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LARGE TIME BEHAVIOR OF SOLUTIONS TO A CLASS OF  
DOUBLY NONLINEAR PARABOLIC EQUATIONS\*

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*Abstract.* We study the large time asymptotic behavior of solutions of the doubly degenerate parabolic equation  $u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - u^q$  with an initial condition  $u(x, 0) = u_0(x)$ . Here the exponents  $m, p$  and  $q$  satisfy  $m + p \geq 3$ ,  $p > 1$  and  $q > m + p - 2$ .

*Keywords:* degenerate parabolic equation, large time asymptotic behavior

*MSC 2010:* 35K55, 35K65, 35B40

## 1. INTRODUCTION

The objective of this article is to study the large time asymptotic behavior of weak solutions of nonlinear parabolic equations of the type

$$(1.1) \quad u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - u^q \quad \text{in } S = \mathbb{R}^N \times (0, \infty),$$

$$(1.2) \quad u(x, 0) = u_0(x) \quad \text{on } \mathbb{R}^N.$$

Here  $p > 1$ ,  $m(p-1) > 1$ ,  $q > 1$ ,  $N \geq 1$  and  $u_0(x) \in L^1(\mathbb{R}^N)$  is a nonnegative function. Equation (1.1) has been suggested as a mathematical model for a variety of problems in mechanics, physics and biology, one can see [3], [5], [1] etc. The existence of a nonnegative solution of (1.1)–(1.2), defined in some weak sense, is well established (see [12] and [8]). In this paper we are interested in the behavior of solutions as  $t \rightarrow \infty$ . The elliptic method was used in several papers (see e.g. [4], [9]) to study the asymptotic behavior of the solutions of the porous media and the  $p$ -Laplacian equations. Also by the elliptic method, J. Manfredi and V. Vespi studied the large

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time behavior of the solution of the initial boundary problem without absorption  $-u^q$  in [7]. In details the large time behavior of the solution of the problem

$$(1.3) \quad u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in } \Omega \times (0, \infty),$$

$$(1.4) \quad u(x, t) = 0 \quad \text{in } \partial\Omega \times (0, \infty),$$

$$(1.5) \quad u(x, 0) = u_0(x) \quad \text{on } \mathbb{R}^N$$

was considered in [7].

In our paper we will study problem (1.1)–(1.2) in a way different from the elliptic method which is used in [7], namely, we will compare the large time behavior of the general solution of (1.1)–(1.2) to the Barenblatt-type solution of (1.1)–(1.2).

We begin with some preliminaries.

It is not difficult to verify that

$$E_c = t^{-l/\mu} \left\{ \left[ b - \frac{m(p-1)-1}{mp} (N\mu)^{-1/(p-1)} (|x|t^{-l/\mu})^{p/(p-1)} \right]_+ \right\}^{(p-1)/(m(p-1)-1)}$$

is the Barenblatt-type solution of the Cauchy problem

$$(1.6) \quad u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in } S = \mathbb{R}^N \times (0, \infty),$$

$$(1.7) \quad u(x, 0) = c\delta(x) \quad \text{on } \mathbb{R}^N$$

where  $l = (1 + (m-1)/(p-1))^{1-p}$ ,  $\mu = m + p - 3 + p/N$ ,  $c = \int_{\mathbb{R}^N} u_0(x) dx$ ,  $b$  is a constant such that  $b = \int_{\mathbb{R}^N} E_c(x, t) dx$ , and  $\delta$  denotes the Dirac mass centered at the origin.

Let

$$B_R(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < R\}, \quad B_R = \{x \in \mathbb{R}^N : |x| < R\}.$$

**Definition 1.1.** A nonnegative function  $u(x, t)$  is called a solution of (1.1)–(1.2) if  $u$  satisfies

$$(1.8) \quad u \in C(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (\tau, T)), \quad u^{(m-1)/(p-1)} Du \in L_{loc}^p(\mathbb{R}^N \times (0, T)),$$

$$u_t \in L^1(\mathbb{R}^N \times (\tau, T)), \quad \forall \tau > 0;$$

$$(1.9) \quad \int_S [u(x, t)\varphi_t(x, t) - u^{m-1}|Du|^{p-2}Du \cdot D\varphi - u^q\varphi] dx dt = 0, \quad \forall \varphi \in C_0^1(S);$$

$$(1.10) \quad \lim_{t \rightarrow 0} \int_{B_R} |u(x, t) - u_0(x)| dx = 0.$$

**Definition 1.2.** A nonnegative function  $U \in C(\bar{S} \setminus (0))$ ,  $U \neq 0$  is called a very singular solution of (1.1), if  $U$  satisfies (1.1) in the sense of distributions in  $S$  and

$$\lim_{t \rightarrow 0} \int_{B_R} U(x, t) dx = 0, \quad \forall R > 0.$$

Let  $U(x, t) = t^{1/(q-1)} f(|x|t^{-1/\beta})$ . Suppose  $f$  is the solution of the ordinary equation

$$(f^{m-1}|f'|^{p-2}f')' + \frac{1}{\eta}f^{m-1}|f'|^{p-2}f' + \frac{1}{\beta}\eta f' + \frac{1}{q}f - f^q = 0,$$

$$f(\eta) \geq 0, \quad f'(0) = 0, \quad \lim_{\eta \rightarrow \infty} \eta^{p/(q-(m+p-2))} f(\eta) = 0.$$

Then we can prove that  $U(x, t)$  is a very singular solution of (1.1); we will publish this result in another paper.

**Theorem 1.3.** *Let  $m(p-1) > 1$ ,  $q > m+p-2$ . If  $E_c$  is a unique solution of (1.6)–(1.7), then the solution  $u$  of (1.1)–(1.2) satisfies*

$$(1.11) \quad t^{l/\mu}|u(x, t) - E_c(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly on the sets  $\{x \in \mathbb{R}^N : |x| < at^{-l/\mu N}, a > 0\}$ , where

$$c = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^q(x, t) dx dt.$$

**Theorem 1.4.** *Suppose  $m(p-1) > 1$ ,  $q > m+p-2$  and*

$$|x|^\alpha u_0(x) \leq B, \quad \lim_{|x| \rightarrow \infty} |x|^\alpha u_0(x) = C,$$

where  $\alpha, B$  and  $C$  are constants with  $\alpha \in (0, p/(q - (m + p - 2)))$ . Then the solution of (1.1)–(1.2) satisfies

$$t^{1/(q-1)}u(x, t) \rightarrow C^* \quad \text{as } t \rightarrow \infty$$

uniformly on the sets

$$\{x \in \mathbb{R}^N : |x| \leq at^{1/\beta}, a > 0\},$$

where  $C^* = (1/(q-1))^{1/(q-1)}$  and  $\beta = (q-1)/(q-(m+p-2))$ .

**Theorem 1.5.** *Suppose  $1 < m(p-1)$ ,  $m+p-2 < q < m+p-2+p/N$  and*

$$|x|^\alpha u_0(x) \leq B, \quad a > \frac{p}{q-(m+p-2)}, \quad \int_{\mathbb{R}^N} u_0(x) dx > 0.$$

Assume that (1.1) has a unique very singular solution. Then the solution of (1.1)–(1.2) satisfies

$$t^{1/(q-1)}|u(x, t) - U(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly on the sets

$$\{x \in \mathbb{R}^N : |x| \leq at^{1/\beta}\},$$

where  $\beta = (q - 1)/(q - (m + p - 2))$ .

**Remark 1.6.** For  $m = 1$ , the uniqueness of solutions of (1.6)–(1.7) is known (see [2]). For  $m = 1, p = 2$ , the uniqueness of the very singular solution of (1.1) is known, too (see [11]).

## 2. PROOF OF THEOREM 1.3

Let  $u$  be a solution of (1.1). We define the family of functions

$$u_k = k^N u(kx, k^N \mu t), \quad k > 0.$$

It is easy to see that they are solutions of the problems

$$(2.1) \quad u_t = \operatorname{div}(u^{m-1} |Du|^{p-2} Du) - k^{-v} u^q \quad \text{in } S = \mathbb{R}^N \times (0, \infty),$$

$$(2.2) \quad u(x, 0) = u_{0k}(x) \quad \text{on } \mathbb{R}^N,$$

where  $\mu = m + p - 3 + p/N$  as before and  $v = q - m - p + 2 - p/N$ ,  $u_{0k}(x) = k^N u_0(x)$ .

**Lemma 2.1.** For any  $s \in (0, m + p - 2)$ ,  $u_k$  satisfies

$$(2.3) \quad \int_0^T \int_{B_R} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^2 \, dx \, dt \leq c(s, R, |u_0|_{L^1}),$$

$$(2.4) \quad \int_0^T \int_{B_R} u_k^{m+p-2+p/N-s} \, dx \, dt \leq c(s, R, |u_0|_{L^1}).$$

**Proof.** From Definition 1.1, we are able to deduce (see [10]):  $\forall \varphi \in C^1(\bar{S})$ ,  $\varphi = 0$  when  $|x|$  is large enough,

$$(2.5) \quad \int_{\mathbb{R}^N} u_k(x, t) \varphi \, dx - \int_0^T \int_{\mathbb{R}^N} (u_k \varphi_t - u_k^{m-1} |Du_k|^{p-2} Du_k \cdot D\varphi) \, dx \, dt \leq \int_{\mathbb{R}^N} u_{0k}(x) \varphi(x, 0) \, dx.$$

Let

$$(2.6) \quad \psi_R \in C_0^\infty(B_{2R}), \quad 0 \leq \psi_R \leq 1, \quad \psi_R = 1 \text{ on } B_R, \quad |D\psi_R| \leq cR^{-1}.$$

By an approximate procedure we can choose  $\varphi = (u_k^s/(1 + u_k^s))\psi_R^p$  in (2.5); then

$$\begin{aligned}
 (2.7) \quad & \int_{\mathbb{R}^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s} dz \psi_R^p(x) dx \\
 & + s \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \psi_R^p(x) dx d\tau \\
 & \leq -p \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1+u_k^s} |Du_k|^{p-2} \psi_R^{p-1}(x) Du_k \cdot D\psi_R dx d\tau \\
 & + \int_{\mathbb{R}^N} \int_0^{u_k(x,h)} \frac{z^s}{1+z^s} dz \psi_R^p(x) dx,
 \end{aligned}$$

where  $0 < h < t$ . Notice that

$$\begin{aligned}
 (2.8) \quad & \left| \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1+u_k^s} |Du_k|^{p-2} \psi_R^{p-1}(x) Du_k \cdot D\psi_R dx d\tau \right| \\
 & \leq \int_h^t \int_{\mathbb{R}^N} \left[ \varepsilon \left( \frac{u_k^{(s+m-2) \cdot (p-1)/p}}{(1+u_k^s)^{2(p-1)/p}} |Du_k|^{p-1} \psi_R^{p-1} \right)^{p/(p-1)} \right. \\
 & \quad \left. + c(\varepsilon) \left( \frac{u_k^{(s+m-1-(s+m-2) \cdot (p-1)/p}}{(1+u_k^s)^{1-2(p-1)/p}} |D\psi_R| \right)^p \right] dx dt \\
 & = \int_h^t \int_{\mathbb{R}^N} \left[ \varepsilon \left( \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \psi_R^p + c(\varepsilon) \frac{u_k^{p+m-2}}{(1+u_k^s)^{2-p}} |D\psi_R|^p \right) \right] dx dt,
 \end{aligned}$$

$$(2.9) \quad \int_{\mathbb{R}^N} \int_0^{u_k(x,h)} \frac{z^s}{1+z^s} dz \psi_R^p(x) dx \leq \int_{\mathbb{R}^N} u(x, k^{N\mu} h) dx,$$

hence by (2.7)–(2.9) we obtain

$$\begin{aligned}
 (2.10) \quad & \sup_{0 < t < T} \int_{\mathbb{R}^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s} dz dx + \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \psi_R^p dx d\tau \\
 & \leq c \int_{\mathbb{R}^N} u(x, k^{N\mu} h) dx + c \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}} |D\psi_R|^p dx d\tau.
 \end{aligned}$$

Because  $u_k \in L^\infty(\mathbb{R}^N \times (h, T)) \cap L^1(S_T)$ ,  $p + m - 2 > 0$ , we have

$$(2.11) \quad \lim_{R \rightarrow \infty} \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}} |D\psi_R|^p dx d\tau = 0.$$

Let  $R \rightarrow \infty$ ,  $h \rightarrow 0$  in (2.10). Then

$$\begin{aligned}
 (2.12) \quad & \sup_{0 < t < T} \int_{\mathbb{R}^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s} dz dx + \iint_{S_t} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p dx d\tau \\
 & \leq c \int_{\mathbb{R}^N} u_0(x) dx.
 \end{aligned}$$

Thus

$$(2.13) \quad \sup_{0 < t < T} \int_{B_{2R}} u_k(x, t) \, dx + \int_0^T \int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \, dx \, d\tau \leq c(R).$$

Let

$$u_1 = \max\{u_k(x, t), 1\}, \quad w = u_1^{(m+p-2-s)/p}.$$

By Sobolev's imbedding inequality (see [6]), for  $\xi \in C_0^1(B_{2R})$ ,  $\xi \geq 0$ , we have

$$\begin{aligned} & \left( \int_{\mathbb{R}^N} \xi^p w^r \, dx \right)^{1/r} \\ & \leq c \left( \int_{\mathbb{R}^N} |D(\xi w)|^p \right)^{s/p} \left( \int_{B_{2R}} w^{p/(m+p-2-s)} \, dx \right)^{((1-\theta)(m+p-2-s))/p}, \end{aligned}$$

where

$$\begin{aligned} \theta &= \left( \frac{m+p-2-s}{p} - \frac{1}{r} \right) \left( \frac{1}{N} - \frac{1}{p} + \frac{m+p-2-s}{p} \right)^{-1}, \\ r &= \frac{p(m+p-2+p/N-s)}{m+p-2-s}. \end{aligned}$$

It follows that

$$(2.14) \quad \begin{aligned} & \iint_{S_T} \xi^p w^r \, dx \, dt \\ & \leq c \iint_{S_T} |D(\xi w)|^p \, dx \, dt \\ & \quad \times \sup_{t \in (0, T)} \left( \int_{B_{2R}} w^{p/(m+p-2-s)} \, dx \right)^{(r-p)(m+p-2-s)/p}. \end{aligned}$$

Since

$$|Dw|^p \leq c \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \quad \text{a.e. on } \{u_k \geq 1\} \quad \text{and} \quad |Dw| = 0 \quad \text{on } \{u_k \leq 1\},$$

we have

$$(2.15) \quad \begin{aligned} \iint_{S_T} |D(\xi w)|^p \, dx \, dt & \leq c \iint_{S_T} (\xi^p |Dw|^p + w^p |D\xi|^p) \, dx \, dt \\ & \leq c \left[ \iint_{S_T} |D\xi|^p u_1^{p+m-2-s} \, dx \, dt \right. \\ & \quad \left. + \int_0^T \int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \, dx \, dt \right]. \end{aligned}$$

Hence, by (2.14), (2.15) and (2.13), we get

$$\iint_{S_T} \xi^p u_1^{m+p-2+p/N-s} dx dt \leq c(s, R, |u_0|_{L^1}) \left( 1 + \iint_{S_T} |D\xi|^p u_1^{p+m-2-s} dx dt \right).$$

Let  $\xi = \psi_R^b$ , where  $\psi_R$  is the function satisfying (2.6) and  $b = N(m + p - 2 - s)/p$ . Then

$$\begin{aligned} & \iint_{S_T} \psi_R^{pb} u_1^{m+p-2+p/N-s} dx dt \\ & \leq c(s, R, |u_0|_{L^1}) \left( 1 + \iint_{S_T} \psi_R^{pb} u_1^{p+m-2+p/N-s} dx dt \right)^{(m+p-2-s)/(m+p-2+p/N-s)}, \end{aligned}$$

which implies (2.4) is true.  $\square$

Let  $Q_\varrho = B_\varrho(x_0) \times (t_0 - \varrho^p, t_0)$  with  $t_0 > (2\varrho)^p$  and  $u_{k1} = \max\{u_k, 1\}$ .

**Lemma 2.2.** *Each  $u_k$  satisfies*

$$(2.16) \quad \sup_{Q_\varrho} u_k \leq c(\varrho, s_1) \left( \iint_{Q_{2\varrho}} u_{k1}^{p+m-3+s_1} dx dt \right)^{1/s_1},$$

where  $c(\varrho, s_1)$  depends on  $\varrho$  and  $s_1$ , and  $s_1$  can be any number satisfying  $0 < s_1 < 1 + p/N$ .

**Lemma 2.3.** *Each  $u_k$  satisfies*

$$(2.17) \quad \int_\tau^T \int_{B_R} u_k^{m-1} |Du_k|^p dx dt \leq c(\tau, R), \quad \int_\tau^T \int_{B_R} |u_{kt}|^p dx dt \leq c(\tau, R).$$

*Proof.* By Lemma 2.1 and 2.2,  $u_k$  are uniformly bounded on every compact set  $K \subset S_T$ . Let  $\psi_R$  be a function satisfying (2.6) and let  $\xi \in C_0^1(0, T + 1)$  with  $0 \leq \xi \leq 1$ ,  $\xi = 1$  if  $t \in (\tau, T)$ . We choose  $\eta = \psi_R^p \xi u_k$  in (2.5) to obtain

$$(2.18) \quad \begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} u_k^2(x, T) \psi_R^p dx + \iint_{S_T} u_k^{m-1} |Du_k|^p \psi_R^p \xi dx dt \\ & \leq \frac{1}{2} \iint_{S_T} u_k^2 \xi' \psi_R^p dx dt - p \iint_{S_T} u_k^m |Du_k|^{p-2} Du_k \cdot D\psi_R \psi_R^{p-1} \xi dx dt. \end{aligned}$$

Notice that

$$(2.19) \quad \begin{aligned} & \iint_{S_T} u_k^m |Du_k|^{p-1} |D\psi_R| \psi_R^{p-1} \xi dx dt \\ & \leq \varepsilon \iint_{S_T} u_k^{m-1} |Du_k|^p \psi_R^p \xi dx dt + c(\varepsilon) \iint_{S_T} u_k^{p+m-1} |D\psi_R|^p \xi dx dt. \end{aligned}$$

By (2.18), (2.19), one knows that the first inequality of (2.17) is true.



Now we will prove the second inequality of (2.17). Let

$$v(x, t) = u_{kr}(x, t) = ru_k(x, r^{m+p-3}t), \quad r \in (0, 1).$$

Then

$$(2.20) \quad v_t(x, t) = \operatorname{div}(v^{m-1}|Dv|^{p-2}Dv) - r^{m+p-2-q}k^{-v}v^q,$$

$$(2.21) \quad v(x, 0) = ru_k(x, 0).$$

Notice that  $r^{m+p-2-q}k^{-v} > k^{-v}$  using the argument similar to that in the proof of Theorem 1 of [12], one can prove

$$u_k \geq u_{kr}.$$

It follows that

$$\frac{u_k(x, r^{m+p-3}t) - u_k(x, t)}{(r^{m+p-3} - 1)t} \geq \frac{r - 1}{(1 - r^{m+p-3})t} u_k(x, r^{m+p-3}t).$$

Letting  $r \rightarrow 1$ , we get

$$(2.22) \quad u_{kt} \geq -\frac{u_k}{(m+p-3)t}.$$

Denote  $w = t^\beta u_k(x, t)$ ,  $\beta = 1/(m+p-3)$ . By (2.22),  $w_t \geq 0$ . By (2.1),

$$(2.23) \quad \begin{aligned} & \int_\tau^T \int_{B_{2R}} t^\beta w_t \psi_R \, dx \, dt \\ &= - \int_\tau^T \int_{B_{2R}} u_k^{m-1} |Du_k|^{p-2} Du_k \cdot D\psi_R \, dx \, dt \\ & \quad - \int_\tau^T \int_{B_{2R}} k^{-v} u_k^q \psi_R \, dx \, dt + \beta \int_\tau^T \int_{B_{2R}} t^{-1} u_k(x) \psi_R \, dx \, dt \\ & \leq \frac{\beta}{\tau} \int_\tau^T \int_{B_{2R}} u_k \, dx \, dt \\ & \quad + \left( \int_\tau^T \int_{B_{2R}} u_k^{m-1} |Du_k|^p \, dx \, dt \right)^{(p-1)/p} \left( \int_\tau^T \int_{B_{2R}} |D\psi_R|^p \, dx \, dt \right)^{1/p}. \end{aligned}$$

From (2.13), (2.16) and (2.23) we obtain (2.17).  $\square$

Proof of Theorem 1.3.

By Lemmas 2.1–2.3 and [2], there exists a subsequence  $\{u_{k_j}\}$  of  $\{u_k\}$  and a function  $v$  such that on every compact set  $K \subset S$

$$u_{k_j} \rightarrow v \text{ in } C(K), \quad Du_{k_j}^m \rightharpoonup Dv^m \text{ in } L_{\text{loc}}^p(S_T), \quad |u_{kt}|_{L_{\text{loc}}^1(S_T)} \leq c.$$

Similar to what was done in the proof of Theorem 2 in [12], we can prove that  $v$  satisfies (1.1) in the sense of distributions.

We now prove  $v(x, 0) = c\delta(x)$ . Let  $\chi \in C_0^1(B_R)$ . Then we have

$$(2.24) \quad \int_{\mathbb{R}^N} u_k(x, t)\chi \, dx - \int_{\mathbb{R}^N} \varphi_k \chi \, dx \\ = - \int_0^t \int_{\mathbb{R}^N} u_k^{m-1} |Du_k|^{p-2} Du_k \cdot D\chi \, dx \, ds - k^{-\nu} \int_0^t \int_{\mathbb{R}^N} u_k^q \chi \, dx \, ds.$$

To estimate  $\int_0^t \int_{\mathbb{R}^N} u_k^{m-1} |Du_k|^{p-2} Du_k \cdot D\chi \, dx \, ds$ , without losing generality, one can assume that  $u_k > 0$ . By Hölder inequality and Lemma 2.1,

$$(2.25) \quad \left| \int_0^t \int_{\mathbb{R}^N} u_k^{m-1} |Du_k|^{p-2} Du_k \cdot D\chi \, dx \, ds \right| \\ \leq c \left( \int_0^T \int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \, dx \, dt \right)^{(p-1)/p} \\ \times \left( \int_0^T \int_{B_{2R}} (1+u_k^s)^{2(p-1)} u_k^{(p-1)(2-s-m)} \, dx \, d\tau \right)^{1/p} \\ \leq c \left( \int_0^t \int_{B_{2R}} (u_{k1}^{(p-1)(2-s-m)} + u_{k1}^{(p-1)(2+s-m)}) \, dx \, d\tau \right)^{1/p} \\ \leq c \left( \int_0^t \int_{B_{2R}} (u_{k1}^{(p-1)(2-s-m)})^{\frac{m+p-2+p/N-s}{(p-1)(s+2-m)}} \, dx \, dt \right)^{\frac{(p-1)(s-2-m)}{m+p-2+p/N-s} \frac{1}{p}} t^d,$$

where  $s \in (0, 1/N)$ ,  $d = ((m-s-1)Np + (s-2)N + p - s + 2) / ((m+p-2)N + p - s) < 1$  because  $p > (N+3)/(2N+1)$ ,  $u_{k1} = \max(u_k, 1)$ .

Hence from (2.24) we get

$$(2.26) \quad \left| \int_{\mathbb{R}^N} u_k(x, t)\chi \, dx - \int_{\mathbb{R}^N} \varphi_k \chi \, dx + k^{-\nu} \int_0^t \int_{\mathbb{R}^N} u_k^q \chi \, dx \, ds \right| \\ = \left| \int_{\mathbb{R}^N} u_k(x, t)\chi \, dx - \int_{\mathbb{R}^N} \varphi_k \chi(k^{-1}x) \, dx + \int_0^{N\mu t} \int_{\mathbb{R}^N} u_k^q \chi(k^{-1}x) \, dx \, d\tau \right| \leq ct^d.$$

Letting now  $k \rightarrow \infty$ ,  $t \rightarrow 0$ , we obtain

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} v(x, t)\chi \, dx = \chi(0) \left( \int_{\mathbb{R}^N} \varphi(x) \, dx - \int_0^\infty \int_{\mathbb{R}^N} u^q \, dx \, dt \right).$$

Thus

$$v(x, 0) = c\delta(x), \quad c = \int_{\mathbb{R}^N} \varphi(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^q dx dt,$$

$v(x, t)$  is a solution of (1.3)–(1.4). By the assumption on uniqueness of solution, we have  $v(x, t) = E_c(x, t)$  and the whole sequence  $\{u_k\}$  converges to  $E_c$  as  $k \rightarrow \infty$ . Set  $t = 1$ . Then

$$u_k(x, 1) = k^N u(kx, k^{N\mu}) \rightarrow E_c(x, 1)$$

uniformly on every compact subset of  $\mathbb{R}^N$ . Thus writing  $kx = k'$ ,  $k^{N\mu} = t'$ , and dropping the prime again, we see that

$$t^{1/\mu} u(x, t) \rightarrow E_c(xt^{1/(N\mu)}, 1) = t^{1/\mu} E_c(x, t)$$

uniformly on the sets  $\{x \in \mathbb{R}^N : |x| \leq at^{1/(N\mu)}\}$ ,  $a > 0$ . Thus Theorem 1.3 is true.  $\square$

### 3. PROOFS OF THEOREM 1.4 AND 1.5

Let  $u$  be a solution of (1.1)–(1.2) and let  $u_k(x, t) = k^\delta u(kx, k^\beta t)$ ,  $k > 0$ . If  $\delta = 1/(q - (m + p - 2))$ ,  $\beta = (q - 1)/(q - (m + p - 2))$ , then

$$(3.1) \quad u_{kt} = \operatorname{div}(u_k^{m-1} |Du_k|^{p-2} Du_k) - u_k^q,$$

$$(3.2) \quad u_k(x, 0) = \varphi_k(x) = k^\delta \varphi(kx).$$

**Lemma 3.1.** *The solution  $u_k$  of (3.1)–(3.2) satisfies*

$$(3.3) \quad u_k(x, t) \leq C^* t^{-1/(q-1)}, \quad C^* = \left(\frac{1}{q-1}\right)^{1/(q-1)}.$$

*Proof.* We consider the regularized problem of (3.1), namely,

$$(3.4) \quad u_{kt} = \operatorname{div}((u_k^{m-1} + \varepsilon)(|Du_k|^2 + \varepsilon)^{(p-2)/2} Du_k) - u_k^q.$$

By the uniqueness of the solution of (3.1)–(3.2), we can prove that

$$u_{k\varepsilon} \rightarrow u_k \quad \text{as } \varepsilon \rightarrow 0 \text{ in } C(K)$$

on every compact set  $K \subset S$ , where  $u_{k\varepsilon}$  are the solutions of (3.4), (3.2). By computation, it is easy to show that  $C^*(t - t_0)^{-1/(q-1)}$  is a solution of (3.4) in  $\mathbb{R}^N \times (t_0, \infty)$ ,  $t_0 > 0$ . For any  $\delta_1 > 0$ , we choose  $\delta_0 \in (0, \delta_1)$  such that

$$|u_{k\varepsilon}(x, \delta_1)|_{L^\infty(\mathbb{R}^N)} \leq C^*(\delta_1 - \delta_0)^{-1/(q-1)}.$$

Hence by the comparison principle, we have

$$u_{k\varepsilon}(x, t) \leq C^*(t - t_0)^{-1/(q-1)}, \quad t > \delta_1.$$

The proof of Lemma 3.1 is completed by letting  $\delta_1 \rightarrow 0$  and  $\varepsilon \rightarrow 0$ . □

**Lemma 3.2.** *Each  $u_k$  satisfies*

$$(3.5) \quad \int_{\tau}^T \int_{B_R} |Du_k|^p \leq c(\tau, R), \quad \int_{\tau}^T \int_{B_R} |u_{kt}| \, dx \, dt \leq c(\tau, R),$$

where  $\tau \in (0, T)$ .

*Proof.* The proof of Lemma 3.2 is similar to that of Lemma 2.3. □

*Proof of Theorem 1.4.* By Lemma 3.1,  $\{u_k\}$  are uniformly bounded on every compact set of  $S$ . Hence by [2], there exists a subsequence  $\{u_{k_j}\}$  and a function  $U \in C(S)$  such that

$$u_{k_j} \rightarrow U \quad \text{in } C(K)$$

and

$$U(x, t) \leq C^*t^{-1/(q-1)}.$$

We now prove that  $U(x, t) = C^*t^{-1/(q-1)}$ . Let us introduce the function

$$(3.6) \quad \varphi_k^A = \min\{\varphi_k, A\}$$

and denote by  $V_{K\varepsilon}^A$  the solution of (3.4) with initial value (3.6). By the comparison principle,

$$(3.7) \quad V_{K\varepsilon}^A \leq u_{k\varepsilon},$$

where  $u_{k\varepsilon}$  is the solution of (3.4), (3.2).

Define

$$V_A = C^* \left( t + \frac{A^{1-q}}{q-1} \right)^{-1/(q-1)},$$

which is the solution of (3.4) with initial value

$$(3.8) \quad V_A(x, 0) = A.$$

Notice that

$$\lim_{k \rightarrow \infty} \varphi_k^A(x) = \lim_{k \rightarrow \infty} \min \left\{ A, \frac{\varphi(kx)|kx|^\alpha k^{\delta-\alpha}}{|x|^\alpha} \right\} = A.$$

Using the uniqueness of solution of (3.4), (3.8), we can prove (see [6])

$$V_{k\varepsilon}^A \rightarrow V_A \quad \text{as } k \rightarrow \infty \text{ in } C(K),$$

where  $K$  is a compact set in  $S$ . Moreover, by [2] and [12]

$$V_{k\varepsilon}^A \rightarrow V_k^A u_{k\varepsilon} \rightarrow u_k \quad \text{as } k \rightarrow \infty \text{ in } C(K)$$

uniformly in  $K$ , where  $V_k^A$  is the solution of (1.1) with initial value (3.6). It follows that

$$V_k^A \rightarrow V_A \quad \text{as } k \rightarrow \infty \text{ in } C(K).$$

Letting  $\varepsilon \rightarrow 0$  and  $k \rightarrow \infty$  in turn in (3.7), we get

$$V_A(x, t) \leq V_\infty(x, t) = C^* t^{-1/(q-1)} \quad \text{in } S.$$

Since the lower bound holds for every  $A > 0$ , we conclude that

$$U(x, t) = V_\infty(x, t) = C^* t^{-1/(q-1)} \quad \text{in } S.$$

Thus

$$k^{p/(q-(m+p-2))} u(kx, k^\beta t) \rightarrow C^* t^{-1/(q-1)} \quad \text{as } k \rightarrow \infty.$$

Set  $t = 1$ . Then

$$k^{p/(q-(m+p-2))} u(kx, k^\beta) \rightarrow C^* \quad \text{as } k \rightarrow \infty$$

uniformly on every compact subset of  $\mathbb{R}^N$ . Therefore if we set  $kx = x'$ ,  $k^\beta = t'$ , and omit the primes, we obtain

$$t^{1/(q-1)} u(x, t) \rightarrow C^* \quad \text{as } t \rightarrow \infty$$

uniformly on sets  $\{x \in \mathbb{R}^N : |x| \leq \alpha t^{1/\beta}\}$  with  $\alpha > 0$  for  $t > 0$  and so Theorem 1.4 is proved.  $\square$

**P r o o f** of Theorem 1.5. By Lemma 3.1 and [2], there exist a subsequence  $\{u_{k_j}\}$  and a function  $U \in C(S)$  such that

$$(3.9) \quad u_{k_j} \rightarrow U \quad \text{in } C(K).$$

By Lemma 3.2, we can prove that  $U$  satisfies (1.1) in the sense of distributions in a manner similar to Theorem 2 of [12].  $\square$

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