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LARGE TIME BEHAVIOR OF SOLUTIONS TO A CLASS OF
DOUBLY NONLINEAR PARABOLIC EQUATIONS*

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Abstract. We study the large time asymptotic behavior of solutions of
the doubly degenerate parabolic equation $u_t = \text{div}(u^{m-1}|Du|^{p-2}Du) - u^q$ with
an initial condition $u(x,0) = u_0(x)$. Here the exponents $m, p$ and $q$ satisfy
$m + p \geq 3$, $p > 1$ and $q > m + p - 2$.

Keywords: degenerate parabolic equation, large time asymptotic behavior

MSC 2010: 35K55, 35K65, 35B40

1. Introduction

The objective of this article is to study the large time asymptotic behavior of weak
solutions of nonlinear parabolic equations of the type

\begin{align}
(1.1) & \quad u_t = \text{div}(u^{m-1}|Du|^{p-2}Du) - u^q \quad \text{in} \quad S = \mathbb{R}^N \times (0, \infty), \\
(1.2) & \quad u(x,0) = u_0(x) \quad \text{on} \quad \mathbb{R}^N.
\end{align}

Here $p > 1$, $m(p-1) > 1$, $q > 1$, $N \geq 1$ and $u_0(x) \in L^1(\mathbb{R}^N)$ is a nonnegative function.
Equation (1.1) has been suggested as a mathematical model for a variety of problems
in mechanics, physics and biology, one can see [3], [5], [1] etc. The existence of a
nonnegative solution of (1.1)–(1.2), defined in some weak sense, is well established
(see [12] and [8]). In this paper we are interested in the behavior of solutions as
$t \to \infty$. The elliptic method was used in several papers (see e.g. [4], [9]) to study
the asymptotic behavior of the solutions of the porous media and the $p$-Laplacian
equations. Also by the elliptic method, J. Manfredi and V. Vespri studied the large

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time behavior of the solution of the initial boundary problem without absorption \(-u^q\)
in [7]. In details the large time behavior of the solution of the problem

\begin{align}
    u_t &= \text{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in} \ \Omega \times (0, \infty), \\
    u(x, t) &= 0 \quad \text{in} \ \partial \Omega \times (0, \infty), \\
    u(x, 0) &= u_0(x) \quad \text{on} \ \mathbb{R}^N
\end{align}

was considered in [7].

In our paper we will study problem (1.1)–(1.2) in a way different from the elliptic
method which is used in [7], namely, we will compare the large time behavior of the
general solution of (1.1)–(1.2) to the Barenblatt-type solution of (1.1)–(1.2).

We begin with some preliminaries.

It is not difficult to verify that

\[ E_c = t^{-l/\mu} \left\{ b - \frac{m(p - 1) - 1}{mp} (N\mu)^{-1/(p-1)} (|x| t^{-l/\mu})^{p/(p-1)} \right\}^{(p-1)/(m(p-1)-1)} \]

is the Barenblatt-type solution of the Cauchy problem

\begin{align}
    u_t &= \text{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in} \ \mathbb{R}^N \times (0, \infty), \\
    u(x, 0) &= c\delta(x) \quad \text{on} \ \mathbb{R}^N
\end{align}

where \( l = (1 + (m - 1)/(p - 1))^{1-p} \), \( \mu = m + p - 3 + p/N \), \( c = \int_{\mathbb{R}^N} u_0(x) \, dx \), \( b \) is a constant such that \( b = \int_{\mathbb{R}^N} E_c(x, t) \, dx \), and \( \delta \) denotes the Dirac mass centered at the origin.

Let

\[ B_R(x_0) = \{ x \in \mathbb{R}^N : |x - x_0| < R \}, \quad B_R = \{ x \in \mathbb{R}^N : |x| < R \}. \]

**Definition 1.1.** A nonnegative function \( u(x, t) \) is called a solution of (1.1)–(1.2) if \( u \) satisfies

\begin{align}
    u &\in C(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (\tau, T)), \quad u^{(m-1)/(p-1)}Du \in L^p_{\text{loc}}(\mathbb{R}^N \times (0, T)), \\
    u_t &\in L^1(\mathbb{R}^N \times (\tau, T)), \quad \forall \tau > 0; \\
    \int_S [u(x, t)\varphi_t(x, t) - u^{m-1}|Du|^{p-2}Du \cdot D\varphi - u^q \varphi] \, dx \, dt = 0, \quad \forall \varphi \in C_0^1(S); \\
    \lim_{t \to 0} |u(x, t) - u_0(x)| \, dx = 0.
\end{align}

**Definition 1.2.** A nonnegative function \( U \in C(S \setminus (0)) \), \( U \neq 0 \) is called a very singular solution of (1.1), if \( U \) satisfies (1.1) in the sense of distributions in \( S \) and

\[ \lim_{t \to 0} \int_{B_R} U(x, t) \, dx = 0, \quad \forall R > 0. \]
Let \( U(x,t) = t^{1/(q-1)} f(|x| t^{-1/\beta}) \). Suppose \( f \) is the solution of the ordinary equation
\[
(f^{m-1} |f'|^{p-2} f')' + \frac{1}{\eta} f^{m-1} |f'|^{p-2} f' + \frac{1}{\beta} \eta f' + \frac{1}{q} f - f^q = 0,
\]
\[
f(\eta) \geq 0, \quad f'(0) = 0, \quad \lim_{\eta \to \infty} \eta^{p/(q-(m+p-2))} f(\eta) = 0.
\]
Then we can prove that \( U(x,t) \) is a very singular solution of (1.1); we will publish this result in another paper.

**Theorem 1.3.** Let \( m(p-1) > 1, q > m + p - 2 \). If \( E_c \) is a unique solution of (1.6)–(1.7), then the solution \( u \) of (1.1)–(1.2) satisfies
\[
t^{1/\mu} |u(x,t) - E_c(x,t)| \to 0 \quad \text{as } t \to \infty
\]
uniformly on the sets \( \{x \in \mathbb{R}^N: \ |x| < at^{-1/\mu N}, \ a > 0\} \), where
\[
c = \int_{\mathbb{R}^N} u_0(x) \, dx - \int_0^\infty \int_{\mathbb{R}^N} u^q(x,t) \, dx \, dt.
\]

**Theorem 1.4.** Suppose \( m(p-1) > 1, q > m + p - 2 \) and
\[
|x|^{\alpha} u_0(x) \leq B, \quad \lim_{|x| \to \infty} |x|^{\alpha} u_0(x) = C,
\]
where \( \alpha, B \) and \( C \) are constants with \( \alpha \in (0, p/(q-(m+p-2))) \). Then the solution of (1.1)–(1.2) satisfies
\[
t^{1/(q-1)} u(x,t) \to C^* \quad \text{as } t \to \infty
\]
uniformly on the sets
\[
\{x \in \mathbb{R}^N: \ |x| \leq at^{1/\beta}, \ a > 0\},
\]
where \( C^* = (1/(q-1))^{1/(q-1)} \) and \( \beta = (q-1)/(q-(m+p-2)) \).

**Theorem 1.5.** Suppose \( 1 < m(p-1), m + p - 2 < q < m + p - 2 + p/N \) and
\[
|x|^{\alpha} u_0(x) \leq B, \quad a > \frac{p}{q-(m+p-2)}, \quad \int_{\mathbb{R}^N} u_0(x) \, dx > 0.
\]
Assume that (1.1) has a unique very singular solution. Then the solution of (1.1)–(1.2) satisfies
\[
t^{1/(q-1)} |u(x,t) - U(x,t)| \to 0 \quad \text{as } t \to \infty
\]
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uniformly on the sets 
\[ \{ x \in \mathbb{R}^N : |x| \leq at^{1/\beta} \} , \]
where \( \beta = (q - 1)/(q - (m + p - 2)) \).

Remark 1.6. For \( m = 1 \), the uniqueness of solutions of (1.6)–(1.7) is known (see [2]). For \( m = 1, p = 2 \), the uniqueness of the very singular solution of (1.1) is known, too (see [11]).

2. Proof of Theorem 1.3

Let \( u \) be a solution of (1.1). We define the family of functions 
\[ u_k = k^N u(kx, k^N \mu t), \quad k > 0. \]

It is easy to see that they are solutions of the problems

\begin{align*}
(2.1) \quad u_t &= \text{div}(u^{m-1}|Du|^{p-2}Du) - k^{-v}u^q \quad \text{in} \ S = \mathbb{R}^N \times (0, \infty), \\
(2.2) \quad u(x, 0) &= u_{0k}(x) \quad \text{on} \ \mathbb{R}^N,
\end{align*}

where \( \mu = m + p - 3 + p/N \) as before and \( v = q - m - p + 2 - p/N \), \( u_{0k}(x) = k^N u_0(x) \).

Lemma 2.1. For any \( s \in (0, m + p - 2) \), \( u_k \) satisfies

\begin{align*}
(2.3) \quad &\int_0^T \int_{B_R} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^2 \, dx \, dt \leq c(s, R, |u_0|_{L^1}), \\
(2.4) \quad &\int_0^T \int_{B_R} u_k^{m+p-2+p/N-s} \, dx \, dt \leq c(s, R, |u_0|_{L^1}).
\end{align*}

Proof. From Definition 1.1, we are able to deduce (see [10]): \( \forall \varphi \in C^1(S) \), \( \varphi = 0 \) when \( |x| \) is large enough,

\begin{align*}
(2.5) \quad &\int_{\mathbb{R}^N} u_k(x, t) \varphi \, dx - \int_0^T \int_{\mathbb{R}^N} (u_k \varphi_t - u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\varphi) \, dx \, dt \\
&\leq \int_{\mathbb{R}^N} u_{0k}(x) \varphi(x, 0) \, dx.
\end{align*}

Let \( \psi_R \in C^\infty_0(B_{2R}) \), \( 0 \leq \psi_R \leq 1 \), \( \psi_R = 1 \) on \( B_R \), \( |D\psi_R| \leq cR^{-1} \).
By an approximate procedure we can choose \( \varphi = (u_k^s/(1 + u_k^s))\psi_R^p \) in (2.5); then

\[
(2.7) \quad \int_{\mathbb{R}^N} \int_0^{u_k(x,t)} \frac{z^s}{1 + z^s} \, dz \psi_R^p(x) \, dx + s \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p \psi_R^p(x) \, dx \, d\tau
\]

\[
\leq -p \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1 + u_k^s} |Du_k|^{p-2}\psi_R^{p-1}(x)Du_k \cdot D\psi_R \, dx \, d\tau + \int_{\mathbb{R}^N} \int_0^{u_k(x,h)} \frac{z^s}{1 + z^s} \, dz \psi_R^p(x) \, dx,
\]

where \( 0 < h < t \). Notice that

\[
(2.8) \quad \left| \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1 + u_k^s} |Du_k|^{p-2}\psi_R^{p-1}(x)Du_k \cdot D\psi_R \, dx \, d\tau \right|
\]

\[
\leq \int_h^t \int_{\mathbb{R}^N} \left[ \varepsilon \left( \frac{u_k^{(s+m-2)(p-1)/p}}{(1 + u_k^s)^2(p-1)/p} |Du_k|^{p-1}\psi_R^{p-1} \right)^{p/(p-1)} + c(\varepsilon) \frac{u_k^{p+m-2}}{(1 + u_k^s)^{2(p-1)/p}} |D\psi_R|^p \right] \, dx \, dt
\]

\[
= \int_h^t \int_{\mathbb{R}^N} \left[ \varepsilon \left( \frac{u_k^{(s+m-2)(p-1)/p}}{(1 + u_k^s)^2(p-1)/p} |Du_k|^{p-1}\psi_R^{p-1} \right)^{p/(p-1)} + c(\varepsilon) \frac{u_k^{p+m-2}}{(1 + u_k^s)^{2(p-1)/p}} |D\psi_R|^p \right] \, dx \, dt,
\]

\[
(2.9) \quad \int_{\mathbb{R}^N} \int_0^{u_k(x,h)} \frac{z^s}{1 + z^s} \, dz \psi_R^p(x) \, dx \leq \int_{\mathbb{R}^N} u(x,k^N\mu_\varepsilon h) \, dx,
\]

hence by (2.7)–(2.9) we obtain

\[
(2.10) \quad \sup_{0 < t < T} \int_{\mathbb{R}^N} \int_0^{u_k(x,t)} \frac{z^s}{1 + z^s} \, dz \, dx + \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p \psi_R^p \, dx \, d\tau
\]

\[
\leq c \int_{\mathbb{R}^N} u(x,k^N\mu_\varepsilon h) \, dx + c \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{p+m-2}}{(1 + u_k^s)^{2-p}} |D\psi_R|^p \, dx \, d\tau.
\]

Because \( u_k \in L^\infty(\mathbb{R}^N \times (h, T)) \cap L^1(S_T), p + m - 2 > 0 \), we have

\[
(2.11) \quad \lim_{R \to \infty} \int_h^t \int_{\mathbb{R}^N} \frac{u_k^{p+m-2}}{(1 + u_k^s)^{2-p}} |D\psi_R|^p \, dx \, d\tau = 0.
\]

Let \( R \to \infty, h \to 0 \) in (2.10). Then

\[
(2.12) \quad \sup_{0 < t < T} \int_{\mathbb{R}^N} \int_0^{u_k(x,t)} \frac{z^s}{1 + z^s} \, dz \, dx + \int_{S_t} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p \, dx \, d\tau
\]

\[
\leq c \int_{\mathbb{R}^N} u_0(x) \, dx.
\]
Thus

\[ (2.13) \quad \sup_{0 < t < T} \int_{B_{2R}} u_k(x, t) \, dx + \int_0^T \int_{B_{2R}} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p \, dx \, d\tau \leq c(R). \]

Let

\[ u_1 = \max\{u_k(x, t), 1\}, \quad w = u_1^{(m+p-2-s)/p}. \]

By Sobolev’s imbedding inequality (see [6]), for \( \xi \in C^1_0(B_{2R}) \), \( \xi \geq 0 \), we have

\[ \left( \int_{\mathbb{R}^N} \xi^pw^r \, dx \right)^{1/r} \leq c \left( \int_{\mathbb{R}^N} |D(\xi w)|^p \right)^{s/p} \left( \int_{B_{2R}} u^{p/(m+p-2-s)} \, dx \right)^{((1-\theta)(m+p-2-s)/p}, \]

where

\[ \theta = \frac{m+p-2-s}{p} - \frac{1}{r} \left( \frac{1}{N} - \frac{1}{p} + \frac{m+p-2-s}{p} \right)^{-1}, \]

\[ r = \frac{p(m+p-2+p/N-s)}{m+p-2-s}. \]

It follows that

\[ (2.14) \quad \iint_{S_T} \xi^pw^r \, dx \, dt \]

\[ \leq c \iint_{S_T} |D(\xi w)|^p \, dx \, dt \]

\[ \times \sup_{t \in (0, T)} \left( \int_{B_{2R}} u^{p/(m+p-2-s)} \, dx \right)^{(r-p)(m+p-2-s)/p}. \]

Since

\[ |Dw|^p \leq c \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p \quad \text{a.e. on} \quad \{u_k \geq 1\} \quad \text{and} \quad |Dw| = 0 \quad \text{on} \quad \{u_k \leq 1\}, \]

we have

\[ (2.15) \quad \iint_{S_T} |D(\xi w)|^p \, dx \, dt \leq c \iint_{S_T} (\xi^p |Dw|^p + w^p |D\xi|^p) \, dx \, dt \]

\[ \leq c \left[ \iint_{S_T} |D\xi|^p u_1^{p+m-2-s} \, dx \, dt \right. \]

\[ + \left. \int_0^T \int_{B_{2R}} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p \, dx \, d\tau \right]. \]
Hence, by (2.14), (2.15) and (2.13), we get
\[
\iint_{S_T} \xi^p u_1^{m+p-2+p/N-s} \, dx \, dt \leq c(s, R, |u_0|_{L^1}) \left( 1 + \iint_{S_T} |D\xi|^p u_1^{p+m-2-s} \, dx \, dt \right).
\]

Let \( \xi = \psi^b_R \), where \( \psi_R \) is the function satisfying (2.6) and \( b = N(m + p - 2 - s)/p \). Then
\[
\iint_{S_T} \psi^b_R u_1^{m+p-2+p/N-s} \, dx \, dt
\]
\[
\leq c(s, R, |u_0|_{L^1}) \left( 1 + \iint_{S_T} \psi^b_R u_1^{p+m-2+2s/p} \, dx \, dt \right)^{(m+p-2-s)/(m+p-2+p/N-s)},
\]
which implies (2.4) is true. \( \square \)

Let \( Q_\varrho = B_\varrho(x_0) \times (t_0 - \varrho^p, t_0) \) with \( t_0 > (2\varrho)^p \) and \( u_{k1} = \max\{u_k, 1\} \).

**Lemma 2.2.** Each \( u_k \) satisfies

(2.16) \[
\sup_{Q_\varrho} u_k \leq c(\varrho, s_1) \left( \iint_{Q_{2\varrho}} u_{k1}^{p+m-3+s_1} \, dx \, dt \right)^{1/s_1},
\]

where \( c(\varrho, s_1) \) depends on \( \varrho \) and \( s_1 \), and \( s_1 \) can be any number satisfying \( 0 < s_1 < 1 + p/N \).

**Lemma 2.3.** Each \( u_k \) satisfies

(2.17) \[
\int_T^\tau \int_{B_R} u_k^{m-1} |Du_k|^p \, dx \, dt \leq c(\tau, R), \quad \int_T^\tau \int_{B_R} |u_{kt}|^p \, dx \, dt \leq c(\tau, R).
\]

**Proof.** By Lemma 2.1 and 2.2, \( u_k \) are uniformly bounded on every compact set \( K \subset S_T \). Let \( \psi_R \) be a function satisfying (2.6) and let \( \xi \in C_0^\infty(0, T + 1) \) with \( 0 \leq \xi \leq 1, \xi = 1 \) if \( t \in (\tau, T) \). We choose \( \eta = \psi^R_R \xi u_k \) in (2.5) to obtain

(2.18) \[
\frac{1}{2} \iint_{S_T} u_k^p(x, T) \psi^R_R \, dx + \iint_{S_T} u_k^{m-1} |Du_k|^p \psi^p_R \xi \, dx \, dt
\]
\[
\leq \frac{1}{2} \iint_{S_T} u_k^p \xi \psi^R_R \, dx \, dt - p \iint_{S_T} u_k^m |Du_k|^{p-2} Du_k \cdot D\psi^p_R \psi^R_R \xi \, dx \, dt.
\]

Notice that

(2.19) \[
\iint_{S_T} u_k^m |Du_k|^{p-1} |D\psi^R_R| \psi^{p-1}_R \xi \, dx \, dt
\]
\[
\leq \varepsilon \iint_{S_T} u_k^{m-1} |Du_k|^p \psi^p_R \psi^R_R \xi \, dx \, dt + c(\varepsilon) \iint_{S_T} u_k^{p+m-1} |D\psi^R_R|^p \xi \, dx \, dt.
\]

By (2.18), (2.19), one knows that the first inequality of (2.17) is true.
Now we will prove the second inequality of (2.17). Let

\[ v(x, t) = u_{kr}(x, t) = ru_k(x, r^{m+p-3}t), \quad r \in (0, 1). \]

Then

\begin{align*}
(2.20) & \quad v_t(x, t) = \text{div}(v^{m-1} |Du|^{p-2}Du) - r^{m+p-2-q}k^{-v}u^q, \\
(2.21) & \quad v(x, 0) = ru_k(x, 0).
\end{align*}

Notice that \( r^{m+p-2-q}k^{-v} > k^{-v} \) using the argument similar to that in the proof of Theorem 1 of [12], one can prove

\[ u_k \geq u_{kr}. \]

It follows that

\[ \frac{u_k(x, r^{m+p-3}t) - u_k(x, t)}{(r^{m+p-3} - 1)t} \geq \frac{r - 1}{(1 - r^{m+p-3})t} u_k(x, r^{m+p-3}t). \]

Letting \( r \to 1 \), we get

\begin{equation}
(2.22) \quad u_{kt} \geq -\frac{u_k}{(m + p - 3)t}.
\end{equation}

Denote \( w = t^\beta u_k(x, t), \beta = 1/(m + p - 3) \). By (2.22), \( w_t \geq 0 \). By (2.1),

\begin{align*}
(2.23) & \quad \int_\tau^T \int_{B_{2R}} t^\beta w_t \psi_R \, dx \, dt \\
& \quad = -\int_\tau^T \int_{B_{2R}} u_k^{m-1} |Du_k|^{p-2}Du_k \cdot D\psi_R \, dx \, dt \\
& \quad - \int_\tau^T \int_{B_{2R}} k^{-v}u_k^q \psi_R \, dx \, dt + \beta \int_\tau^T \int_{B_{2R}} t^{-1}u_k(x)\psi_R \, dx \, dt \\
& \quad \leq \frac{\beta}{\tau} \int_\tau^T \int_{B_{2R}} u_k \, dx \, dt \\
& \quad + \left( \int_\tau^T \int_{B_{2R}} u_k^{m-1} |Du_k|^{p} \, dx \, dt \right)^{(p-1)/p} \left( \int_\tau^T \int_{B_{2R}} |D\psi_R|^{p} \, dx \, dt \right)^{1/p}.
\end{align*}

From (2.13), (2.16) and (2.23) we obtain (2.17). \( \square \)
Proof of Theorem 1.3.

By Lemmas 2.1–2.3 and [2], there exists a subsequence \{u_{k_j}\} of \{u_k\} and a function \(v\) such that on every compact set \(K \subset S\)
\[
u_{k_j} \rightarrow v \quad \text{in } C(K), \quad Du_{k_j}^m \rightarrow Du^m \quad \text{in } L^p_{\text{loc}}(S_T), \quad |u_{k}t|_{L^1_{\text{loc}}(S_T)} \leq c.
\]
Similar to what was done in the proof of Theorem 2 in [12], we can prove that \(v\) satisfies (1.1) in the sense of distributions.

We now prove \(v(x,0) = c\delta(x)\). Let \(\chi \in C_0^1(B_R)\). Then we have
\[
\int_{\mathbb{R}^N} u_k(x,t)\chi dx - \int_{\mathbb{R}^N} \varphi_k \chi dx = - \int_0^t \int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\chi dx ds - k^{-v} \int_0^t \int_{\mathbb{R}^N} u_k^q \chi dx ds.
\]
To estimate \(\int_0^t \int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\chi dx ds\), without losing generality, one can assume that \(u_k > 0\). By Hölder inequality and Lemma 2.1,
\[
\left| \int_0^t \int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\chi dx ds \right| 
\leq c \left( \int_0^T \int_{B_{2R}} \frac{u_k^{s+m-2}}{(1 + u_k^s)^2} |Du_k|^p dx dt \right)^{(p-1)/p}
\times \left( \int_0^T \int_{B_{2R}} (1 + u_k^s)^{2(p-1)} u_k^{(p-1)(2-s-m)} dx dt \right)^{1/p}
\leq c \left( \int_0^T \int_{B_{2R}} \left( u_k^{(p-1)(2-s-m)} + u_k^{(p-1)(2+s-m)} \right) dx dt \right)^{1/p}
\leq c \left( \int_0^T \int_{B_{2R}} \left( u_k^{(p-1)(2-s-m)} \right)^{\frac{m+p-2+m/N}{(p-1)(s-2-m)}} dx dt \right)^{\frac{1}{1/p}}
\]
where \(s \in (0,1/N), d = ((m-s-1)Np+(s-2)N+p-s+2)/((m+p-2)N+p-s) < 1\) because \(p > (N+3)/(2N+1), u_{k1} = \max(u_k,1)\).

Hence from (2.24) we get
\[
\left| \int_{\mathbb{R}^N} u_k(x,t)\chi dx - \int_{\mathbb{R}^N} \varphi_k \chi dx + k^{-v} \int_0^t \int_{\mathbb{R}^N} u_k^q \chi dx ds \right|
\leq \left| \int_{\mathbb{R}^N} u_k(x,t)\chi dx - \int_{\mathbb{R}^N} \varphi_k \chi(k^{-1}x) dx + \int_0^T \int_{\mathbb{R}^N} u_k^q \chi(k^{-1}x) dx dt \right| \leq ct^d.
\]
Letting now \(k \rightarrow \infty, t \rightarrow 0\), we obtain
\[
\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} v(x,t)\chi dx = \chi(0) \left( \int_{\mathbb{R}^N} \varphi(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^q dx dt \right).
\]
Thus
\[ v(x, 0) = c\delta(x), \quad c = \int_{\mathbb{R}^N} \varphi(x) \, dx - \int_0^\infty \int_{\mathbb{R}^N} u^q \, dx \, dt, \]
v(x, t) is a solution of (1.3)–(1.4). By the assumption on uniqueness of solution, we have \( v(x, t) = E_c(x, t) \) and the whole sequence \( \{u_k\} \) converges to \( E_c \) as \( k \to \infty \). Set \( t = 1 \). Then
\[ u_k(x, 1) = k^N u(kx, k^{N\mu}) \to E_c(x, 1) \]
uniformly on every compact subset of \( \mathbb{R}^N \). Thus writing \( kx = k', k^{N\mu} = t' \), and dropping the prime again, we see that
\[ t^{1/\mu} u(x, t) \to E_c(xt^{1/(N\mu)}, 1) = t^{1/\mu} E_c(x, t) \]
uniformly on the sets \( \{x \in \mathbb{R}^N : |x| \leq at^{1/(N\mu)}\} \), \( a > 0 \). Thus Theorem 1.3 is true. \( \square \)

3. PROOFS OF THEOREM 1.4 AND 1.5

Let \( u \) be a solution of (1.1)–(1.2) and let \( u_k(x, t) = k^\delta u(kx, k^\beta t), k > 0 \). If \( \delta = 1/(q - (m + p - 2)), \beta = (q - 1)/(q - (m + p - 2)) \), then
\[ u_k(t) = \text{div}(u_k^{m-1}|Du_k|^{p-2}Du_k) - u_k^q, \]
\[ u_k(x, 0) = \varphi_k(x) = k^{\delta}\varphi(kx). \]

**Lemma 3.1.** The solution \( u_k \) of (3.1)–(3.2) satisfies
\[ u_k(x, t) \leq C^* t^{-1/(q-1)}, \quad C^* = \left(\frac{1}{q-1}\right)^{1/(q-1)}. \]

**Proof.** We consider the regularized problem of (3.1), namely,
\[ u_{kt} = \text{div}((u_k^{m-1} + \varepsilon)|Du_k|^2 + \varepsilon)^{(p-2)/2}Du_k) - u_k^q. \]
By the uniqueness of the solution of (3.1)–(3.2), we can prove that
\[ u_{k\varepsilon} \to u_k \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad C(K) \]
on every compact set \( K \subset S \), where \( u_{k\varepsilon} \) are the solutions of (3.4), (3.2). By computation, it is easy to show that \( C^*(t - t_0)^{-1/(q-1)} \) is a solution of (3.4) in \( \mathbb{R}^N \times (t_0, \infty) \), \( t_0 > 0 \). For any \( \delta_1 > 0 \), we choose \( \delta_0 \in (0, \delta_1) \) such that
\[ |u_{k\varepsilon}(x, \delta_1)|_{L^\infty(\mathbb{R}^N)} \leq C^*(\delta_1 - \delta_0)^{-1/(q-1)}. \]
Hence by the comparison principle, we have

\[ u_{k\varepsilon}(x, t) \leq C^*(t - t_0)^{-1/(q-1)}, \quad t > \delta_1. \]

The proof of Lemma 3.1 is completed by letting \( \delta_1 \to 0 \) and \( \varepsilon \to 0 \). \( \square \)

**Lemma 3.2.** Each \( u_k \) satisfies

\[ \int_{\tau}^{T} \int_{B_R} |Du_k|^p \leq c(\tau, R), \quad \int_{\tau}^{T} \int_{B_R} |u_{kt}| \, dx \, dt \leq c(\tau, R), \]

where \( \tau \in (0, T) \).

**Proof.** The proof of Lemma 3.2 is similar to that of Lemma 2.3. \( \square \)

**Proof of Theorem 1.4.** By Lemma 3.1, \( \{u_k\} \) are uniformly bounded on every compact set of \( S \). Hence by [2], there exists a subsequence \( \{u_{k_j}\} \) and a function \( U \in C(S) \) such that

\[ u_{k_j} \to U \quad \text{in} \quad C(K) \]

and

\[ U(x, t) \leq C^* t^{-1/(q-1)}. \]

We now prove that \( U(x, t) = C^* t^{-1/(q-1)} \). Let us introduce the function

\[ \varphi_A^k = \min\{\varphi_k, A\} \]

and denote by \( V_{K\varepsilon}^A \) the solution of (3.4) with initial value (3.6). By the comparison principle,

\[ V_{K\varepsilon}^A \leq u_{k\varepsilon}, \]

where \( u_{k\varepsilon} \) is the solution of (3.4), (3.2).

Define

\[ V_A = C^* \left(t + \frac{A^{1-q}}{q-1}\right)^{-1/(q-1)}, \]

which is the solution of (3.4) with initial value

\[ V_A(x, 0) = A. \]

Notice that

\[ \lim_{k \to \infty} \varphi_A^k(x) = \lim_{k \to \infty} \min\left\{ A, \varphi(kx) \frac{|kx|^{\alpha} k^{\delta-\alpha}}{|x|^{\alpha}} \right\} = A. \]
Using the uniqueness of solution of (3.4), (3.8), we can prove (see [6])
\[ V^A_{k\varepsilon} \to V_A \quad \text{as} \quad k \to \infty \quad \text{in} \quad C(K), \]
where \( K \) is a compact set in \( S \). Moreover, by [2] and [12]
\[ V^A_{k\varepsilon} \to V^A_k u_{k\varepsilon} \to u_k \quad \text{as} \quad k \to \infty \quad \text{in} \quad C(K) \]
uniformly in \( K \), where \( V^A_k \) is the solution of (1.1) with initial value (3.6). It follows that
\[ V^A_k \to V_A \quad \text{as} \quad k \to \infty \quad \text{in} \quad C(K). \]

Letting \( \varepsilon \to 0 \) and \( k \to \infty \) in turn in (3.7), we get
\[ V_A(x,t) \leq V_\infty(x,t) = C^* t^{-1/(q-1)} \quad \text{in} \quad S. \]
Since the lower bound holds for every \( A > 0 \), we conclude that
\[ U(x,t) = V_\infty(x,t) = C^* t^{-1/(q-1)} \quad \text{in} \quad S. \]
Thus
\[ k^{p/(q-(m+p-2))} u(kx, k^\beta t) \to C^* t^{-1/(q-1)} \quad \text{as} \quad k \to \infty. \]

Set \( t = 1 \). Then
\[ k^{p/(q-(m+p-2))} u(kx, k^\beta) \to C^* \quad \text{as} \quad k \to \infty \]
uniformly on every compact subset of \( \mathbb{R}^N \). Therefore if we set \( kx = x', \ k^\beta = t' \), and omit the primes, we obtain
\[ t^{1/(q-1)} u(x,t) \to C^* \quad \text{as} \quad t \to \infty \]
uniformly on sets \( \{ x \in \mathbb{R}^N : |x| \leq \alpha t^{1/\beta} \} \) with \( \alpha > 0 \) for \( t > 0 \) and so Theorem 1.4 is proved. \( \square \)

Proof of Theorem 1.5. By Lemma 3.1 and [2], there exist a subsequence \( \{ u_{k_j} \} \) and a function \( U \in C(S) \) such that
\[ u_{k_j} \to U \quad \text{in} \quad C(K). \]
By Lemma 3.2, we can prove that \( U \) satisfies (1.1) in the sense of distributions in a manner similar to Theorem 2 of [12]. \( \square \)
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References


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