|^2 \quad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\mathbf{u}_{tt} - \operatorname{div}((1-\chi)\varepsilon(\mathbf{u}) + \chi\varepsilon(\mathbf{u}_t)) = \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{1.3}$$

which has been recently introduced by Michel Frémond in [15] in order to model phase transition phenomena in a viscoelastic material occupying a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N = 1, 2, 3$, subject to thermal fluctuations during a time interval $[0, T]$. The state variables are the *absolute* temperature $\vartheta$ of the system ($\vartheta_c$ being the equilibrium temperature), and the order parameter $\chi$ (see [14, p. 5]), standing for the local proportion of one of the two phases, which have different viscous/elastic features. For example, in a melting-solidification process one has $\chi = 0$ in the (elastic) solid phase, and $\chi = 1$ in the (viscous) liquid phase. The symbol $\mathbf{u}$ denotes the vector of the small displacements.

We refer to [15] and [26, Sect. 2] for the derivation of the system (1.1–1.3) according to Frémond's modelling approach to phase change phenomena in mechanics. Without going into details, here we just point out that (1.1) is the internal energy balance equation, $g$ being a known heat source. Likewise, the equation (1.3) is the classical balance equation for macroscopic movements (also known as *stress-strain relation*), and accounts for accelerations as well. As usual, by $\varepsilon(\mathbf{u})$ we denote the linearized symmetric strain tensor, given by $\varepsilon_{ij}(\mathbf{u}) := (u_{i,x_j} + u_{j,x_i})/2$, $i, j = 1, 2, 3$ (here $(\cdot)_x$ stands for the space derivative of $(\cdot)$), while the symbol div denotes the vectorial divergence operator. Further, the term $\mathbf{f}$ on the right-hand side may be interpreted as an exterior volume force applied to the body. Following Frémond's perspective, (1.1) and (1.3) are coupled with (1.2), the equation of microscopic movements for the phase variable $\chi$, in which $W'$ is the derivative of a generally nonconvex energy potential, and $|\varepsilon(\mathbf{u})|^2$ is a short-hand for the colon product $\varepsilon(\mathbf{u})\colon \varepsilon(\mathbf{u})$.

The latter term, on the one hand, and the terms in $\chi$ occurring in the stress-strain relation, on the other hand, give the coupling between (1.2) and (1.3). What is more, they highlight what we believe is the most peculiar feature of the system (1.1)–(1.3) in comparison with other phase change models: namely, that the viscous and elastic properties of the physical system are given distinguished role, in order to account for their influence on the phase transition. From the analytical point of view, one of the elliptic operators in (1.3) becomes degenerate when either the viscous or elastic effects prevail and, accordingly, $\chi = 1$ or $\chi = 0$. This, as we shall see later, is one of the main mathematical difficulties attached to the analysis of (1.1)–(1.3).

Indeed, in most models for phase change phenomena the equation for macroscopic movements is neglected, even within the modelling approach developed by Frémond (see [9], [29], [23])). Instead, in the papers [4], [5] the focus is on the analysis of a model for thermoviscoelastic systems not subject to phase transitions: the related (highly nonlinear) PDE system couples a linear viscoelastic equation for the displacement $\mathbf{u}$ and an internal energy balance equation for $\vartheta$, while the equation of microscopic movements is not considered. The model is analyzed in [20] and in [21] pertains to nonlinear thermoviscoplasticity: in the one-dimensional (in space) case, the authors prove the global well-posedness of a PDE system, incorporating both hysteresis effects and modelling phase change, which however does not display a degenerating character. Degenerating phase parameters appear in models for damaging phenomena, see [6], [7], [8]. In this case, the phase variable $\chi$ is related to the local proportion of damaged material. Hence, $\chi$ is forced to take values in $[0, 1]$, with the convention that $\chi = 0$ when the body is completely damaged, and $\chi = 1$ in the damage-free case. Hence, in [7], [8] the equation for macroscopic movements has a degenerating character related to the parameter $\chi$, which is however different from the one in (1.3). For, in their case the coefficients of the elliptic operators in the stress-strain relation vanish only as $\chi \searrow 0$, contrary to the twofold degeneracy of the equation (1.6 c), which we shall further comment later on. Within this framework, in [7], [8] *local in time* well-posedness results are proved for the resulting PDE system.

In the very recent paper [26], we proved a *local in time* well-posedness result for the initial-boundary value problem associated with (1.1)–(1.3), in the three-dimensional case, but neglecting the quadratic dissipative contributions on the right-hand side in (1.1). Hence, the internal energy balance equation we considered in [26] reads

$$\vartheta_t + \chi_t \vartheta - \Delta\vartheta = g \quad \text{in } \Omega \times (0, T). \tag{1.4}$$

Let us note that this simplification is completely justified in the framework of the so-called *small perturbations assumption*, see [16]. Further, in [26] we also proved a *global in time* well-posedness result for system (1.2)–(1.4) in the *one-dimensional case*.

In this paper, we carry on the investigation initiated in [26]: again, we restrict our analysis to the $1D$ case, thus assuming that

$$\Omega = (0, \ell), \quad \ell > 0, \tag{1.5}$$

but we consider the *full* system (1.1)–(1.3) and prove a *global in time* well-posedness result for the related initial-boundary value problem, in a suitable functional framework. Next, we investigate the long-time behavior of the solutions to the *simplified* PDE system ((1.2), (1.3), (1.4)).

In the framework of (1.5), we shall hereafter suppose that the displacement variable $\mathbf{u}$ is scalar, just for the sake of simplifying the notation throughout the paper. Indeed, as the calculations carried out in [26] (where we kept the variable $\mathbf{u}$ vectorial in the one-dimensional case as well) show, in the analysis of the stress-strain relation there is no significant difference between the vectorial and the scalar case. Hence, we end up with the following PDE system

$$\vartheta_t + \chi_t\vartheta - \partial^2_{xx}\vartheta = g + |\chi_t|^2 + \chi|\partial_x u_t|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{1.6 a}$$

$$\chi_t - \partial^2_{xx}\chi + W'(\chi) = \vartheta + \frac{1}{2}|\partial_x u|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{1.6 b}$$

$$u_{tt} - \partial_x(\chi\partial_x u_t + (1-\chi)\partial_x u) = f \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{1.6 c}$$

which we complement with the boundary conditions

$$u(0) = u(\ell) = 0 \quad \text{on } (0,T), \tag{1.7}$$

$$\partial_x\chi(0) = \partial_x\chi(\ell) = 0 \quad \text{on } (0,T), \tag{1.8}$$

$$\partial_x\vartheta(0) = \partial_x\vartheta(\ell) = 0 \quad \text{on } (0,T), \tag{1.9}$$

and the initial conditions

$$\vartheta(0) = \vartheta_0 \quad \text{in } (0,\ell), \tag{1.10}$$

$$\chi(0) = \chi_0 \quad \text{in } (0,\ell), \tag{1.11}$$

$$u(0) = u_0, \ u_t(0) = v_0 \quad \text{in } (0,\ell), \tag{1.12}$$

$\vartheta_0$, $\chi_0$, $u_0$, and $v_0$ being suitable known initial data for the problem.

As we mentioned before, the main challenges in the analysis of the system (1.6 a)–(1.6 c) are related to the *twofold* degenerating character of the equation (1.6 c) and to the nonlinear features of the equations (1.6 a)–(1.6 b), given by the quadratic terms $|\chi_t|^2$, $\chi|\partial_x u_t|^2$, $\chi_t\,\vartheta$, $\frac{1}{2}|\partial_x u|^2$, and by the nonmonotone, possibly nonsmooth term $W'(\chi)$. In particular, throughout the paper we shall suppose that the potential $W$ is given by the sum of a smooth nonconvex function $\widehat{\gamma}$ and of a convex function $\widehat{\beta}$, with domain contained in $[0,1]$ and differentiable in $(0,1)$. Note that, in this way, the values outside $[0,1]$ (which indeed are not physically meaningful for the order parameter $\chi$, denoting a phase proportion), are excluded. A crucial assumption for our analysis shall be that $\widehat{\beta}$ is "sufficiently" coercive at the barriers 0 and 1, namely

$$\lim_{r\to 0^+}\widehat{\beta}'(r) = -\infty, \qquad \lim_{r\to 1^-}\widehat{\beta}'(r) = +\infty. \tag{1.13}$$

Typical examples of functionals which we can include in our analysis are the logarithmic potential

$$W(r) := r\ln(r) + (1-r)\ln(1-r) - c_1 r^2 - c_2 r - c_3, \quad \forall\, r \in (0,1), \tag{1.14}$$

where $c_1$ and $c_2$ are positive constants. Exploiting (1.13) and suitable comparison and maximum principle techniques (see [24] and Lemma 3.1 later on), we shall prove that, if the initial datum $\chi_0$ of the phase parameter is separated from both potential barriers, namely

$$\min_{x\in[0,\ell]} \chi_0(x) > 0, \quad \max_{x\in[0,\ell]} \chi_0(x) < 1,$$

then there exist constants $\delta, \zeta_T \in (0,1)$ ($\zeta_T$ also depending on the final time $T > 0$) such that

$$\chi(x,t) \geqslant \delta, \quad \chi(x,t) \leqslant \zeta_T \quad \forall\,(x,t) \in [0,\ell] \times [0,T], \tag{1.15}$$

i.e. the solution component $\chi$ stays *globally* away from the potential barriers. Separation inequalities of this kind have been obtained in other papers on the analysis of phase transition models, see [17], [18], [24]. In fact, (1.15) rules out the degeneracy of both elliptic operators in (1.6 c) and allows us to prove our global existence and uniqueness Theorem 1 by carefully combining (1.15) with an extension procedure for local solutions to (1.6 a)–(1.6 c).

Due to the troublesome nonlinearities on the right-hand side of (1.6 a), we are able to perform the long-time analysis of the simplified system (1.6 b), (1.6 c), (1.16), where the internal energy balance is

$$\vartheta_t + \chi_t\vartheta - \partial^2_{xx}\vartheta = g \quad \text{a.e. in } (0,\ell) \times (0,T). \tag{1.16}$$

In Theorem 2, we shall prove that the $\omega$-limit set (i.e., the set of the cluster points in some suitable topology) of the solution trajectories $(\vartheta(t), \chi(t), u(t))_{t\in(0,+\infty)}$ is nonempty, connected and compact, and that its elements solve the stationary system corresponding to (1.6 b), (1.6 c), (1.16). A crucial step for showing this consists in observing that the first inequality in (1.15) extends to $(0,+\infty)$, namely that

$$\chi(x,t) \geqslant \delta \quad \forall\,(x,t) \in [0,\ell] \times [0,+\infty), \tag{1.17}$$

see Proposition 4.1 later on. On the contrary, the second part of (1.15) does not hold globally on $(0,+\infty)$, see Remark 4.2. This is the main technical point preventing us from improving our long-time results for (1.6 b), (1.6 c), (1.16). Indeed, in order to prove the existence of the *global attractor* for bundle of trajectories, rather than for a single trajectory, of the system (1.6 b), (1.6 c), (1.16), we would need to strengthen our large-time a priori estimates on the solution component $u$ (cf. Proposition 4.1). However, it seems to us that better large-time estimates on $u$ cannot be obtained, if one relies solely on (1.7). The same technical drawback makes it difficult to implement Łojasiewicz-Simon procedures (cf. [28] and [13]) in order to prove the convergence as $t \to +\infty$ of the whole trajectories $(\vartheta(t), \chi(t), u(t))_{t\in(0,+\infty)}$

to the elements of their $\omega$-limit. Note that such techniques have been successfully exploited in the study of the convergence to equilibrium of some phase field systems, see e.g. [1], [12], [17]. In fact, we believe that the study of the long-time behavior of (1.6 b), (1.6 c), (1.16), both in the direction of global attractors and of Łojasiewicz-Simon-type results, is an interesting open problem.

**Plan of the paper.** In Section 2, we set up some notation and introduce a suitable variational formulation of the initial boundary-value problem for the full PDE system (1.6 a)–(1.12) (cf. Problem 1). Hence, we state our main results, regarding the global well-posedness (in finite time) of the latter problem (which we prove in Section 3), and the analysis of the $\omega$-limit associated with the solution trajectories of the simplified system (1.6 b), (1.6 c), (1.16), which we perform in Section 4.

## 2. Main results

Notation. Given $k \in \{1,2\}$, we shall consider the Sobolev spaces

$$H_0^k(0,\ell) := \{v \in W^{k,2}(0,\ell)\colon\ v(0) = v(\ell) = 0\},$$
$$H_N^k(0,\ell) := \{v \in W^{k,2}(0,\ell)\colon\ \partial_x v(0) = \partial_x v(\ell) = 0\},$$

both endowed with the norms of $W^{k,2}(0,\ell)$. Furthermore, we shall identify $L^2(0,\ell)$ with its dual space $L^2(0,\ell)'$, so that $H^1(0,\ell) \hookrightarrow L^2(0,\ell) \hookrightarrow H^1(0,\ell)'$ with dense and continuous embeddings. We shall use the symbols $\|\cdot\|$ and $(\cdot,\cdot)$ for the norm and the scalar product on $L^2(0,\ell)$, while $\langle\cdot,\cdot\rangle$ shall stand both for the duality pairing between $H^1(0,\ell)'$ and $H^1(0,\ell)$ and for the duality between $H^{-1}(0,\ell)$ and $H_0^1(0,\ell)$. Given $v \in H^1(0,\ell)'$, we shall use the notation $m(v) := \langle v, 1\rangle$ for its mean value. Finally, we recall that

$$H^1(0,\ell) \Subset L^\infty(0,\ell) \quad \text{with a compact embedding,} \tag{2.1}$$
$$H^2(0,\ell) \Subset W^{1,\infty}(0,\ell) \quad \text{with a compact embedding.} \tag{2.2}$$

Combining this with the continuous embeddings

$$L^\infty(0,\ell) \subset L^2(0,\ell) \quad \text{and} \quad W^{1,\infty}(0,\ell) \subset H^1(0,\ell)$$

and recalling [22, Lemma 5.1, p. 58], one has

$$\forall\,\varepsilon > 0\ \exists\,C_\varepsilon > 0\colon\ \forall\,v \in H^1(0,\ell)\ \ \|v\|_{L^\infty(0,\ell)} \leqslant \varepsilon\|v\|_{H^1(0,\ell)} + C_\varepsilon\|v\|, \tag{2.3}$$
$$\forall\,\varepsilon > 0\ \exists\,C_\varepsilon > 0\colon\ \forall\,w \in H^2(0,\ell)\ \ \|w\|_{W^{1,\infty}(0,\ell)} \leqslant \varepsilon\|w\|_{H^2(0,\ell)} + C_\varepsilon\|w\|_{H^1(0,\ell)}. \tag{2.4}$$

In order to state the variational formulation of the Cauchy problem for (1.6 a)–(1.9), we need to introduce the following operators:

1. for a given measurable function $\eta\colon\ (0,\ell)\to[0,1]$, we define $\mathcal{H}(\eta\,\cdot)\colon\ H^1_0(0,\ell)\to H^{-1}(0,\ell)$ by

$$\langle\mathcal{H}(\eta v),w\rangle:=\int_0^\ell \eta\partial_x v\partial_x w\quad\forall\, v,w\in H_0(0,\ell);$$

2. $A\colon\ H^1(0,\ell)\to H^1(0,\ell)'$, realizing the Laplace operator $-\partial^2_{xx}$ with homogeneous Neumann boundary conditions, defined by

$$\langle Au,v\rangle:=(\partial_x u,\partial_x v)\quad\forall\, u,v\in H^1(0,\ell);$$

3. the duality operator $J:=A+I\colon\ H^1(0,\ell)\to H^1(0,\ell)'$ ($I$ being the identity operator): in the sequel, we shall make use of the relations

$$\begin{aligned}\langle Ju,u\rangle&=\|u\|^2_{H^1(0,\ell)}\quad\forall\, u\in H^1(0,\ell),\\ \langle J^{-1}v,v\rangle&=\|v\|^2_{H^1(0,\ell)'}\quad\forall\, v\in H^1(0,\ell)'.\end{aligned}\tag{2.5}$$

### 2.1. A global well-posedness result in $[0,T]$ for the full system

We now list our assumptions on the problem data:

$$\begin{gathered}g\in L^2(0,T;L^2(0,\ell))\cap L^\infty(0,T;H^1(0,\ell)'),\\ g(x,t)\geqslant 0\quad\text{for a.e. } (x,t)\in(0,\ell)\times(0,T),\end{gathered}\tag{2.6}$$

$$f\in L^2(0,T;L^2(0,\ell)),\tag{2.7}$$

$$\vartheta_0\in H^1(0,\ell)\quad\text{and}\quad\min_{x\in[0,\ell]}\vartheta_0(x)>0,\tag{2.8}$$

$$\chi_0\in H^2_N(0,\ell),\tag{2.9}$$

$$u_0\in H^2_0(0,\ell),\quad v_0\in H^1_0(0,\ell).\tag{2.10}$$

As for the potential $W$, we require that

$$W=\widehat{\beta}+\widehat{\gamma},\tag{W1}$$

where

$$\widehat{\gamma}\in \mathrm{C}^2([0,1]),\quad\text{with derivative } \gamma:=\widehat{\gamma}',\tag{W2}$$

(W3) $\overline{\mathrm{dom}(\widehat{\beta})}=[0,1]$, and $\widehat{\beta}\colon\ \mathrm{dom}(\widehat{\beta})\to\mathbb{R}$ l.s.c., convex, differentiable in $(0,1)$.

Henceforth, we shall denote by $\beta$ the derivative $\widehat{\beta}'$, and further assume the "coercivity" conditions

$$\text{(W4)} \qquad \lim_{x\to 0^+} \beta(x) = -\infty, \quad \lim_{x\to 1^-} \beta(x) = +\infty,$$

and that

$$\text{(W5)} \qquad \text{for all } \varrho > 0, \ \ \beta \text{ is a Lipschitz continuous function on } [\varrho, 1-\varrho].$$

Note that (W4) (which corresponds to the *strong coercivity condition* of [17]) in fact rules out the case in which $\widehat{\beta}$ is the indicator function of $[0,1]$, but is fulfilled in the case of the logarithmic potential (1.14).

Our next assumption then ensures that the initial datum $\chi_0$ is "separated from the potential barriers":

$$\min_{x\in[0,\ell]} \chi_0(x) > 0, \quad \max_{x\in[0,\ell]} \chi_0(x) < 1. \tag{2.11}$$

We are now in a position to state the variational formulation of the initial-boundary value problem for (1.6 a)–(1.6 c).

Problem 1. Find $(\vartheta, \chi, u)$ with

$$\vartheta \in L^2(0,T;H^2_N(0,\ell)) \cap L^\infty(0,T;H^1(0,\ell)) \cap H^1(0,T;L^2(0,\ell)) \cap W^{1,\infty}(0,T;H^1(0,\ell)'), \tag{2.12}$$

$$\chi \in L^\infty(0,T;H^2_N(0,\ell)) \cap H^1(0,T;H^1(0,\ell)) \cap W^{1,\infty}(0,T;L^2(0,\ell)), \tag{2.13}$$

$$u \in H^1(0,T;H^2_0(0,\ell)) \cap W^{1,\infty}(0,T;H^1_0(0,\ell)) \cap H^2(0,T;L^2(0,\ell)), \tag{2.14}$$

complying with the initial conditions (1.10)–(1.12), with the equations

$$\vartheta_t + \chi_t\vartheta + A\vartheta = g + |\chi_t|^2 + \chi|\partial_x u_t|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{2.15}$$

$$\chi_t + A\chi + \beta(\chi) + \gamma(\chi) = \vartheta + \frac{1}{2}|\partial_x u|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{2.16}$$

$$u_{tt} + \mathcal{H}((1-\chi)u) + \mathcal{H}(\chi u_t) = f \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{2.17}$$

and such that

$$\min_{(x,t)\in[0,\ell]\times[0,T]} \vartheta(x,t) > 0, \quad \min_{(x,t)\in[0,\ell]\times[0,T]} \chi(x,t) > 0, \quad \max_{(x,t)\in[0,\ell]\times[0,T]} \chi(x,t) < 1. \tag{2.18}$$

The following result, stating the *global well-posedness* of Problem 1 on the interval $[0,T]$, extends [26, Thm. 2].

**Theorem 1.** *Assume* (2.6)–(2.10), (W1)–(W5), *and* (2.11). *Then, there exist constants*

1. $\delta \in (0,1)$, *depending on the potential* $W$ *and on the initial datum* $\chi_0$,
2. $\zeta_T \in (0,1)$, *depending on* $W$, *on* $\chi_0$, *and on the final time* $T$,
3. $\theta^*_T > 0$, *depending* $T$ *and on the problem data,*

*and a triple* $(\vartheta, \chi, u)$ *solving Problem* 1, *such that the* $\vartheta$ *and* $\chi$ *components fulfil*

$$\chi(x,t) \geqslant \delta > 0 \quad \forall\,(x,t) \in [0,\ell] \times [0,T], \tag{2.19}$$

$$\chi(x,t) \leqslant \zeta_T < 1 \quad \forall\,(x,t) \in [0,\ell] \times [0,T], \tag{2.20}$$

$$\vartheta(x,t) \geqslant \theta^*_T > 0 \quad \forall\,(x,t) \in [0,\ell] \times [0,T]. \tag{2.21}$$

*Moreover, the triple* $(\vartheta, \chi, u)$ *is the unique solution to Problem* 1 *and depends continuously on the initial data, on* $g$, *and on* $f$, *in the sense specified by Proposition* 3.8. *Finally,* $\chi$ *has the further regularity*

$$\chi \in H^2(0,T; H^1(0,\ell)'). \tag{2.22}$$

### 2.2. Results on the long-time behavior of solutions

We now state our main result on the long-time behavior of the solutions to (the initial-boundary value problem for) the system (1.6 b), (1.6 c), (1.16), whose variational formulation reads

Problem 2. Find $(\vartheta, \chi, u)$, with the regularity (2.12)–(2.14), fulfilling (2.18), the initial conditions (1.10)–(1.12), the equations (2.16)–(2.17), and

$$\vartheta_t + \chi_t \vartheta + A\vartheta = g \quad \text{a.e. in } (0,\ell) \times (0,T). \tag{2.23}$$

Henceforth, we shall further require that

$$g \in L^2(0,+\infty; L^2(0,\ell)), \quad g(x,t) \geqslant 0 \text{ for a.e. } (x,t) \in (0,\ell) \times (0,+\infty), \tag{2.24}$$

$$f \in L^2(0,+\infty; L^2(0,\ell)). \tag{2.25}$$

Remark 2.1. Theorem 1 and (2.24)–(2.25) guarantee that for every triple $(\vartheta_0, \chi_0, u_0)$ fulfilling (2.8)–(2.10) and (2.11) there exists a unique solution $(\vartheta, \chi, u)$: $[0,+\infty) \to H^1(0,\ell) \times H^1(0,\ell) \times H^1_0(0,\ell)$ to Problem 2 starting from $(\vartheta_0, \chi_0, u_0)$.

Given an initial triple $(\vartheta_0, \chi_0, u_0)$ fulfilling (2.8)-(2.10) and (2.11), we recall the definition of the $\omega$-limit of the associated solution trajectory $(\vartheta(t), \chi(t), u(t))$ in the space $H^1(0,\ell) \times H^1(0,\ell) \times H^1_0(0,\ell)$:

$$\begin{aligned} \omega(\vartheta_0, \chi_0, u_0) := \{&(\vartheta_\infty, \chi_\infty, u_\infty) \in H^1(0,\ell) \times H^1(0,\ell) \times H^1_0(0,\ell)\colon \\ &\exists\, t_n \nearrow \infty\colon\ (\vartheta(t_n), \chi(t_n), u(t_n)) \to (\vartheta_\infty, \chi_\infty, u_\infty) \\ &\text{in } H^{1-\nu}(0,\ell) \times H^{1-\nu}(0,\ell) \times H^{1-\nu}_0(0,\ell)\ \ \forall \nu \in (0,1)\}. \end{aligned} \tag{2.26}$$

Our next results concerns the structure of the set (2.26).

**Theorem 2.** *Assume* (W1)–(W5) *and* (2.24)–(2.25). *Let* $(\vartheta_0, \chi_0, u_0)$ *be an initial triple fulfilling* (2.8)–(2.10) *and* (2.11). *Then,*

1. $\omega(\vartheta_0, \chi_0, u_0)$ *is a nonempty, compact, and connected subset of the following product space*

$$H^{1-\nu}(0,\ell) \times H^{1-\nu}(0,\ell) \times H_0^{1-\nu}(0,\ell)$$

*for all* $\nu \in (0,1)$. *Moreover, there exists a constant* $\zeta_\infty \in (0,1)$ *such that every* $(\vartheta_\infty, \chi_\infty, u_\infty) \in \omega(\vartheta_0, \chi_0, u_0)$ *solves the stationary problem*

$$A\vartheta_\infty = 0 \quad \text{a.e. in } (0,\ell), \tag{2.27}$$

$$A\chi_\infty + \beta(\chi_\infty) + \gamma(\chi_\infty) = \vartheta_\infty \quad \text{a.e. in } (0,\ell), \tag{2.28}$$

$$u_\infty = 0 \quad \text{a.e. in } (0,\ell), \tag{2.29}$$

*and fulfils*

$$\min_{x\in[0,\ell]} \vartheta_\infty(x) \geqslant 0, \quad \min_{x\in[0,\ell]} \chi_\infty(x) \geqslant \delta, \quad \max_{x\in[0,\ell]} \chi_\infty(x) \leqslant \zeta_\infty. \tag{2.30}$$

*In particular,*

$$\exists\, \bar\vartheta_\infty \in [0,+\infty) \text{ such that } \vartheta_\infty(x) = \bar\vartheta_\infty \text{ for all } x \in [0,\ell]. \tag{2.31}$$

2. *In addition, if*

$$W' = \beta + \gamma \text{ is strictly increasing in } (0,1), \tag{2.32}$$

*for every* $(\vartheta_\infty, \chi_\infty, 0) \in \omega(\vartheta_0, \chi_0, u_0)$, *the component* $\chi_\infty$ *is also constant on* $(0,\ell)$ *and*

$$\chi_\infty(x) = (\beta+\gamma)^{-1}(\bar\vartheta_\infty) \quad \forall\, x \in [0,\ell]. \tag{2.33}$$

R e m a r k 2.2 (Further regularity of solutions). If, besides (2.8) and (2.24), we further supposed that

$$\vartheta_0 \in H_N^2(0,\ell), \quad g \in L^\infty(0,+\infty; L^2(0,\ell)), \quad g_t \in L^1(0,+\infty; L^2(0,\ell)), \tag{2.34}$$

arguing in the same way as in the proof of [26, Lemma 5.6] we could obtain an estimate for $\|\vartheta\|_{L^\infty(0,+\infty;H_N^2(0,\ell))}$ and $\|\vartheta_t\|_{L^2(0,+\infty;H^1(0,\ell))}$. In this case, we would conclude that the projection on the first component of the set $\omega(\vartheta_0, \chi_0, u_0)$ is in fact in $H_N^2(0,\ell)$, and that the elements of $\omega(\vartheta_0, \chi_0, u_0)$ are cluster points for the trajectory w.r.t. the topology of $H^{2-\nu}(0,\ell) \times H^{1-\nu}(0,\ell) \times H_0^{1-\nu}(0,\ell)$ for all $\nu \in (0,1)$. Moreover, let us note that it is possible to prove (by means of a bootstrap argument) that the solutions $\vartheta_\infty$ and $\chi_\infty$ to the stationary problem (2.27)–(2.28) are of class $C^\infty([0,\ell])$, and so the equations (2.27) and (2.28) hold true for all $x \in [0,\ell]$.

## 3. Proof of well-posedness for Problem 1 on $[0,T]$

### 3.1. Strategy of the proof of Theorem 1

In order to prove the existence of a global solution to Problem 1, we shall combine a Schauder fixed point argument (yielding the existence of a local solution) with a careful extension procedure.

**First step: existence of a local solution.** Using (2.11) and (W4), we fix a constant $\delta > 0$ such that

$$\chi_0(x) \geqslant \delta > 0 \quad \text{and} \quad W'(\delta) < 0, \tag{3.1}$$

and we consider the truncated PDE system

$$\vartheta_t + \chi_t \vartheta + A\vartheta = g + |\chi_t|^2 + \chi|\partial_x u_t|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{3.2 a}$$

$$\chi_t + A\chi + \beta(\chi) + \gamma(\chi) = \vartheta + \frac{1}{2}|\partial_x u|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{3.2 b}$$

$$u_{tt} + \mathcal{H}((1-\chi)u) + \mathcal{H}(T_\delta(\chi)u_t) = f \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{3.2 c}$$

where $T_\delta$ is the truncation operator defined by

$$T_\delta(r) := \max\{r, \delta\} \quad \forall\, r \in \mathbb{R}. \tag{3.3}$$

Note that here the degeneracy of the *main* part of the elliptic operator is ruled out.

Using the Schauder fixed point theorem, we shall prove the existence of a local solution $(\widehat{\vartheta}, \widehat{\chi}, \widehat{u})$ to the Cauchy problem for the system (3.2) on some interval $[0,T_0]$, enjoying the regularity (2.12)–(2.14), the positivity properties (2.18), and fulfilling

$$\widehat{\chi}(x,t) \geqslant \delta > 0 \quad \forall\, (x,t) \in [0,\ell]\times[0,T_0], \tag{3.4}$$

$$\max_{(x,t)\in[0,\ell]\times[0,T_0]} \widehat{\chi}(x,t) < 1. \tag{3.5}$$

As we shall see (cf. Remark 3.2 later on), the two separation inequalities have a different character. In particular, the second one depends on the $L^\infty(0,\ell)$ norm of the right-hand side of (3.2 b), hence it forces an appropriate choice of the functional setting for the fixed point argument, see (3.7).

Thanks to (3.4), we shall conclude that the triple $(\widehat{\vartheta}, \widehat{\chi}, \widehat{u})$ is in particular a local solution to Problem 1.

**Second step: extension procedure.** After finding a local solution $(\widehat{\vartheta}, \widehat{\chi}, \widehat{u})$ to Problem 1, we shall prove the existence of a *global* solution by a technique which is essentially tailored to extending to the whole interval $[0,T]$ the local solution $(\widehat{\vartheta},\widehat{\chi},\widehat{u})$ *along with* the separation inequalities (2.19)–(3.5) (see the notion of *$\delta$-separated solution* later on). This procedure shall be developed at length in Section 3.3 and substantially relies on some *global estimates*, proved in Lemma 3.7.

### 3.2. The Schauder fixed point argument

We construct the Schauder operator associated with system (3.2) in three phases: we start by solving (3.2 b) with $\vartheta$ and $u$ fixed, then proceed to solving (3.2 c) with $\chi$ fixed, and finally tackle (3.2 a) with data $\chi$ and $u$ from the previous steps.

We fix $\bar{t}\in(0,T]$ and $R>0$, with

$$R > 2\|\vartheta_0\|_{L^\infty(0,\ell)}, \tag{3.6}$$

and consider the balls

$$\begin{aligned}\mathcal{O}_{\bar{t}} := \{\vartheta \in L^\infty(0,\bar{t};L^\infty(0,\ell))\cap H^1(0,\bar{t};H^1(0,\ell)')\colon \\ \|\vartheta\|_{L^\infty(0,\bar{t};L^\infty(0,\ell))\cap H^1(0,\bar{t};H^1(0,\ell)')}\leqslant R, \\ \vartheta>0 \text{ a.e. in } (0,\ell)\times(0,\bar{t})\}, \\ \mathcal{U}_{\bar{t}} := \{u\in H^1(0,\bar{t};W^{1,\infty}_0(0,\ell))\colon\ \|u\|_{H^1(0,\bar{t};W^{1,\infty}_0(0,\ell))}\leqslant R\}.\end{aligned} \tag{3.7}$$

Henceforth, we shall denote most of the positive constants, occurring in the calculations and depending on the problem data but not on $\bar{t}$, by the same symbol $C$ (whose meaning may vary even within the same line).

First, we have the following result.

**Lemma 3.1.** *Assume* (2.9), (W1)–(W5), *and* (2.11). *Then, there exist constants $M_1>0$ and $\zeta_R\in(0,1)$, depending on $R$ and on the problem data but independent of $\bar{t}\in(0,T]$, such that for all $(\overline{\vartheta},\overline{u})\in\mathcal{O}_{\bar{t}}\times\mathcal{U}_{\bar{t}}$ there exists a unique $\chi$ with*

$$\begin{aligned}\chi\in \mathrm{C}^0([0,\bar{t}];H^2_N(0,\ell))\cap H^1(0,\bar{t};H^1(0,\ell))\cap W^{1,\infty}(0,\bar{t};L^2(0,\ell)) \\ \cap H^2(0,\bar{t};H^1(0,\ell)'),\end{aligned} \tag{3.8}$$

*fulfilling the initial condition* (1.11), *the equation*

$$\chi_t + A\chi + \beta(\chi)+\gamma(\chi) = \overline{\vartheta} + \frac{1}{2}|\partial_x\overline{u}|^2 \quad \text{a.e. in } (0,\ell)\times(0,\bar{t}), \tag{3.9}$$

*the separation inequalities*

$$\chi(x,t)\geqslant \delta>0 \qquad \forall\,(x,t)\in[0,\ell]\times[0,\bar{t}], \tag{3.10}$$

$$\chi(x,t)\leqslant \zeta_R<1 \quad \forall\,(x,t)\in[0,\ell]\times[0,\bar{t}], \tag{3.11}$$

*and the estimate*

$$\|\chi\|_{L^\infty(0,\bar t;H^2_N(0,\ell))\cap H^1(0,\bar t;H^1(0,\ell))\cap W^{1,\infty}(0,\bar t;L^2(0,\ell))\cap H^2(0,\bar t;H^1(0,\ell)')} \leqslant M_1. \tag{3.12}$$

P r o o f. In view of our choice (3.7) of the fixed point setup, for every $(\overline{\vartheta},\overline{u}) \in \mathcal{O}_{\bar t}\times\mathcal{U}_{\bar t}$ the term on the right-hand side of (3.9) (which shall be denoted as $\omega$ throughout this proof) is in $L^\infty(0,\bar t;L^\infty(0,\ell))$. Hence, we may for instance apply [11, Lemma 3.3] (based on the theory of maximal monotone operators in Hilbert and Banach spaces, see [10], [3]) and conclude that the Cauchy problem for (3.9) has a unique solution

$$\chi \in L^2(0,\bar t;H^2_N(0,\ell)) \cap \mathrm{C}^0([0,\bar t];H^1(0,\ell)) \cap H^1(0,\bar t;L^2(0,\ell)). \tag{3.13}$$

Our argument for proving (3.11) is the same as in the proof of [24, Corollary 2.1], nonetheless we shall detail it here for the sake of readability. We set

$$H^*(t) := \|\overline{\vartheta}(t) + \frac{1}{2}|\partial_x\overline{u}(t)|^2\|_{L^\infty(0,\ell)}, \quad t\in(0,\bar t)$$

and consider the first order ODE associated with (3.9), having $H^*$ on the right-hand side,

$$y'(t) + \beta(y(t)) + \gamma(y(t)) = H^*(t), \quad t\in(0,\bar t), \tag{3.14}$$

which we supplement with the condition

$$y(0) = \max_{x\in[0,\ell]} \chi_0(x). \tag{3.15}$$

We fix a constant $\zeta_R \in (0,1)$ such that

$$\max_{x\in[0,\ell]} \chi_0(x) \leqslant \zeta_R, \tag{3.16}$$

$$W'(\zeta_R) \geqslant \frac{1}{2}\|u_0\|^2_{W^{1,\infty}(0,\ell)} + R + R^2 \tag{3.17}$$

(note that such a constant exists thanks to (W4) and the second part of (2.11)), and multiply (3.14) by $(y(t)-\zeta_R)^+$. Upon integrating in time, we find that for all $t\in(0,\bar t)$

$$\begin{aligned} &\frac{1}{2}\|(y(t)-\zeta_R)^+\|^2 + \int_0^t (\beta(y(s))-\beta(\zeta_R))(y(s)-\zeta_R)^+\,\mathrm{d}s \\ &\quad = \frac{1}{2}\|(y(0)-\zeta_R)^+\|^2 - \int_0^t (\gamma(y(s))-\gamma(\zeta_R))(y(s)-\zeta_R)^+\,\mathrm{d}s \\ &\qquad + \int_0^t (H^*(s)-W'(\zeta_R))(y(s)-\zeta_R)^+\,\mathrm{d}s. \end{aligned} \tag{3.18}$$

Note that the second term on the left-hand side of (3.18) is nonnegative by the monotonicity of $\beta$. On the other hand, for all $t \in [0, \bar{t}]$ we have

$$
\begin{aligned}
H^*(t) &= \Big\| \overline{\vartheta}(t) + \frac{1}{2}|\partial_x \overline{u}(t)|^2 \Big\|_{L^\infty(0,\ell)} \\
&\leqslant \|\overline{\vartheta}(t)\|_{L^\infty(0,\ell)} + \frac{1}{2}\|\overline{u}(t)\|^2_{W^{1,\infty}(0,\ell)} \\
&\leqslant R + \frac{1}{2}\|u_0\|^2_{W^{1,\infty}(0,\ell)} + \int_0^t \|\overline{u}_t\|_{W^{1,\infty}(0,\ell)}\|\overline{u}\|_{W^{1,\infty}(0,\ell)} \\
&\leqslant R + \frac{1}{2}\|u_0\|^2_{W^{1,\infty}(0,\ell)} + R^2 \leqslant W'(\zeta_R),
\end{aligned}
$$

due to (3.17), so that the last term on the right-hand side of (3.18) is nonpositive. Finally, by (3.15) and (3.16) we find $(y(0) - \zeta_R)^+ = 0$. Hence, using $\gamma \in \mathrm{C}^1([0,1])$ we deduce from (3.18) that

$$
(3.19) \qquad \frac{1}{2}\|(y(t) - \zeta_R)^+\|^2 \leqslant \|\gamma'\|_{L^\infty(0,1)} \int_0^t \|(y(s) - \zeta_R)^+\|^2 \,\mathrm{d}s \quad \forall\, t \in [0, \bar{t}],
$$

and the Gronwall Lemma leads to $(y(t) - \zeta_R)^+ = 0$, i.e. $y(t) \leqslant \zeta_R$ for all $t \in [0, \bar{t}]$. On the other hand, by the comparison principle for second-order parabolic equations, we conclude

$$
\zeta_R \geqslant y(t) \geqslant \chi(x,t) \quad \forall\, (x,t) \in [0,\ell] \times [0,\bar{t}],
$$

and (3.11) ensues.

Although all calculations are developed in [26, Lemma 5.6], we shall sketch the proof of (3.10), in order to highlight the differences against the argument for (3.11). First of all, it does not rely on the comparison with an auxiliary ODE: we just directly test (3.9) by $-(\chi - \delta)^-$ and integrate in time, concluding

$$
\begin{aligned}
(3.20) \quad \frac{1}{2}\|(\chi(t) - \delta)^-\|^2 &+ \int_0^t \|(\partial_x(\chi - \delta)^-)\|^2 - \int_0^t\int_0^\ell (\beta(\chi) - \beta(\delta))(\chi - \delta)^- \\
&= \frac{1}{2}\|(\chi_0 - \delta)^-\|^2 + \int_0^t\int_0^\ell (\gamma(\chi) - \gamma(\delta))(\chi - \delta)^- \\
&\quad - \int_0^t\int_0^\ell \Big(\overline{\vartheta} + \frac{1}{2}\|\partial_x \overline{u}\|^2 - W'(\delta)\Big)(\chi - \delta)^-.
\end{aligned}
$$

Again, the third term on the left-hand side of (3.20) is nonnegative due to the monotonicity of $\beta$, whereas the first summand on the right-hand side is zero thanks to the first part of (3.1), and the second term can be estimated in the same way as in (3.19). Finally, now the last term in (3.20) is nonpositive simply thanks to the second part of (3.1) and the positivity of $\overline{\vartheta} + \frac{1}{2}\|\partial_x \overline{u}\|^2$. By the Gronwall Lemma, we likewise infer $|(\chi(t) - \delta)^-| = 0$ for all $t \in [0, \bar{t}]$, whence (3.10).

In order to conclude (3.12), we differentiate (3.9), add $\chi_t$ to both sides of $(3.9)_t$, test it by $J^{-1}(\chi_{tt})$, and integrate in time. Taking into account (2.5), by standard calculations we obtain

$$
\begin{aligned}
(3.21)\qquad \int_0^t \|\chi_{tt}\|^2_{H^1(0,\ell)'} + \frac{1}{2}\|\chi_t(t)\|^2 \\
&\leqslant \frac{1}{2}\|\chi_t(0)\|^2 + \left|\int_0^t\int_0^\ell (\beta'(\chi)+\gamma'(\chi))\chi_t J^{-1}(\chi_{tt})\right| \\
&\quad + \left|\int_0^t\int_0^\ell \chi_t J^{-1}(\chi_{tt})\right| + \left|\int_0^t\int_0^\ell \overline{\vartheta}_t J^{-1}(\chi_{tt})\right| \\
&\quad + \left|\int_0^t\int_0^\ell \partial_x\overline{u}\partial_x\overline{u}_t J^{-1}(\chi_{tt})\right|.
\end{aligned}
$$

Now, the first summand on the right-hand side of (3.21) is estimated via

$$\|\chi_t(0)\|^2 \leqslant C(\|A\chi_0\| + \|\beta(\chi_0)+\gamma(\chi_0)\| + \|\overline{\vartheta}(0)\| + \|\overline{u}(0)\|) \leqslant C$$

thanks to (2.9) and (3.7); the second term is estimated by (3.13), while, in order to deal with the third term, we exploit (3.10)–(3.11), which, joint with (W5), yield an estimate for $\beta'(\chi)+\gamma'(\chi)$ in $L^\infty(0,\bar{t};L^\infty(0,\ell))$. The last two summands are estimated via $\|\overline{\vartheta}_t\|_{L^2(0,\bar{t};H^1(0,\ell)')}$ and $\|\overline{u}\|_{H^1(0,\bar{t};W_0^{1,4}(0,\ell))}$. For all details, we refer the reader to the proof of [26, Proposition 4.5]. Hence, we infer

$$\|\chi\|_{H^2(0,\bar{t};H^1(0,\ell)')\cap W^{1,\infty}(0,\bar{t};L^2(0,\ell))} \leqslant C.$$

Arguing by comparison in $(3.9)_t$, we also obtain an estimate for $A\chi_t$ in $L^2(0,\bar{t};H^1(0,\ell)')$, hence for $\chi_t$ in $L^2(0,\bar{t};H^1(0,\ell))$. Finally, a comparison in (3.9) yields that $A\chi+\beta(\chi)\in L^\infty(0,\bar{t};L^2(0,\ell))$. The monotonicity of $\beta$ entails an estimate for $A\chi$ in $L^\infty(0,\bar{t};L^2(0,\ell))$, hence, by elliptic regularity, $\chi$ is in $L^\infty(0,\bar{t};H^2_N(0,\ell))$. □

Remark 3.2. As we have already mentioned, the inequalities (3.10) and (3.11) have a different character. The first one is a direct consequence of the weak maximum principle for parabolic equations and essentially relies on the positivity of the right-hand side of (3.9). The second one has a *quantitative nature*, i.e. it holds with a constant $\zeta_R$ depending on the $L^\infty(0,\bar{t};L^\infty(0,\ell))$-norm of the right-hand side of (3.9), and thus, ultimately, on the time interval in which (3.9) is considered. Instead, the constant (3.10) is fixed right from the beginning and, as we shall see, invariant in time: in this sense, we may say that (3.10) is a *global in time* separation inequality.

Thanks to Lemma 3.1, for all $\bar{t} \in (0,T]$ the solution operator $\mathcal{T}_1$ associated with the Cauchy problem for (3.9) is well-defined on $\mathcal{O}_{\bar{t}} \times \mathcal{U}_{\bar{t}}$, and takes values in the set

$$\begin{aligned}\mathcal{X}_{\bar{t}} := \{\chi \in L^\infty(0,\bar{t}; H^2_N(0,\ell)) \cap H^1(0,\bar{t}; H^1(0,\ell)) \\ \cap W^{1,\infty}(0,\bar{t}; L^2(0,\ell)) \cap H^2(0,\bar{t}; H^1(0,\ell)') \text{ such that} \\ \|\chi\|_{L^\infty(0,\bar{t};H^2_N(0,\ell))\cap H^1(0,\bar{t};H^1(0,\ell))\cap W^{1,\infty}(0,\bar{t};L^2(0,\ell))\cap H^2(0,\bar{t};H^1(0,\ell)')} \leqslant M_1\}.\end{aligned}$$

We now solve (the Cauchy problem for) (3.2 c), with fixed datum $\overline{\chi} \in \mathcal{X}_{\bar{t}}$ (see [8, Lemma 3.4] and [26, Lemma 5.3] for the proof).

**Lemma 3.3.** *Assume* (2.7) *and* (2.10). *Then, there exists a constant* $M_2 > 0$, *only depending on* $R$, *on the constant* $\delta$ *specified by* (3.1), *and on the problem data, but independent of* $\bar{t} \in (0,T]$, *such that for all* $\overline{\chi} \in \mathcal{X}_{\bar{t}}$ *there exists a unique* $u$ *with the regularity* (2.14), *fulfilling* (1.2),

$$u_{tt} + \mathcal{H}((1-\overline{\chi})u) + \mathcal{H}(T_\delta(\overline{\chi})u_t) = f \quad \text{a.e. in } (0,\ell) \times (0,\bar{t}), \tag{3.22}$$

*and the estimate*

$$\|u\|_{H^1(0,\bar{t};H^2_0(0,\ell))\cap W^{1,\infty}(0,\bar{t};H^1_0(0,\ell))\cap H^2(0,\bar{t};L^2(0,\ell))} \leqslant M_2. \tag{3.23}$$

Thus, also in view of (2.1), it is possible to associate with the Cauchy problem for (3.22) a solution operator

$$\begin{aligned}\mathcal{T}_2\colon\ \mathcal{X}_{\bar{t}} \to \mathcal{U}^\sharp_{\bar{t}} := \{u \in H^1(0,\bar{t}; H^2_0(0,\ell)) \cap W^{1,\infty}(0,\bar{t}; H^1_0(0,\ell)) \cap H^2(0,\bar{t}; L^2(0,\ell))\colon \\ \|u\|_{H^1(0,\bar{t};H^2_0(0,\ell))\cap W^{1,\infty}(0,\bar{t};H^1_0(0,\ell))\cap H^2(0,\bar{t};L^2(0,\ell))} \leqslant M_2\}.\end{aligned}$$

Our last auxiliary result concerns (the Cauchy problem for) (3.2 a), with fixed $\overline{\chi} \in \mathcal{X}_{\bar{t}}$ and $\tilde{u} \in \mathcal{U}^\sharp_{\bar{t}}$.

**Lemma 3.4.** *Assume* (2.6) *and* (2.8). *Then, there exist constants* $M_3, \theta_R > 0$, *only depending on* $R$, *on* $M_1$, $M_2$, *and on the problem data, but independent of* $\bar{t} \in (0,T]$, *such that for all* $(\overline{\chi}, \tilde{u}) \in \mathcal{X}_{\bar{t}} \times \mathcal{U}^\sharp_{\bar{t}}$ *there exists a unique function* $\vartheta$, *with the regularity* (2.12), *fulfilling* (1.10) *and*

$$\vartheta_t + \overline{\chi}_t\vartheta + A\vartheta = g + |\overline{\chi}_t|^2 + \overline{\chi}|\partial_x \tilde{u}_t|^2 \quad \text{a.e. in } (0,\ell)\times(0,\bar{t}), \tag{3.24}$$

$$\vartheta(x,t) > \theta_R > 0 \quad \forall\,(x,t) \in [0,\ell]\times[0,\bar{t}], \tag{3.25}$$

$$\|\vartheta\|_{L^2(0,\bar{t};H^2_N(0,\ell))\cap L^\infty(0,\bar{t};H^1(0,\ell))\cap H^1(0,\bar{t};L^2(0,\ell))\cap W^{1,\infty}(0,\bar{t};H^1(0,\ell)')} \leqslant M_3. \tag{3.26}$$

P r o o f. Since

$$\overline{\chi}_t \in L^2(0,\bar{t};H^1(0,\ell)) \cap L^\infty(0,\bar{t};L^2(0,\ell)), \tag{3.27}$$

one has $|\overline{\chi}_t|^2 \in L^1(0,\bar{t};L^\infty(0,\ell)) \cap L^\infty(0,\bar{t};L^1(0,\ell))$, so that, by an interpolation argument and trivial estimates,

$$|\overline{\chi}_t|^2 \in L^2(0,\bar{t};L^2(0,\ell)) \cap L^\infty(0,\bar{t};H^1(0,\ell)'). \tag{3.28}$$

In the same way, $\tilde{u}_t \in L^2(0,\bar{t};H_0^2(0,\ell)) \cap L^\infty(0,\bar{t};H_0^1(0,\ell))$ implies that

$$|\partial_x \tilde{u}_t|^2 \in L^1(0,\bar{t};L^\infty(0,\ell)) \cap L^\infty(0,\bar{t};L^1(0,\ell)),$$

whence, using that $\|\overline{\chi}\|_{L^\infty(0,\bar{t};L^\infty(0,\ell))} \leqslant C$, we easily deduce

$$\overline{\chi}|\partial_x \tilde{u}_t|^2 \in L^2(0,\bar{t};L^2(0,\ell)) \cap L^\infty(0,\bar{t};H^1(0,\ell)'). \tag{3.29}$$

In view of (3.27)–(3.29) and (2.6)–(2.8), the assumptions of [2, Theorem 3.2] are satisfied, hence there exists a unique

$$\vartheta \in L^2(0,\bar{t};H^1(0,\ell)) \cap \mathrm{C}^0([0,\bar{t}];L^2(0,\ell)) \cap H^1(0,\bar{t};H^1(0,\ell)') \tag{3.30}$$

satisfying (a suitably weak formulation of) the equation (3.24), supplemented with the initial condition (1.10). To infer the further regularity (2.12), we test (3.24) by $\vartheta_t$, integrate on $(0,t)$, and add the term $\frac{1}{2}\|\vartheta(t)\|^2$ to both sides. By straightforward calculations, we get

$$\begin{aligned}\int_0^t \|\vartheta_t\|^2 + \frac{1}{2}\|\vartheta(t)\|^2_{H^1(0,\ell)} &\leqslant \frac{1}{2}\|\vartheta_0\|^2_{H^1(0,\ell)} + \frac{1}{2}\|\vartheta\|^2_{L^\infty(0,\bar{t};L^2(0,\ell))} \\ &\quad + 2\|g\|^2_{L^2(0,T;L^2(0,\ell))} + \frac{1}{4}\int_0^t \|\vartheta_t\|^2 + I_1 + I_2 + I_3,\end{aligned} \tag{3.31}$$

where, exploiting (2.1), (3.28), and (3.29), we get

$$\begin{aligned} I_1 &= \int_0^t \|\overline{\chi}_t\|\|\vartheta\|_{L^\infty(0,\ell)}\|\vartheta_t\| \leqslant C\int_0^t \|\overline{\chi}_t\|^2\|\vartheta\|^2_{H^1(0,\ell)} + \frac{1}{4}\int_0^t \|\vartheta_t\|^2, \\ I_2 &= \int_0^t \||\overline{\chi}_t|^2\|\|\vartheta_t\| \leqslant \frac{1}{4}\int_0^t \|\vartheta_t\|^2 + C\||\overline{\chi}_t|^2\|^2_{L^2(0,t;L^2(0,\ell))}, \\ I_3 &= \int_0^t \|\overline{\chi}|\partial_x \tilde{u}_t|^2\|\|\vartheta_t\| \leqslant C + \frac{1}{4}\int_0^t \|\vartheta_t\|^2. \end{aligned} \tag{3.32}$$

Collecting (3.31)–(3.32) and applying the Gronwall Lemma, we arrive at an estimate for $\vartheta \in L^\infty(0,\bar{t};H^1(0,\ell)) \cap H^1(0,\bar{t};L^2(0,\ell))$. With an easy comparison argument (taking into account (3.28)–(3.29) and (2.6)) we first find an estimate for $A\vartheta$ in $L^2(0,\bar t;L^2(0,\ell))$ (hence for $\vartheta$ in $L^2(0,\bar t;H^2_N(0,\ell))$), and secondly for $\vartheta_t$ in $L^\infty(0,\bar t;H^1(0,\ell)')$. Therefore, (3.26) ensues.

We conclude (3.25) by invoking a refined version of the maximum principle for parabolic equations proved in [19, Proposition 3.6, p. 10], which yields

$$\vartheta(x,t) \geqslant \min_{x\in[0,\ell]} \vartheta_0(x)\exp\Big(-\int_0^{\bar t} \|\overline{\chi}_t(s)\|_{L^\infty(0,\ell)}\,\mathrm{d}s\Big). \tag{3.33}$$

□

Therefore, we may introduce the associated solution operator

$$\begin{aligned}\mathcal{T}_3\colon\ &\mathcal{X}_{\bar t}\times\mathcal{U}^\sharp_{\bar t}\to\mathcal{O}^\sharp_{\bar t}\\ &:=\{\vartheta\in L^2(0,\bar t;H^2_N(0,\ell))\\ &\qquad\cap L^\infty(0,\bar t;H^1(0,\ell))\cap H^1(0,\bar t;L^2(0,\ell))\\ &\qquad\cap W^{1,\infty}(0,\bar t;H^1(0,\ell)')\colon\ \vartheta>0\ \text{ a.e. in }\ (0,\ell)\times(0,\bar t)\\ &\qquad\|\vartheta\|_{L^2(0,\bar t;H^2_N(0,\ell))\cap L^\infty(0,\bar t;H^1(0,\ell))\cap H^1(0,\bar t;L^2(0,\ell))\cap W^{1,\infty}(0,\bar t;H^1(0,\ell)')}\leqslant M_3\}.\end{aligned}$$

The solution operator for the system (3.2) is hence

$$\mathcal{T}:\mathcal{O}_{\bar t}\times\mathcal{U}_{\bar t}\to\mathcal{O}^\sharp_{\bar t}\times\mathcal{U}^\sharp_{\bar t},$$

defined by

$$\mathcal{T}(\vartheta,u):=(\mathcal{T}_3(\mathcal{T}_1(\vartheta,u),\mathcal{T}_2(\mathcal{T}_1(\vartheta,u))),\mathcal{T}_2(\mathcal{T}_1(\vartheta,u)))\quad\text{for all }\ (\vartheta,u)\in\mathcal{O}_{\bar t}\times\mathcal{U}_{\bar t}.$$

**Proposition 3.5.** *Assume* (2.6)–(2.10), (W1)–(W5), *and* (2.11). *Then, there exists* $0<T_0\leqslant T$ *such that*

$$\text{the operator } \mathcal{T} \text{ maps } \mathcal{O}_{T_0}\times\mathcal{U}_{T_0} \text{ into itself.} \tag{3.34}$$

*Further,* $\mathcal{T}\colon\ \mathcal{O}_{T_0}\times\mathcal{U}_{T_0}\to\mathcal{O}_{T_0}\times\mathcal{U}_{T_0}$ *is compact and continuous with respect to the topology of* $(L^\infty(0,T_0;L^\infty(0,\ell))\cap H^1(0,T_0;H^1(0,\ell)'))\times H^1(0,T_0;W^{1,\infty}_0(0,\ell))$.

Proof. For all $\bar t\in(0,T]$ and $(\overline{\vartheta},\overline{u})\in\mathcal{O}_{\bar t}\times\mathcal{U}_{\bar t}$, we set $(\vartheta,u):=\mathcal{T}(\overline{\vartheta},\overline{u})$. It follows from (2.4) and (3.23) that

$$\begin{aligned}\forall\,\varepsilon>0\ \exists\,C_\varepsilon>0\colon\ \|u\|_{H^1(0,\bar t;W^{1,\infty}_0(0,\ell))}&\leqslant\varepsilon\|u\|_{H^1(0,\bar t;H^2_0(0,\ell))}+C_\varepsilon\|u\|_{H^1(0,\bar t;H^1_0(0,\ell))}\\ &\leqslant\varepsilon M_2+C_\varepsilon\bar t^{1/2}\|u\|_{W^{1,\infty}(0,\bar t;H^1_0(0,\ell))}\\ &\leqslant\varepsilon M_2+C_\varepsilon\bar t^{1/2}M_2.\end{aligned}$$

Thus, choosing $\varepsilon \leqslant R/(4M_2)$ and, setting, accordingly, $T_0^1 := R^2/(4M_2^2C_\varepsilon^2)$, we conclude that

$$\forall\, (\overline{\vartheta},\overline{u}) \in \mathcal{O}_{T_0^1} \times \mathcal{U}_{T_0^1} \quad \|u\|_{H^1(0,T_0^1;W_0^{1,\infty}(0,\ell))} \leqslant R.$$

In the same way, by (3.26) we have

$$\|\vartheta\|_{H^1(0,\bar{t};H^1(0,\ell)')} \leqslant \bar{t}^{\,1/2}\|\vartheta\|_{W^{1,\infty}(0,\bar{t};H^1(0,\ell)')} \leqslant \bar{t}^{\,1/2}M_3, \tag{3.35}$$

and, in view of (2.3), for every $\varepsilon > 0$ there exists $C_\varepsilon$ such that

(3.36)

$$\begin{aligned}
\|\vartheta(t)-\vartheta_0\|_{L^\infty(0,\ell)} &\leqslant \varepsilon\|\vartheta(t)-\vartheta_0\|_{H^1(0,\ell)} + C_\varepsilon\|\vartheta(t)-\vartheta_0\| \\
&\leqslant \varepsilon(\|\vartheta\|_{L^\infty(0,\bar{t};H^1(0,\ell))} + \|\vartheta_0\|_{H^1(0,\ell)} + C_\varepsilon\bar{t}^{1/2}\|\vartheta\|_{H^1(0,\bar{t};L^2(0,\ell))} \\
&\leqslant \varepsilon(M_3 + \|\vartheta_0\|_{H^1(0,\ell)}) + C_\varepsilon\bar{t}^{1/2}M_3,
\end{aligned}$$

the last inequality ensuing from (3.36). Arguing as in the above lines and exploiting (3.6), in view of (3.35)–(3.36) we find $T_0^2 \in (0,T]$ such that

$$\forall\, (\overline{\vartheta},\overline{u}) \in \mathcal{O}_{T_0^2} \times \mathcal{U}_{T_0^2} \quad \|\vartheta\|_{L^\infty(0,T_0^2;L^\infty(0,\ell))\cap H^1(0,T_0^2;H^1(0,\ell)')} \leqslant R.$$

Therefore, (3.34) ensues upon taking $T_0 := \min\{T_0^1, T_0^2\}$.

In order to prove that $\mathcal{T}$ is compact, we fix a bounded sequence $\{(\overline{\vartheta}_n,\overline{u}_n)\} \subset \mathcal{O}_{T_0} \times \mathcal{U}_{T_0}$. Using the Ascoli theorem and other standard compactness results (see, e.g., [27]), we find a subsequence and a limit pair $(\overline{\vartheta},\overline{u})$ such that the following convergences hold as $k \nearrow \infty$

$$\begin{aligned}
\overline{\vartheta}_{n_k} &\rightharpoonup^* \overline{\vartheta} \quad \text{in } L^\infty(0,T_0;L^\infty(0,\ell)) \cap H^1(0,T_0;H^1(0,\ell)'),\\
\overline{\vartheta}_{n_k} &\to \overline{\vartheta} \quad \text{in } C^0(0,T_0;H^1(0,\ell)'),\\
\overline{u}_{n_k} &\rightharpoonup^* \overline{u} \quad \text{in } H^1(0,T_0;W^{1,\infty}(0,\ell)),\\
\overline{u}_{n_k} &\to \overline{u} \quad \text{in } C^0(0,T_0;W^{1-\varrho,\infty}(0,\ell)) \ \text{ for all } \ \varrho \in (0,1].
\end{aligned} \tag{3.37}$$

On the other hand, setting $\overline{\chi}_{n_k} := \mathcal{T}_1(\overline{\vartheta}_{n_k},\overline{u}_{n_k})$ for every $k \in \mathbb{N}$, due to (3.12) we find that $\{\overline{\chi}_{n_k}\}$ is bounded in $L^\infty(0,T_0;H^2_N(0,\ell)) \cap H^1(0,T_0;H^1(0,\ell)) \cap W^{1,\infty}(0,T_0;L^2(0,\ell)) \cap H^2(0,T_0;H^1(0,\ell)')$. Therefore, there exists $\overline{\chi}$ such that, up to a further subsequence,

$$\begin{aligned}
\overline{\chi}_{n_k} &\rightharpoonup^* \overline{\chi} \quad \text{in } L^\infty(0,T_0;H^2_N(0,\ell)) \cap H^1(0,T_0;H^1(0,\ell))\\
&\qquad\qquad \cap W^{1,\infty}(0,T_0;L^2(0,\ell)) \cap H^2(0,T_0;H^1(0,\ell)'),\\
\overline{\chi}_{n_k} &\to \overline{\chi} \quad \text{in } \mathrm{C}^0(0,T_0;H^{2-\varrho}(0,\ell)) \cap W^{1,p}(0,T_0;L^2(0,\ell))\\
&\qquad\qquad \cap \mathrm{C}^1(0,T_0;H^1(0,\ell)')
\end{aligned} \tag{3.38}$$

for all $\varrho \in (0,2]$ and $1 \leqslant p < \infty$. Combining (3.37) and (3.38) it is not difficult to infer that (see the proof of [26, Proposition. 5.4] for details) $\overline{\chi} = \mathcal{T}_1(\overline{\vartheta}, \overline{u})$, so that (3.38) holds along the whole sequence $\{\overline{\chi}_{n_k}\}$.

In the same way, we set $\tilde{u}_{n_k} := \mathcal{T}_2(\overline{\chi}_{n_k})$ for all $k \in \mathbb{N}$: taking into account (3.38) as well as (3.23), we find that the sequence $\{\tilde{u}_{n_k}\}$ is bounded in $H^1(0,T_0; H_0^2(0,\ell)) \cap W^{1,\infty}(0,T_0;H_0^1(0,\ell)) \cap H^2(0,T_0;L^2(0,\ell))$. Hence, again by the Ascoli theorem and [27] there exists $\tilde{u}$ such that

$$
\begin{aligned}
(3.39)\quad & \tilde{u}_{n_k} \rightharpoonup^* \tilde{u} \\
& \text{in } H^2(0,T_0;L^2(0,\ell)) \cap W^{1,\infty}(0,T_0;H^1(0,\ell)) \cap H^1(0,T_0;H_0^2(0,\ell)), \\
& \tilde{u}_{n_k} \to \tilde{u} \\
& \text{in } H^1(0,T_0;H^{2-\varrho}(0,\ell)) \cap W^{1,p}(0,T_0;H^1(0,\ell)) \cap \mathrm{C}^1(0,T_0;H^{1-\varrho/2}(0,\ell)),
\end{aligned}
$$

for all $\varrho \in (0,2]$, $1 \leqslant p < \infty$. In particular,

$$
\tilde{u}_{n_k} \to \tilde{u} \quad \text{in } H^1(0,T_0;W^{1,\infty}(0,\ell)). \tag{3.40}
$$

Arguing as in [26, Proposition 5.4], we conclude that $\tilde{u} = \mathcal{T}_2(\overline{\chi}) = \mathcal{T}_2(\mathcal{T}_1(\overline{\vartheta}, \overline{u}))$.

Finally, we set $\vartheta_{n_k} := \mathcal{T}_3(\overline{\chi}_{n_k}, \tilde{u}_{n_k})$ for all $k \in \mathbb{N}$ and $\vartheta = \mathcal{T}_3(\overline{\chi}, \tilde{u})$: we are now going to show that, up to a further subsequence,

$$
\vartheta_{n_k} \to \vartheta \quad \text{in } L^\infty(0,T_0;L^\infty(0,\ell)), \tag{3.41}
$$

$$
\vartheta_{n_k} \to \vartheta \quad \text{in } H^1(0,T_0;H^1(0,\ell)'). \tag{3.42}
$$

To this aim, we subtract the equation (3.24) from (3.24), written for the triple $(\vartheta_{n_k}, \overline{\chi}_{n_k}, \tilde{u}_{n_k})$, thus getting

$$
\begin{aligned}
(3.43)\quad & (\vartheta_{n_k} - \vartheta)_t + \overline{\chi}_{n_k,t}(\vartheta_{n_k} - \vartheta) + \vartheta(\overline{\chi}_{n_k,t} - \overline{\chi}_t) + A(\vartheta_{n_k} - \vartheta) \\
& \quad = |\overline{\chi}_{n_k,t}|^2 - |\overline{\chi}_t|^2 + (\overline{\chi}_{n_k} - \overline{\chi})|\partial_x \tilde{u}_{n_k,t}|^2 + \overline{\chi}(|\partial_x \tilde{u}_{n_k,t}|^2 - |\partial_x \tilde{u}_t|^2).
\end{aligned}
$$

We then add $(\vartheta_{n_k} - \vartheta)$ to both sides of (3.43) and test it by $J^{-1}((\vartheta_{n_k} - \vartheta)_t)$. We obtain

$$
\int_0^t \|J^{-1}((\vartheta_{n_k} - \vartheta)_t)\|^2_{H^1(0,\ell)} + \frac{1}{2}\|\vartheta_{n_k}(t) - \vartheta(t)\|^2 = I_4 + I_5 + I_6 + I_7 + I_8, \tag{3.44}
$$

where, also exploiting (2.1),

$$
\begin{aligned}
I_4 &= \int_0^t\int_0^\ell \overline{\chi}_{n_k,t}(\vartheta_{n_k}-\vartheta)J^{-1}((\vartheta_{n_k}-\vartheta)_t)\\
&\leqslant \frac{1}{8}\int_0^t \|J^{-1}((\vartheta_{n_k}-\vartheta)_t)\|^2_{H^1(0,\ell)} + C\int_0^t \|\overline{\chi}_{n_k,t}\|^2\|\vartheta_{n_k}-\vartheta\|^2,\\
I_5 &= \int_0^t\int_0^\ell \vartheta(\overline{\chi}_{n_k,t}-\overline{\chi}_t)J^{-1}((\vartheta_{n_k}-\vartheta)_t)\\
&\leqslant \frac{1}{8}\int_0^t \|J^{-1}((\vartheta_{n_k}-\vartheta)_t)\|^2_{H^1(0,\ell)} + C\int_0^t \|\vartheta\|^2\|\overline{\chi}_{n_k,t}-\overline{\chi}_t\|^2,\\
I_6 &\leqslant \int_0^t\int_0^\ell |\overline{\chi}_{n_k,t}+\overline{\chi}_t||\overline{\chi}_{n_k,t}-\overline{\chi}_t|\|J^{-1}((\vartheta_{n_k}-\vartheta)_t)\|\\
&\leqslant \frac{1}{8}\int_0^t \|J^{-1}((\vartheta_{n_k}-\vartheta)_t)\|^2_{H^1(0,\ell)} + C\int_0^t \|\overline{\chi}_{n_k,t}+\overline{\chi}_t\|^2\|\overline{\chi}_{n_k,t}-\overline{\chi}_t\|^2,\\
I_7 &= \int_0^t\int_0^\ell (\overline{\chi}_{n_k}-\overline{\chi})|\partial_x\tilde{u}_{n_k,t}|^2J^{-1}((\vartheta_{n_k}-\vartheta)_t)\\
&\leqslant \frac{1}{8}\int_0^t \|J^{-1}((\vartheta_{n_k}-\vartheta)_t)\|^2_{H^1(0,\ell)} + C\int_0^t \|\overline{\chi}_{n_k}-\overline{\chi}\|^2\|\partial_x\tilde{u}_{n_k,t}\|^4_{L^4(0,\ell)},\\
I_8 &= \int_0^t\int_0^\ell \overline{\chi}(|\partial_x\tilde{u}_{n_k,t}|^2-|\partial_x\tilde{u}_t|^2)J^{-1}((\vartheta_{n_k}-\vartheta)_t)\\
&\leqslant \frac{1}{8}\int_0^t \|J^{-1}((\vartheta_{n_k}-\vartheta)_t)\|^2_{H^1(0,\ell)}\\
&\quad + C\int_0^t \|\overline{\chi}\|^2_{H^1(0,\ell)}\|\partial_x\tilde{u}_{n_k,t}+\partial_x\tilde{u}_t\|^2\|\partial_x\tilde{u}_{n_k,t}-\partial_x\tilde{u}_t\|^2.
\end{aligned}
$$

Using now the convergences (3.38) and (3.39), it is easy to deduce from (3.44) that (3.42) holds, as well as $\vartheta_{n_k}\to\vartheta$ in $L^2(0,T_0;L^2(0,\ell))$. On the other hand, taking into account (3.26) we find that $\{\vartheta_{n_k}\}$ is bounded in $L^2(0,T_0;H^2_N(0,\ell))\cap L^\infty(0,T_0;H^1(0,\ell))\cap H^1(0,T_0;L^2(0,\ell))\cap W^{1,\infty}(0,T_0;H^1(0,\ell)')$. Hence, again due to (2.1) and [27] we find that $\{\vartheta_{n_k}\}$ is compact in $L^\infty(0,T_0;L^\infty(0,\ell))$, so that (3.41) ensues.

Collecting (3.40) and (3.41)–(3.42), we conclude that $\mathcal{T}\colon\ \mathcal{O}_{T_0}\times\mathcal{U}_{T_0}\to\mathcal{O}_{T_0}\times\mathcal{U}_{T_0}$ is compact. In fact, the very same arguments we have developed yield that $\mathcal{T}$ is continuous, too. □

### 3.3. Global estimates

**The notion of** ***δ-separated solution*****.** It follows from Proposition 3.5 that the Cauchy problem for the system (3.2) admits a solution $(\widehat{\vartheta},\widehat{\chi},\widehat{u})$, in fact fulfilling (3.4)–(3.5). Therefore, $(\widehat{\vartheta},\widehat{\chi},\widehat{u})$ is in fact a solution of Problem 1 in $(0,T_0)$.

We now introduce a notion of local solution to Problem 1 which generalizes the properties of $(\widehat{\vartheta},\widehat{\chi},\widehat{u})$, in particular retaining (3.4)–(3.5).

**Definition 3.6.** We say that a triple $(\vartheta, \chi, u)$ is a $\delta$*-separated solution* of Problem 1 on some interval $(0, \bar{t})$, $0 < \bar{t} \leqslant T$, if

$$
\begin{gathered}
(\vartheta, \chi, u) \text{ solves Problem 1 on } (0, \ell) \times (0, \bar{t}), \\
\chi \text{ satisfies (3.4)–(3.5) on } (0, \bar{t}), \text{ and } \min_{(x,t) \in [0,\ell] \times [0,\bar{t}]} \vartheta(x, t) > 0.
\end{gathered}
$$

**Global estimates for $\delta$-separated solutions**

**Lemma 3.7** (Global estimates)**.** *Under the assumptions of Theorem* 1*, there exist a constant $M_4 > 0$ only depending on the problem data but independent of $\bar{t} \in (0, T]$, and a constant $M_5(T) > 0$, depending on the problem data and on $T$, but not on $\bar{t} \in (0, T)$, such that for any $\delta$-separated solution $(\vartheta, \chi, u)$ of Problem* 1 *on the interval $(0, \bar{t})$ there holds*

$$
\begin{aligned}
(3.45) \quad & \|\chi\|_{L^\infty(0,\bar{t};H^1(0,\ell))} + \|\chi_t\|_{L^2(0,\bar{t};L^2(0,\ell))} + \|\vartheta\|_{L^\infty(0,\bar{t};L^1(0,\ell))} + \|u\|_{L^\infty(0,\bar{t};H^1_0(0,\ell))} \\
& \quad + \|u_t\|_{L^\infty(0,\bar{t};L^2(0,\ell)) \cap L^2(0,\bar{t};H^1_0(0,\ell))} \\
& \quad + \|(1-\chi)^{1/2} \partial_x u\|_{L^2(0,\bar{t};L^2(0,\ell)) \cap L^\infty(0,\bar{t};L^2(0,\ell))} \\
& \leqslant M_4 (1 + \|\chi_0\|_{H^1(0,\ell)} + \|\vartheta_0\|_{L^1(0,\ell)} + \|u_0\|_{H^1_0(0,\ell)} + \|v_0\| \\
& \qquad + \|g\|_{L^1(0,T;L^1(0,\ell))} + \|f\|_{L^2(0,T;L^2(0,\ell))}),
\end{aligned}
$$

$$
\begin{aligned}
(3.46) \quad & \|\chi\|_{L^\infty(0,\bar{t};H^2_N(0,\ell)) \cap H^1(0,\bar{t};H^1(0,\ell)) \cap W^{1,\infty}(0,\bar{t};L^2(0,\ell))} \\
& + \|\vartheta\|_{L^2(0,\bar{t};H^2_N(0,\ell)) \cap L^\infty(0,\bar{t};H^1(0,\ell)) \cap H^1(0,\bar{t};L^2(0,\ell))} \\
& + \|u\|_{H^1(0,\bar{t};H^2_0(0,\ell)) \cap W^{1,\infty}(0,\bar{t};H^1_0(0,\ell)) \cap H^2(0,\bar{t};L^2(0,\ell))} \\
& \leqslant M_5(T) (1 + \|\chi_0\|_{H^2_N(0,\ell)} + \|\vartheta_0\|_{H^1(0,\ell)} + \|u_0\|_{H^2_0(0,\ell)} + \|v_0\|_{H^1_0(0,\ell)} \\
& + \|g\|_{L^2(0,T;L^2(0,\ell)) \cap L^\infty(0,T;H^1(0,\ell)')} + \|f\|_{L^2(0,T;L^2(0,\ell))}).
\end{aligned}
$$

P r o o f. Throughout this proof, we shall denote by $C$ and by $S_i$, $i = 1, \ldots$, some positive constants only depending on $\delta$ and on the quantities $\|\chi_0\|_{H^2_N(0,\ell)}$, $\|\vartheta_0\|_{H^1(0,\ell)}$, $\|u_0\|_{H^2_0(0,\ell)}$, $\|v_0\|_{H^1_0(0,\ell)}$, $\|g\|_{L^2(0,T;L^2(0,\ell)) \cap L^\infty(0,T;H^1(0,\ell)')}$, and $\|f\|_{L^2(0,T;L^2(0,\ell))}$, but independent of $T$. We will explicitly denote their dependence on $T$ by the symbol $C(T)$ and $S_i(T)$, $i = 1, \ldots$, whenever it occurs.

**First estimate.** We test (2.17) by $u_t$, (2.16) by $\chi_t$, and (2.15) by 1. We add them up and integrate the resulting equation in time. Some terms cancel out and, recalling the positivity of $\vartheta$ (according to Definition 3.6), by elementary computations we find that

$$\begin{aligned}
&\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|(1-\chi(t))^{1/2}\partial_x u(t)\|^2 + \|\vartheta(t)\|_{L^1(0,\ell)} + \frac{1}{2}\|\partial_x\chi(t)\|^2 \\
&\quad + \int_0^\ell (\widehat{\beta}(\chi(t)) + \widehat{\gamma}(\chi(t))) \\
&\leqslant \frac{1}{2}\|u_0\|^2_{H^1(0,\ell)} + \frac{1}{2}\|v_0\|^2 + \|\vartheta_0\|_{L^1(0,\ell)} + C\|\chi_0\|^2_{H^1(0,\ell)} \\
&\qquad + \|\widehat{\beta}(\chi_0)\|_{L^1(0,\ell)} + \|g\|_{L^1(0,T;L^1(0,\ell))} + \frac{1}{2}\|f\|^2_{L^2(0,T;L^2(0,\ell))}.
\end{aligned}$$

On the other hand, by the convexity of $\widehat{\beta}$ and due to (W2)–(W3), there exists a positive constant $C$ such that

$$\int_0^\ell (\widehat{\beta}(\chi(t)) + \widehat{\gamma}(\chi(t))) \geqslant -C.$$

Since, by (W5) and (2.11), $\|\widehat{\beta}(\chi_0)\|_{L^1(0,\ell)} \leqslant C\|\chi_0\|_{H^1(0,\ell)}$, we ultimately find a positive constant $S_1$ such that

$$\begin{aligned}
\|u_t\|_{L^\infty(0,\bar{t};L^2(0,\ell))} + \|(1-\chi)^{1/2}\partial_x u\|_{L^\infty(0,\bar{t};L^2(0,\ell))} + \|\vartheta\|_{L^\infty(0,\bar{t};L^1(0,\ell))}& \\
+ \|\chi\|_{L^\infty(0,\bar{t};H^1(0,\ell))} \leqslant S_1.&
\end{aligned} \tag{3.47}$$

**Second estimate.** We test (2.17) by $u_t + u$, integrate in time. Integrating the first integral on the right-hand side by parts in time, using Poincaré inequality, and exploiting the previous estimate (3.47), we get

$$\begin{aligned}
&\frac{1}{2}\|u_t(t)\|^2 + \int_0^t\int_0^\ell (1-\chi)|\partial_x u|^2 + C\|u(t)\|^2_{H_0^1(0,\ell)} + \int_0^t\int_0^\ell \chi|\partial_x u_t|^2 \\
&\quad\leqslant C(\|v_0\|^2 + \|u_0\|^2_{H_0^1(0,\ell)} + \|f\|_{L^2(0,T;L^2(0,\ell))}) \\
&\qquad + \int_0^\ell |u_t(t)u(t)| + \int_0^t\int_0^\ell |u_t|^2 \\
&\quad\leqslant C(\|v_0\|^2 + \|u_0\|^2_{H_0^1(0,\ell)} + \|f\|_{L^2(0,T;L^2(0,\ell))}) \\
&\qquad + \|u_t(t)\|^2 + \frac{1}{4}\|u(t)\|^2 + \int_0^t\int_0^\ell |u_t|^2.
\end{aligned}$$

Applying the Gronwall Lemma and taking into account (3.47) and the fact that $\chi \geqslant \delta$, we deduce that for some positive constant $S_2$,

$$\|u_t\|_{L^2(0,\bar{t};H^1(0,\ell))} + \|u\|_{L^\infty(0,\bar{t};H_0^1(0,\ell))} + \|(1-\chi)^{1/2}\partial_x u\|_{L^2(0,\bar{t};L^2(0,\ell))} \leqslant S_2. \tag{3.48}$$

Hence, by comparison in (2.17), we get

$$\|u_{tt}\|_{L^2(0,\bar{t};H^1(0,\ell)')} \leqslant S_3. \tag{3.49}$$

**Third estimate.** Following the outline of [29, Section 5], we multiply equation (2.15) by $-\vartheta^{-1}$ (note that this is admissible as $\min_{[0,\ell]\times[0,t]} \vartheta > 0$) and integrate it over $(0,\ell)\times(0,t)$. Using (2.8), (2.6) and (3.47), we get

$$\begin{aligned} -\int_\Omega \log(\vartheta(t)) + \int_0^t\int_0^\ell \Big(\frac{|\partial_x\vartheta|^2}{\vartheta^2} + \frac{|\chi_t|^2}{\vartheta} + \frac{|\partial_x u_t|^2}{\vartheta}\Big) \\ \leqslant -\int_0^\ell \log\vartheta_0 + \int_0^t\int_0^\ell \chi_t - \int_0^t\int_0^\ell \frac{g}{\vartheta} \leqslant C + \frac{1}{2}\int_0^t\int_0^\ell \frac{|\chi_t|^2}{\vartheta}. \end{aligned} \tag{3.50}$$

Since, due to (3.47), the first term on the left-hand side in (3.50) is bounded from below, we get

$$\int_0^t\int_0^\ell \Big(\frac{|\partial_x\vartheta|^2}{\vartheta^2} + \frac{|\chi_t|^2}{\vartheta} + \frac{|\partial_x u_t|^2}{\vartheta}\Big) \leqslant C. \tag{3.51}$$

Using now (2.1), (3.47), and (3.50), we obtain

$$\begin{aligned} \int_0^t \|\vartheta\|_{L^\infty(0,\ell)} &= \int_0^t \|\vartheta^{1/2}\|^2_{L^\infty(0,\ell)} \\ &\leqslant C(T)\int_0^t (\|\partial_x(\vartheta^{1/2})\|^2_{L^1((0,\ell))} + \|\vartheta\|_{L^1((0,\ell))}) \\ &\leqslant C(T)\Big(1 + \int_0^t\Big(\int_0^\ell\Big|\frac{\partial_x\vartheta}{\vartheta^{1/2}}\Big|\Big)^2\Big) \\ &\leqslant C(T)\Big(1 + \int_0^t\Big(\Big\|\frac{\partial_x\vartheta}{\vartheta}\Big\|\|\vartheta^{1/2}\|\Big)^2\Big) \leqslant C(T). \end{aligned}$$

Hence, we have

$$\|\vartheta\|_{L^1(0,\bar{t};L^\infty(0,\ell))} \leqslant C(T), \tag{3.52}$$

and so, interpolating between (3.47) and (3.52), we get

$$\|\vartheta\|_{L^2(0,\bar{t};L^2(0,\ell))} \leqslant S_4(T). \tag{3.53}$$

**Fourth estimate.** We test now equation (2.16) by $\chi_t$. With standard computations, also relying on (3.47)–(3.48), (3.53), and integrating by parts in time the term $1/2\int_0^t\int_0^\ell \chi_t|\partial_x u|^2$, we get

$$\|\chi\|_{L^\infty(0,\bar{t};H^1(0,\ell))\cap H^1(0,\bar{t};L^2(0,\ell))} \leqslant S_5(T). \tag{3.54}$$

Arguing exactly in the same way as in the proof of [26, Lemma 5.9], we find by comparison with (2.16) that

$$\|\chi\|_{L^2(0,\bar{t};W^{2,1}(0,\ell))} \leqslant S_6(T), \tag{3.55}$$

which, via the Gagliardo-Nirenberg inequality [25, p. 125], leads to

$$\|\chi\|_{L^4(0,\bar{t};W^{1,4}(0,\ell))} \leqslant S_7(T). \tag{3.56}$$

**Fifth estimate.** We test (2.17) by $-\partial^2_{xx}u_t$ and integrate in time. Referring to the proof of [26, Lemma 5.3, Lemma 5.9] for all the detailed computations (which rely on (3.47), (3.48), (3.49), and (3.56)), we get

$$\|u\|_{H^1(0,\bar{t};H^2_0(0,\ell))\cap W^{1,\infty}(0,\bar{t};H^1(0,\ell))\cap H^2(0,\bar{t};L^2(0,\ell))} \leqslant S_8(T). \tag{3.57}$$

**Sixth estimate.** We take the time derivative of (2.16) (note that this procedure is justified by (W5) and (3.4)–(3.5)) and multiply it by $\chi_t/4$. We test equation (2.15) by $J^{-1}\vartheta_t$ (cf. (2.5)). We sum up the two resulting equations and integrate over $(0,t)$. Using (W3) and (2.8), (2.9), we get

$$\|\vartheta_t\|^2_{L^2(0,t;H^1(0,\ell)')}+\frac{1}{2}\|\vartheta(t)\|^2+\frac{1}{8}\|\chi_t(t)\|^2+\frac{1}{4}\|\partial_x\chi_t\|^2_{L^2(0,t;L^2(0,\ell))} \leqslant C+\sum_{i=9}^{14} I_i, \tag{3.58}$$

where, using (2.1), (W2), and (3.54), we have

$$\begin{aligned}
I_9 &:= -\int_0^t\int_0^\ell \vartheta\chi_t J^{-1}(\vartheta_t) \leqslant \frac{1}{4}\|\vartheta_t\|^2_{L^2(0,t;H^1(0,\ell)')} + \int_0^t \|\vartheta\|^2\|\chi_t\|^2, \\
I_{10} &:= \int_0^t\int_0^\ell |\chi_t|^2 J^{-1}(\vartheta_t) \leqslant \frac{1}{4}\|\vartheta_t\|^2_{L^2(0,t;H^1(0,\ell)')} + \int_0^t \|\chi_t\|^2\|\chi_t\|^2, \\
I_{11} &:= \int_0^t\int_0^\ell \chi|\partial_x u_t| J^{-1}(\vartheta_t) \leqslant \frac{1}{4}\|\vartheta_t\|^2_{L^2(0,t;H^1(0,\ell)')} + \int_0^t \|\chi\|^2_{L^\infty(0,\ell)}\|\partial_x u_t\|^2, \\
I_{12} &:= -1/4\int_0^t\int_0^\ell \gamma'(\chi)\chi_t^2 \leqslant C\|\chi_t\|^2_{L^2(0,\bar{t};L^2(0,\ell))} \leqslant C(T), \\
I_{13} &:= 1/4\int_0^t \langle\vartheta_t,\chi_t\rangle \leqslant \frac{1}{8}\|\vartheta_t\|^2_{L^2(0,t;H^1(0,\ell)')} + \frac{1}{8}\|\chi_t\|^2_{L^2(0,t;H^1(0,\ell))}, \\
I_{14} &:= 1/4\int_0^t\int_0^\ell \partial_x u\partial_x u_t\chi_t \leqslant C\|\partial_x u\|^2_{L^2(0,t;H^1_0(0,\ell))} + \int_0^t \|\partial_x u_t\|^2\|\chi_t\|^2.
\end{aligned} \tag{3.59}$$

Using Gronwall lemma along with estimates (3.53)–(3.54) and (3.57), we get

$$\|\vartheta\|_{L^\infty(0,\bar{t};L^2(0,\ell))\cap H^1(0,\bar{t};H^1(0,\ell)')}+\|\chi\|_{H^1(0,\bar{t};H^1(0,\ell))\cap W^{1,\infty}(0,\bar{t};L^2(0,\ell))} \leqslant S_9(T). \tag{3.60}$$

**Seventh estimate.** We multiply (2.15) by $\vartheta_t$ and integrate in time. In view of the estimates (3.57) and (3.60), we may argue exactly in the same way as in the proof of Lemma 3.4, and thus we find

$$\|\vartheta\|_{L^2(0,\bar t;H^2_N(0,\ell))\cap L^\infty(0,\bar t;H^1(0,\ell))\cap H^1(0,\bar t;L^2(0,\ell))\cap W^{1,\infty}(0,\bar t;H^1(0,\ell)')} \leqslant S_{10}(T). \tag{3.61}$$

**Eighth estimate.** We test (2.16) by $(A\chi+\beta(\chi))_t$ and integrate in time. Hence, we develop the very same computations as in the proof of [26, Lemma 4.2]: using (3.57) and (3.61), we get

$$\|\chi\|_{L^\infty(0,\bar t;H^2_N(0,\ell))\cap H^1(0,\bar t;H^1(0,\ell))\cap W^{1,\infty}(0,\bar t;L^2(0,\ell))} \leqslant S_{11}(T). \tag{3.62}$$

□

### 3.4. Conclusion of the proof of global existence for Problem 1

We now introduce the set (cf. with Definition 3.6)

$$\mathscr{T} := \{\bar t \in (0,T]\colon \text{ there exists a } \delta\textit{-separated solution on } (0,\bar t)\}. \tag{3.63}$$

Of course, $\mathscr{T} \neq \emptyset$, as $T_0 \in \mathscr{T}$ thanks to Proposition 3.5. We let $T^* = \sup \mathscr{T}$ and, without loss of generality, suppose that $T^* > T_0$. Following the argument of [26, Section 5.5], (which we shall sketch here in order to make the paper more readable), we are going to show first that

$$T^* \in \mathscr{T}, \tag{3.64}$$

(so that $T^* = \max \mathscr{T}$), and secondly that

$$T^* = T. \tag{3.65}$$

In this way, we shall conclude the existence of a *global* $\delta$-separated solution $(\vartheta,\chi,u)$ to Problem 1 on $(0,T)$.

Proof of (3.64). By definition of $T^*$, there exists a sequence $\{\bar t_n\} \subset (0,T^*]$, with $\bar t_n \nearrow T^*$, such that for all $n \in \mathbb{N}$ there exists a $\delta$-separated solution $(\vartheta_n,\chi_n,u_n)$ on $(0,\bar t_n)$. In view of Proposition 3.8 later on, the triple $(\vartheta_n,\chi_n,u_n)$ is in fact an extension of the local solution $(\widehat\vartheta,\widehat\chi,\widehat u)$. Let us now extend $(\vartheta_n,\chi_n,u_n)$ to the interval $[0,T^*]$ by setting

$$\widetilde\vartheta_n(t) := \begin{cases} \vartheta_n(t), & t \in [0,\bar t_n],\\ \vartheta_n(\bar t_n), & t \in (\bar t_n, T^*], \end{cases}$$
$$\widetilde\chi_n(t) := \begin{cases} \chi_n(t), & t \in [0,\bar t_n],\\ \chi_n(\bar t_n), & t \in (\bar t_n, T^*], \end{cases}$$
$$\widetilde u_n(t) := \begin{cases} u_n(t), & t \in [0,\bar t_n],\\ \partial_t u_n(\bar t_n)(t-\bar t_n) + u_n(\bar t_n), & t \in (\bar t_n, T^*]. \end{cases}$$

The global estimates of Lemma 3.7 yield that there exists a positive constant $M_6(T)$ fulfilling

(3.66)
$$\left.\begin{aligned}\|\widetilde{\vartheta}_n\|_{L^2(0,T^*;H^2_N(0,\ell))\cap L^\infty(0,T^*;H^1(0,\ell))\cap H^1(0,T^*;L^2(0,\ell))\cap W^{1,\infty}(0,T^*;H^1(0,\ell)')}\\ \|\widetilde{\chi}_n\|_{L^\infty(0,T^*;H^2_N(0,\ell))\cap H^1(0,T^*;H^1(0,\ell))\cap W^{1,\infty}(0,T^*;L^2(0,\ell))}\\ \|\widetilde{u}_n\|_{H^1(0,T^*;H^2_0(0,\ell))\cap W^{1,\infty}(0,T^*;H^1_0(0,\ell))\cap H^2(0,T^*;L^2(0,\ell))}\end{aligned}\right\}\leqslant M_6(T).$$

Thus, recalling the proof of Lemma 3.1 (in particular, cf., (3.16)–(3.19)), we conclude that there exists a constant $\zeta^*\in(0,1)$, only depending on $M_6(T)$, such that the functions $\widetilde{\chi}_n$ fulfil the separation inequality

$$\widetilde{\chi}_n(x,t)\leqslant\zeta^*\quad\forall\,(x,t)\in[0,\ell]\times[0,T^*]\ \ \forall\,n\in\mathbb{N}. \tag{3.67}$$

In the same way, again by [19] (see (3.33)) we have

$$\begin{aligned}\widetilde{\vartheta}_n(x,t)&\geqslant\min_{x\in[0,\ell]}\vartheta_0(x)\exp\left(-\int_0^{\bar t}\|\widetilde{\chi}_{n,t}(s)\|_{L^\infty(0,\ell)}\,\mathrm{d}s\right)\\&\geqslant\min_{x\in[0,\ell]}\vartheta_0(x)\exp(-C\sqrt{T^*}M_6(T))\quad\forall\,n\in\mathbb{N},\end{aligned} \tag{3.68}$$

where $C$ depends on the embedding constant in (2.1). Now, (3.66) and [27, Theorem 4, Corollary 5] imply that there exists a triple $(\vartheta^*,\chi^*,u^*)$ such that, along a (not relabelled) subsequence, $(\widetilde{\vartheta}_n,\widetilde{\chi}_n,\widetilde{u}_n)$ converges to $(\vartheta^*,\chi^*,u^*)$ in suitable topologies, all of which we do not specify for the sake of simplicity. In particular, thanks to (2.1) we have

$$\widetilde{\vartheta}_n\to\vartheta^*\quad\text{and}\quad\widetilde{\chi}_n\to\chi^*\ \text{ in }\ \mathrm{C}^0([0,T^*];\mathrm{C}^0([0,\ell])). \tag{3.69}$$

Thus, the triple $(\vartheta^*,\chi^*,u^*)$ solves the Cauchy problem for system (2.15)–(2.17) on the interval $[0,T^*]$ and, combining (3.69) with (3.4) and (3.67)–(3.68), fulfils

$$\begin{gathered}\min_{(x,t)\in[0,\ell]\times[0,T^*]}\vartheta^*(x,t)>0,\qquad\max_{(x,t)\in[0,\ell]\times[0,T^*]}\chi^*(x,t)<1,\\ \chi^*(x,t)\geqslant\delta\quad\forall\,(x,t)\in(0,\ell)\times[0,T^*].\end{gathered} \tag{3.70}$$

Therefore, $(\vartheta^*,\chi^*,u^*)$ is in fact a $\delta$-separated solution to Problem 1 on the interval $(0,T^*)$, and (3.64) ensues. □

P r o o f of (3.65). Suppose now by contradiction that $T^*<T$. Now, by the previous step the quadruple $(\vartheta^*(T^*),\chi^*(T^*),u^*(T^*),\partial_t u^*(T^*))$ complies with (2.8)–(2.10) and (2.11), and, in view of (3.70), provides a set of admissible initial data for

Problem 1. Therefore, arguing in the same way as throughout Sections 3.1–3.2, we find that Problem 1, supplemented with the initial conditions

$$\vartheta(T^*) = \vartheta^*(T^*), \quad \chi(T^*) = \chi^*(T^*), \quad u(T^*) = u^*(T^*) \quad \text{in } (0,\ell),$$

has a $\delta$-separated solution $(\vartheta_\sharp, \chi_\sharp, u_\sharp)$ on some interval $[T^*, T^* + \eta]$. Gluing $(\vartheta^*, \chi^*, u^*)$ and $(\vartheta_\sharp, \chi_\sharp, u_\sharp)$ (see [26, Section 5.5]), we arrive at a contradiction. □

P r o o f of (2.22). Finally, the further regularity (2.22) of $\chi$ can be shown arguing exactly in the same way as in the proof Lemma 3.1, see (3.21) and the ensuing calculations. □

### 3.5. Uniqueness for Problem 1

Our continuous dependence result for Problem 1 holds under weaker regularity requirements for the solution component $\chi$.

**Proposition 3.8.** *Assume* (W1)–(W5), *let* $(g_i, f_i, \vartheta_0^i, \chi_0^i, u_0^i, v_0^i)$, $i = 1,2$, *be two sets of data for Problem* 1, *complying with* (2.6)–(2.10), *and, accordingly, let* $(\vartheta_i, \chi_i, u_i)$, $i = 1,2$, *be two associated solution triples on* $(0,T)$, *such that the functions* $\vartheta_i$, $u_i$, $i = 1,2$, *enjoy the regularity* (2.12), (2.14), *and* $\chi_i$ *fulfil*

$$\chi_i \in H^1(0,T;H^1(0,\ell)) \cap L^\infty(0,T;H^2_N(0,\ell)), \quad i = 1,2. \tag{3.71}$$

*Further, suppose that for some* $\nu > 0$

$$0 < \nu \leqslant \chi_i(x,t) \leqslant 1-\nu \quad \forall\,(x,t) \in (0,\ell)\times[0,T] \quad \text{for } i = 1,2. \tag{3.72}$$

*Set*

$$\mathcal{M} := \max_{i=1,2}\{\|\chi_i\|_{H^1(0,T;H^1(0,\ell))} + \|u_i\|_{H^1(0,T;H^2_0(0,\ell))} + \|\vartheta_i\|_{L^2(0,T;H^2_N(0,\ell))}\}.$$

*Then, there exists a positive constant* $S_0$, *depending on* $\mathcal{M}$, $\nu$, $T$, *and* $\ell$, *such that*

$$\begin{aligned}
&\|u_1 - u_2\|_{W^{1,\infty}(0,T;L^2(0,\ell))\cap H^1(0,T;H^1_0(0,\ell))} \\
&\quad + \|\chi_1 - \chi_2\|_{H^1(0,T;L^2(0,\ell))\cap L^\infty(0,T;H^1(0,\ell))} \\
&\quad + \|\vartheta_1 - \vartheta_2\|_{L^\infty(0,T;L^2(0,\ell))\cap L^2(0,T;H^1(0,\ell))} \\
&\leqslant S_0(\|u_0^1 - u_0^2\|_{H^1(0,\ell)} + \|v_0^1 - v_0^2\| + \|\chi_0^1 - \chi_0^2\|_{H^1(0,\ell)} + \|\vartheta_0^1 - \vartheta_0^2\| \\
&\qquad + \|f_1 - f_2\|_{L^2(0,T;H^{-1}(0,\ell))} + \|g_1 - g_2\|_{L^2(0,T;H^1(0,\ell)')}).
\end{aligned} \tag{3.73}$$

P r o o f. Since some of the computations we are going to develop are similar to the ones contained in the proof of [26, Proposition 4.8] and [8, Thoerem 2.4], we do not detail all of them and refer the reader to [26]. Let $(\vartheta_i, \chi_i, u_i)$, $i = 1, 2$, be two solution triples like in the above statement and set $(\vartheta, \chi, u) := (\vartheta_1 - \vartheta_2, \chi_1 - \chi_2, u_1 - u_2)$. Clearly, the triple $(\vartheta, \chi, u)$ fulfils a.e. in $(0, \ell) \times (0, T)$

$$
\begin{aligned}
&(3.74)\quad \vartheta_t + \chi_{1,t}\vartheta + \chi_t\vartheta_2 + A\vartheta = g_1 - g_2 + \chi_t(\chi_{1,t} + \chi_{2,t}) + \chi|\partial_x u_{1,t}|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \chi_2 \partial_x u_t(\partial_x u_{1,t} + \partial_x u_{2,t}),\\
&(3.75)\quad \chi_t + A\chi + \beta(\chi_1) - \beta(\chi_2) + \gamma(\chi_1) - \gamma(\chi_2) = \vartheta + \frac{1}{2}(|\partial_x u_1|^2 - |\partial_x u_2|^2),\\
&(3.76)\quad u_{tt} + \mathcal{H}((1-\chi_1)u) - \mathcal{H}(\chi u_2) + \mathcal{H}(\chi_1 u_t) - \mathcal{H}(\chi \partial_t u_2) = f_1 - f_2.
\end{aligned}
$$

Now, we test (3.76) by $u_t$ and integrate in time. Proceeding exactly like in [26, Proposition 4.8], we get

$$
\begin{aligned}
(3.77)\quad &\frac{1}{2}\|u_t(t)\|^2 + \frac{\nu}{2}\int_0^t \|u_t\|^2_{H_0^1(0,\ell)}\\
&\leqslant \frac{1}{2}\|v_0^1 - v_0^2\|^2 + C\|f_1 - f_2\|^2_{L^2(0,T;H^{-1}(0,\ell))} + \frac{\nu}{4}\int_0^t \|u_t\|^2_{H^1(0,\ell)}\\
&\quad + C\|u_0^1 - u_0^2\|^2_{H^1(0,\ell)} + C\int_0^t \left(\int_0^s \|u_t(r)\|^2_{H_0^1(0,\ell)}\,\mathrm{d}r\right)\mathrm{d}s\\
&\quad + C\int_0^t (\|u_2\|_{H_0^2(0,\ell)} + \|u_{2,t}\|_{H_0^2(0,\ell)})^2 \|\chi\|^2_{H^1(0,\ell)}.
\end{aligned}
$$

Next, we test (3.75) by $\chi_t$ and integrate the resulting equation in time. By elementary computations, also taking into account the Lipschitz continuity of $\gamma$ and combining (3.72) with (W5), we get

$$
\begin{aligned}
(3.78)\quad &\int_0^t \|\chi_t\|^2 + \|\chi(t)\|^2_{H^1(0,\ell)}\\
&\leqslant C_{\mathcal{M}}\Bigg(\|\chi_0^1 - \chi_0^2\|^2_{H^1(0,\ell)} + \|u_0^1 - u_0^2\|^2_{H_0^1(0,\ell)} + \int_0^t \|\vartheta\|^2\\
&\quad + \int_0^t \|\chi\|^2 + \int_0^t (\|\partial_x u_1\|^2_{H_0^1(0,\ell)} + \|\partial_x u_{2,t}\|^2_{H_0^1(0,\ell)})\|\chi\|^2_{H_1(0,\ell)}\\
&\quad + \int_0^t \|u_t\|^2_{H_0^1(0,\ell)} + \int_0^t\int_0^s \|u_t(r)\|^2_{H_0^1(0,\ell)}\,\mathrm{d}r\,\mathrm{d}s\Bigg),
\end{aligned}
$$

where the constant $C_{\mathcal{M}}$ depends on $\mathcal{M}$ as well. Finally, we test (3.74) by $\vartheta$. We integrate in time and add $\int_0^t \|\vartheta\|^2$ to both sides, thus obtaining

$$
(3.79)\qquad \frac{1}{2}\|\vartheta(t)\|^2 + \int_0^t \|\vartheta\|^2_{H^1(0,\ell)} \leqslant \frac{1}{2}\|\vartheta_0^1 - \vartheta_0^2\|^2 + \int_0^t \|\vartheta\|^2 + \sum_{i=15}^{20} I_i
$$

where, taking into account (2.1), we estimate

$$I_{15} = \int_0^t \int_0^\ell g\vartheta \leqslant \frac{1}{4}\int_0^t \|\vartheta\|^2_{H^1(0,\ell)} + C\int_0^t \|g\|^2_{H^1(0,\ell)'}, \tag{3.80}$$

$$I_{16} = \int_0^t \|\chi_{1,t}\|_{L^\infty(0,\ell)}\|\vartheta\|^2 \leqslant \frac{1}{2}\int_0^t \|\vartheta\|^2 + C\int_0^t \|\chi_{1,t}\|^2_{H^1(0,\ell)}\|\vartheta\|^2, \tag{3.81}$$

$$\begin{aligned} I_{17} &= \int_0^t \|\chi_t\|\|\vartheta_2\|_{L^\infty(0,\ell)}\|\vartheta\| \\ &\leqslant \varpi\int_0^t \|\chi_t\|^2 + C_\varpi\int_0^t \|\vartheta_2\|^2_{H^1(0,\ell)}\|\vartheta\|^2, \end{aligned} \tag{3.82}$$

$$\begin{aligned} I_{18} &= \int_0^t \|\chi_t\|\|\vartheta\|\|\chi_{1,t}+\chi_{2,t}\|_{L^\infty(0,\ell)} \\ &\leqslant \varpi\int_0^t \|\chi_t\|^2 + C_\varpi\int_0^t \|\vartheta\|^2\|\chi_{1,t}+\chi_{2,t}\|^2_{H^1(0,\ell)}, \end{aligned} \tag{3.83}$$

$$\begin{aligned} I_{19} &= \int_0^t \|\chi\|\|\partial_x u_{1,t}\|^2_{L^\infty(0,\ell)}\|\vartheta\| \\ &\leqslant \frac{1}{2}\int_0^t \|\chi\|^2\|\partial_x u_{1,t}\|^2_{L^\infty(0,\ell)} + \frac{1}{2}\int_0^t \|\vartheta\|^2\|\partial_x u_{1,t}\|^2_{L^\infty(0,\ell)}, \end{aligned} \tag{3.84}$$

$$\begin{aligned} I_{20} &= \int_0^t \|\chi_2\|_{L^\infty(0,\ell)}\|\partial_x u_{1,t}+\partial_x u_{2,t}\|_{L^\infty(0,\ell)}\|\partial_x u_t\|\|\vartheta\| \\ &\leqslant \varsigma\int_0^t \|u_t\|^2_{H^1(0,\ell)} + C_\varsigma\int_0^t \|\vartheta\|^2\|\chi_2\|^2_{H^1(0,\ell)}\|\partial_x u_{1,t}+\partial_x u_{2,t}\|^2_{H^1_0(0,\ell)}, \end{aligned} \tag{3.85}$$

for some suitable constants $\varpi, \varsigma > 0$ ($C_\varpi$, $C_\varsigma$ being the related constants via the Young inequality). Finally, we add (3.77), (3.78) (multiplied by a positive constant $m$ such that $mC_{\mathcal{M}} \leqslant \nu/8$), and (3.79), and use (3.80)–(3.85), in which we choose $0 < \varpi < m/8$ and $\varsigma < \nu/16$. Applying the Gronwall Lemma, we arrive at the continuous dependence estimates for $\|\chi_1-\chi_2\|_{H^1(0,T;L^2(0,\ell))\cap L^\infty(0,T;H^1(0,\ell))}$, for $\|\vartheta_1-\vartheta_2\|_{L^\infty(0,T;L^2(0,\ell))\cap L^2(0,T;H^1(0,\ell))}$, and for $\|u_{1,t}-u_{2,t}\|_{L^\infty(0,T;L^2(0,\ell))\cap L^2(0,T;H^1_0(0,\ell))}$. Integrating in time, we obtain the estimate for $\|u_1-u_2\|_{W^{1,\infty}(0,T;L^2(0,\ell))\cap H^1(0,\bar{t};H^1_0(0,\ell))}$ as well, and (3.73) ensues. □

## 4. Proof of Theorem 2

Preliminarily, we collect some estimates on the trajectory $\{(\vartheta(t),\chi(t),u(t))\}_{t\geqslant 0}$ originating from an initial triple $(\vartheta_0,\chi_0,u_0)$ complying with (2.8)–(2.10) and (2.11), see Remark 2.1.

**Proposition 4.1** (Global estimates on $(0,+\infty)$).

*Assume* (W1)–(W5) *and* (2.24)–(2.25)*. Let* $(\vartheta_0,\chi_0,u_0)$ *be an initial triple fulfilling* (1.10)–(1.12) *and* (2.11)*. Then, there exists a constant* $C_\infty>0$ *such that*

$$\begin{aligned}(4.1)\qquad &\|\vartheta\|_{L^\infty(0,+\infty;H^1(0,\ell))}+\|A\vartheta\|_{L^2(0,+\infty;L^2(0,\ell))}+\|\vartheta_t\|_{L^2(0,+\infty;L^2(0,\ell))}\\ &+\|\chi\|_{L^\infty(0,+\infty;W^{2,1}(0,\ell))}+\|\chi_t\|_{L^2(0,+\infty;H^1(0,\ell))\cap L^\infty(0,+\infty;L^2(0,\ell))}\\ &+\|u\|_{L^\infty(0,+\infty;H^1_0(0,\ell))}+\|u_t\|_{L^2(0,+\infty;H^1_0(0,\ell))\cap L^\infty(0,+\infty;L^2(0,\ell))}\\ &+\|u_{tt}\|_{L^2(0,+\infty;H^{-1}(0,\ell))}\\ &+\|(1-\chi)^{1/2}\partial_x u\|_{L^2(0,+\infty;L^2(0,\ell))\cap L^\infty(0,+\infty;L^2(0,\ell))}\leqslant C_\infty,\end{aligned}$$

$$(4.2)\qquad \chi(x,t)\geqslant\delta\quad\forall\,(x,t)\in[0,\ell]\times[0,+\infty).$$

P r o o f. First, we point out that the estimate (3.45) holds on $(0,+\infty)$. The estimates for $u$ are a direct consequence of the ones contained in (3.45). Hence, in order to prove (4.1), we test (2.23) by $\vartheta_t$ and integrate on $(0,t)$, $t\in(0,+\infty)$. By elementary calculations and using Poincaré's inequality and (2.24), we obtain

$$\begin{aligned}\int_0^t\|\vartheta_t\|^2+\frac12\|\partial_x\vartheta(t)\|^2&\leqslant\frac12\|\vartheta_0\|^2_{H^1(0,\ell)}+\int_0^t\|g\|^2+\frac12\int_0^t\|\vartheta_t\|^2\\ &\quad+2\int_0^t\|\chi_t\|^2\|\vartheta-m(\vartheta)\|^2_{L^\infty(0,\ell)}+2\int_0^t\|\chi_t\|^2\|m(\vartheta)\|^2_{L^\infty(0,\ell)}\\ &\leqslant\frac12\int_0^t\|\vartheta_t\|^2+C\Big(1+\int_0^t\|\chi_t\|^2\|\partial_x\vartheta\|^2+\|\vartheta\|^2_{L^\infty(0,+\infty;L^1(0,\ell))}\|\chi_t\|^2_{L^2(0,+\infty;L^2(0,\ell))}\Big).\end{aligned}$$

Taking into account (3.45), and applying Gronwall's Lemma, we obtain the estimates for $\vartheta$ and $\vartheta_t$ stated in (4.1). Next, we differentiate in time (2.16), test it by $\chi_t$, and integrate in time. Thus, we get

$$\frac12\|\chi_t(t)\|^2+\int_0^t\|\partial_x\chi_t\|^2\leqslant C+\sum_{i=21}^{23}I_i,$$

where, using (W2), we have

$$\begin{aligned}I_{21}&:=\int_0^t\int_0^\ell\gamma'(\chi)\chi_t^2\leqslant C\|\chi_t\|^2_{L^2(0,+\infty;L^2(0,\ell))},\\ I_{22}&:=\int_0^t\int_0^\ell\vartheta_t\chi_t\leqslant\frac12\|\vartheta_t\|^2_{L^2(0,+\infty;L^2(0,\ell))}+\frac12\|\chi_t\|^2_{L^2(0,+\infty;L^2(0,\ell))},\\ I_{23}&:=\int_0^t\int_0^\ell|\partial_x u||\partial_x u_t||\chi_t|\leqslant C\|u\|^2_{L^\infty(0,+\infty;H^1_0(0,\ell))}\|u_t\|^2_{L^2(0,+\infty;H^1_0(0,\ell))}\\ &\quad+\frac12\int_0^t\|\chi_t\|^2+\frac12\int_0^t\|\partial_x\chi_t\|^2,\end{aligned}$$

where the constant $C$ in the latter estimate also accounts for the continuous embedding (2.1). Taking into account (4.1) and the previous uniform estimates on $u$ and $u_t$ (cf. (3.45)), we get an estimate for $\|\chi_t\|_{L^2(0,+\infty;H^1(0,\ell))\cap L^\infty(0,+\infty;L^2(0,\ell))}$. Then, by comparison in (2.16), we obtain an estimate for $\|A\chi\|_{L^\infty(0,+\infty;L^1(0,\ell))}$, implying, by means of the monotonicity of $\beta$ and of the standard regularity results for parabolic equations, the boundedness of $\|\chi\|_{L^\infty(0,+\infty;W^{2,1}(0,\ell))}$. Further, we argue by comparison in (2.23): by the previous estimates on $\vartheta$ and $\chi_t$ and (2.1), $\vartheta\chi_t$ is estimated in $L^2(0,+\infty;L^2(0,\ell))$, hence we have the same estimate for $A\vartheta$. Finally, the inequality (2.19) extends to $[0,+\infty)$, as the separation constant $\delta$ depends on $\chi_0$ and on the potential $W'$, but not on the final time $T>0$. Thus, (4.2) holds. $\square$

Remark 4.2. As it is clear from the proof of Lemma 3.1, the separation constant for $\chi$ from the potential barrier 1 in the inequality (2.20) depends on the $L^\infty$-norm (both w.r.t. the time and w.r.t. the space variables) of the right-hand side of the equation (2.16). Thus, in order to extend (2.20) to $[0,+\infty)$, one should prove an estimate for the right-hand side of (2.16) in $L^\infty(0,+\infty;L^\infty(0,\ell))$. In turn, it seems that the latter uniform estimate can be proved only provided one disposes of a uniform in time separation inequality of $\chi$ from 1.

**Conclusion of the proof of Theorem 2.** For every $(\vartheta_\infty,\chi_\infty,u_\infty)\in\omega(\vartheta_0,\chi_0,u_0)$, let $t_n\nearrow\infty$ be such that for all $\nu\in(0,1)$

$$(\vartheta(t_n),\chi(t_n),u(t_n))\to(\vartheta_\infty,\chi_\infty,u_\infty) \quad \text{in } H^{1-\nu}(0,\ell)\times H^{1-\nu}(0,\ell)\times H_0^{1-\nu}(0,\ell). \tag{4.3}$$

Hence, for $n\geqslant 1$ and $t\geqslant 0$, we can define

$$\vartheta_n(t):=\vartheta(t_n+t),\quad \chi_n(t):=\chi(t_n+t),\quad u_n(t):=u(t_n+t).$$

We note that for every $T>0$ the triple $(\vartheta_n,\chi_n,u_n)$ solves the PDE system

$$\vartheta_{n,t}+\chi_{n,t}\vartheta_n+A\vartheta_n=g_n \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{4.4}$$

$$\chi_{n,t}+A\chi_n+\beta(\chi_n)+\gamma(\chi_n)=\vartheta_n+\frac{1}{2}|\partial_x u_n|^2 \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{4.5}$$

$$\partial^2_{tt}u_n+\mathcal{H}((1-\chi_n)u_n)+\mathcal{H}(\chi_n u_{n,t})=f_n \quad \text{a.e. in } (0,\ell)\times(0,T), \tag{4.6}$$

where we have set $g_n(t):=g(t_n+t)$, $f_n(t):=f(t_n+t)$, supplemented with the initial conditions

$$\vartheta_n(0)=\vartheta(t_n),\quad u_n(0)=u(t_n),\quad u_{n,t}(0)=u_t(t_n),\quad \chi_n(0)=\chi(t_n). \tag{4.7}$$

The sequence $\{(\vartheta_n, \chi_n, u_n)\}$ clearly fulfils estimate (4.1) on $(0,T)$, with a constant independent of $T > 0$. Furthermore, due to (3.46), there exists a positive constant $C(T)$, depending on the problem data and on $T$, such that for every $n$

$$\begin{aligned}(4.8)\quad &\|\vartheta_n\|_{L^2(0,T;H^2_N(0,\ell))\cap L^\infty(0,T;H^1(0,\ell))\cap H^1(0,T;L^2(0,\ell))} + \|\chi_n\|_{L^\infty(0,T;H^2_N(0,\ell))} \\ &+\|u_n\|_{H^2(0,T;L^2(0,\ell))\cap W^{1,\infty}(0,T;H^1_0(0,\ell))\cap H^1(0,T;H^2_0(0,\ell))} \leqslant C(T).\end{aligned}$$

From (4.1), with a standard argument we deduce that

$$\begin{aligned}
&(4.9) && u_{n,t} \to 0 \quad \text{in } L^2(0,T;H^1_0(0,\ell)),\\
&(4.10) && \partial^2_{tt} u_n \to 0 \quad \text{in } L^2(0,T;H^{-1}(0,\ell)),\\
&(4.11) && \chi_{n,t} \to 0 \quad \text{in } L^2(0,T;H^1(0,\ell)),\\
&(4.12) && \vartheta_{n,t} \to 0 \quad \text{in } L^2(0,T;L^2(0,\ell)),
\end{aligned}$$

and, likewise, thanks to (2.24)–(2.25) we have

$$(4.13)\qquad g_n \to 0, \quad f_n \to 0 \quad \text{in } L^2(0,T;L^2(0,\ell)).$$

Moreover, by [27, Theorem 4, Corollary 5] and the Ascoli theorem, there exist a limit triple $(\vartheta^\star, \chi^\star, u^\star)$ such that, up to a subsequence, the following convergences hold for all $1 \leqslant p < \infty$ and for all $\varrho \in (0,2]$:

$$\begin{aligned}
(4.14)\quad & \vartheta_n \to \vartheta^\star && \text{in } L^2(0,T;H^{2-\varrho}(0,\ell)) \cap L^p(0,T;H^1(0,\ell)) \\
& && \quad \cap \mathrm{C}^0([0,T];H^{1-\varrho/2}(0,\ell)),\\
& \vartheta_n \rightharpoonup^* \vartheta^\star && \text{in } L^2(0,T;H^2_N(0,\ell)) \cap L^\infty(0,T;H^1(0,\ell)) \cap H^1(0,T;L^2(0,\ell)),\\
(4.15)\quad & \chi_n \to \chi^\star && \text{in } L^p(0,T;H^2_N(0,\ell)) \cap \mathrm{C}^0([0,T];H^{2-\varrho}(0,\ell)) \\
& && \quad \cap H^1(0,T;H^{1-\varrho/2}(0,\ell)) \cap W^{1,p}(0,T;L^2(0,\ell)),\\
& \chi_n \rightharpoonup^* \chi^\star && \text{in } L^\infty(0,T;H^2_N(0,\ell)) \cap H^1(0,T;H^1(0,\ell)) \\
& && \quad \cap W^{1,\infty}(0,T;L^2(0,\ell)),\\
(4.16)\quad & u_n \to u^\star && \text{in } H^1(0,T;H^{2-\varrho}(0,\ell)) \cap W^{1,p}(0,T;H^1(0,\ell)) \\
& && \quad \cap \mathrm{C}^1([0,T];H^{1-\varrho/2}(0,\ell)),\\
& u_n \rightharpoonup^* u^\star && \text{in } H^1(0,T;H^2_0(0,\ell)) \cap W^{1,\infty}(0,T;H^1(0,\ell)) \cap H^2(0,T;L^2(0,\ell)).
\end{aligned}$$

Furthermore, by the strong-weak closedness of the maximal monotone operator induced by $\beta$ on $L^2(0,T;L^2(0,\ell))$, we also conclude that, along the same subsequence,

$$(4.17)\qquad \beta(\chi_n) \rightharpoonup^* \beta(\chi^\star) \quad \text{in } L^2(0,T;L^2(0,\ell)).$$

In view of (4.9)–(4.12), we have that

$$\vartheta_t^\star(x,t) = \chi_t^\star(x,t) = u_t^\star(x,t) = 0 \quad \text{for a.e } (x,t) \in (0,\ell) \times (0,T).$$

Hence, for every $t \in [0,T]$

$$\vartheta^\star(t) = \vartheta^\star(0) = \lim_{n \to +\infty} \vartheta(t_n) = \vartheta_\infty \quad \text{in } L^2(0,\ell).$$

In the same way, we conclude that for all $t \in [0,T]$

$$\chi^\star(t) = \chi_\infty, \quad u^\star(t) = u_\infty \quad \text{in } H^1(0,\ell).$$

Now, using (4.9)–(4.17) and the assumptions (W2)–(W5), we pass to the limit in the equation (4.4)–(4.6), and conclude that the triple $(\vartheta_\infty, \chi_\infty, u_\infty)$ satisfies (2.27)–(2.28), as well as

$$\mathcal{H}((1-\chi_\infty)u_\infty) = 0 \quad \text{a.e. in } (0,\ell). \tag{4.18}$$

Furthermore, combining (4.2) with (4.14)–(4.15) we have that $\vartheta_\infty$ and $\chi_\infty$ fulfil the first two inequalities in (2.30). In particular, (2.31) holds. Now, we first prove

$$\max_{x \in [0,\ell]} \chi_\infty(x) < 1. \tag{4.19}$$

Indeed, (4.19) follows from comparing $\chi_\infty$ with the solution $\bar{\chi}_\infty$ of the Cauchy problem

$$\begin{cases} \bar{\chi}_\infty'(t) + \beta(\bar{\chi}_\infty(t)) + \gamma(\bar{\chi}_\infty(t)) = \bar{\vartheta}_\infty & \forall\, t \in [0,T], \\ \bar{\chi}_\infty(0) = \overline{\zeta}, \end{cases} \tag{4.20}$$

where $0 < \overline{\zeta} < 1$ fulfils $\beta(\overline{\zeta}) + \gamma(\overline{\zeta}) < \bar{\vartheta}_\infty$. By a standard argument (cf. the proof of Lemma 3.1), one finds $\max\limits_{x \in [0,\ell]} \chi_\infty(x) \leqslant \max\limits_{t \in [0,T]} \bar{\chi}_\infty(t) \leqslant \overline{\zeta}$, whence (4.19). Hence, the third part of (2.30) follows from (4.19) and the fact that the set of (the $\chi$ components) of all stationary solutions is compact in $C^0([0,\ell])$. Thus, testing (4.18) by $u_\infty$ yields (2.29).

Finally, under the additional assumption (2.32) we test (2.28) by $A\chi_\infty$: integrating by parts and using (2.27) we find

$$\int_0^\ell |A\chi_\infty|^2 + \int_0^\ell (\beta'(\chi_\infty) + \gamma'(\chi_\infty))|\nabla \chi_\infty|^2 = \int_0^\ell \chi_\infty A\vartheta_\infty = 0.$$

By (2.32), the second term on the left-hand side is nonnegative, hence $A\chi_\infty \equiv 0$ on $(0,\ell)$. Thus, $\chi_\infty$ is constant on $[0,\ell]$, and (2.33) ensues. $\square$

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