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APPROXIMATION AND EIGENVALUE EXTRAPOLATION OF STOKES EIGENVALUE PROBLEM BY NONCONFORMING FINITE ELEMENT METHODS*

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Abstract. In this paper we analyze the stream function-vorticity-pressure method for the Stokes eigenvalue problem. Further, we obtain full order convergence rate of the eigenvalue approximations for the Stokes eigenvalue problem based on asymptotic error expansions for two nonconforming finite elements, $Q_1^{\rm rot}$ and $EQ_1^{\rm rot}$. Using the technique of eigenvalue error expansion, the technique of integral identities and the extrapolation method, we can improve the accuracy of the eigenvalue approximations.

Keywords: Stokes eigenvalue problem, stream function-vorticity-pressure method, asymptotic expansion, extrapolation, a posteriori error estimates, nonconforming finite element methods

MSC 2010: 65N30, 65N25, 35Q30

1. Introduction

There are various approximation methods for solving the Stokes problem, see Bercovier and Pironneau [3], Brezzi et al. [4], Chen and Lin [5], Girault and Raviart [9], Glowinski and Pironneau [10], Han [11], Křížek [13], Mercier et al. [20], Rannacher and Turek [21], Wang and Ye [23], Ye [25], Zhou and Li [26], and references cited therein.

In this article we will study two nonconforming finite elements, $Q_1^{\rm rot}$ and $EQ_1^{\rm rot}$, for the eigenvalue approximations of the Stokes eigenvalue problems. For simplicity

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we consider the model eigenvalue problem stated as follows:

$$
(1) \qquad \begin{cases} -\Delta \mathbf{u} + \mathbf{grad}\, p = \lambda \mathbf{u} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases}
$$

where Ω is a rectangular domain in $\mathbb{R}^2$. For simplicity, we take $\Omega = (0,1) \times (0,1)$.

It is well known that the extrapolation method is an efficient procedure for improving the accuracy of approximation of many problems in numerical analysis. The effectiveness of this technique relies heavily on the existence of an asymptotic expansion for the error. This technique has been well demonstrated in its application to the finite element methods [17], [18] and [22].

The application of the extrapolation method to eigenvalue problems was first proposed by Q. Lin and T. Lü [16], and was analyzed in [14], [15], [17], [19], and [22].

In [5], the bilinear finite element has been analyzed for the Stokes eigenvalue problem and the error asymptotic expansion and the extrapolation formula were given.

This paper is organized in the following way. In Section 2 we present asymptotic expansions for nonconforming finite elements, Q_1^{rot} and EQ_1^{rot}. The analysis for the eigenvalue problem is given in Section 3, and in Section 4 we derive the expansions of the eigenvalue error, using some integral identities and the Bramble-Hilbert lemma in our analysis. Section 5 is devoted to extrapolation and an a posteriori error estimate for eigenvalue approximations. And finally, in Section 6 numerical experiments are reported.

Throughout this article we shall use the standard notation as in Chen [7] and Ciarlet [8], for example, the notation of the Sobolev space, product, norms, seminorms, and discretized norms.

The main technique we use is the eigenvalue error expansion technique first proposed by Lin and Lü [16], and the Bramble-Hilbert lemma [15].

2. Q_1^{rot} and EQ_1^{rot} elements

Let $\mathbf{T}_h = \{e\}$ be a rectangular partition over Ω, where

$$
e = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e],
$$

$h = \max_e \{h_e, k_e\}$. Moreover, $\mathbf{T}_h$ is regular, i.e.

$$
C_0 h^2 \leqslant \operatorname{meas}(e) \leqslant C_1 h^2 \quad \forall\, e \in \mathbf{T}_h,
$$

where $C_i > 0$ $(i = 0, 1)$.

The Q_1^{rot} finite element space V_h is defined as follows:

$$V_h := \{v \in L^2(\Omega);\ v|_e \in \mathrm{span}\{1, x, y, x^2 - y^2\},\ \forall\, e \in \mathbf{T}_h\}$$

with the interpolation $u_I \in V_h$ which is defined by the edge conditions

$$\int_{l_i} u \,\mathrm{d}s = \int_{l_i} u_I \,\mathrm{d}s, \quad i = 1, 2, 3, 4,$$

where l_i $(i = 1, 2, 3, 4)$ are the four edges of e. In addition, set

$$V_{0h} = \{v \in V_h\,;\ v|_{\partial\Omega} = 0\}.$$

The EQ_1^{rot} finite element space W_h is defined as follows:

$$W_h := \{v \in L^2(\Omega);\ v|_e \in \mathrm{span}\{1, x, y, x^2, y^2\},\ \forall\, e \in \mathbf{T}_h\}$$

with the interpolation $u_I \in W_h$ which is defined by the edge-surface conditions

$$\int_{l_i} u \,\mathrm{d}s = \int_{l_i} u_I \,\mathrm{d}s, \quad i = 1, 2, 3, 4,$$
$$\int_{e} u \,\mathrm{d}x \,\mathrm{d}y = \int_{e} u_I \,\mathrm{d}x \,\mathrm{d}y,$$

where l_i $(i = 1, 2, 3, 4)$ are the four edges of e.

In addition, set

$$W_{0h} = \{v \in W_h\,;\ v|_{\partial\Omega} = 0\}.$$

It is obvious that $V_h \not\subseteq H^1(\Omega)$ and $W_h \not\subseteq H^1(\Omega)$.

We have the following integral expansions (see [15]):

Lemma 2.1. *For all $v \in V_h$ or W_h and $u \in H^5(\Omega)$ we have*

$$(2)\qquad \sum_{e\in\mathbf{T}_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} v \,\mathrm{d}s = \sum_{e\in\mathbf{T}_h} \Bigg(\frac{k_e^2}{3} \int_e u_{xxy} v_y \,\mathrm{d}x\,\mathrm{d}y - \frac{4k_e^4}{45} \int_e u_{xxyy} v_{yy} \,\mathrm{d}x\,\mathrm{d}y + \frac{h_e^2}{3} \int_e u_{yyx} v_x \,\mathrm{d}x\,\mathrm{d}y - \frac{4h_e^4}{45} \int_e u_{yyxx} v_{xx} \,\mathrm{d}x\,\mathrm{d}y \Bigg) + O(h^4)|u|_5 |v|_{1,h}$$

and furthermore, if T_h is uniform and $\partial u/\partial \mathbf{n}|_{\partial\Omega} = 0$ or $v|_{\partial\Omega} = 0$, we have

$$(3)\qquad \sum_{e\in\mathbf{T}_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} v \,\mathrm{d}s = -\frac{h^2 + k^2}{3} \int_\Omega u_{xxyy} v \,\mathrm{d}x\,\mathrm{d}y + O(h^4)\|u\|_5 \|v\|_{2,h},$$

where h and k are the mesh step sizes in x- and y-directions, respectively.

Lemma 2.2. *If $v \in V_h$, $\mathbf{T}_h$ is a square mesh and $u_I \in V_h$ then*

$$\sum_{e\in\mathbf{T}_h} \int_e \nabla(u-u_I)\nabla v \,\mathrm{d}x\,\mathrm{d}y = 0, \tag{4}$$

$$\int_\Omega (u-u_I)v \,\mathrm{d}x\,\mathrm{d}y = -\frac{h^2}{6}\int_\Omega (u_{xx}+u_{yy})v\,\mathrm{d}x\,\mathrm{d}y + O(h^4)\|u\|_4\|v\|_{1,h}. \tag{5}$$

Lemma 2.3. *For all $v \in W_h$, $u_I \in W_h$ we have*

$$\sum_{e\in\mathbf{T}_h} \int_e \nabla(u-u_I)\nabla v \,\mathrm{d}x\,\mathrm{d}y = 0, \tag{6}$$

$$\int_\Omega (u-u_I)v \,\mathrm{d}x\,\mathrm{d}y = O(h^4)\|u\|_4\|v\|_{1,h}. \tag{7}$$

3. Approximation of the Stokes eigenvalue problem

In this section we consider a stream function-vorticity-pressure method to solve the eigenvalue problem (1).

We introduce the stream function ψ for the velocity ($\mathbf{u} = \mathbf{curl}\,\psi = (\partial_2\psi, -\partial_1\psi)$), based on the identities ([4], [8])

$$\begin{aligned}\mathbf{curl}(\mathrm{curl}\,\mathbf{u}) &= -\Delta\mathbf{u} + \mathbf{grad}(\mathrm{div}\,\mathbf{u}),\\ \mathrm{curl}(\mathbf{curl}\,\psi) &= -\Delta\psi\end{aligned}$$

where $\mathrm{curl}\,\mathbf{u} = -\partial_2 u_1 + \partial_1 u_2$. Problem (1) can be expressed as the following buckling plate problem:

Find λ, ψ satisfying

$$\begin{cases} -\Delta^2\psi = \lambda\Delta\psi & \text{in } \Omega,\\ \psi = \dfrac{\partial\psi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $\mathbf{n}$ is the outward unit normal.

Then we can obtain the following weak mixed formulation for (8) which seeks $\lambda \in \mathbb{R}$, $(\psi, \omega) \in H_0^1(\Omega) \times H^1(\Omega)$ satisfying $s(\psi,\psi) = 1$ and

$$\begin{cases} a(\omega,\theta) + b(\theta,\psi) = 0 & \forall\,\theta \in H^1(\Omega),\\ b(\omega,\varphi) = -\lambda s(\psi,\varphi) & \forall\,\varphi \in H_0^1(\Omega), \end{cases} \tag{9}$$

and find $p \in H^1(\Omega)$ such that

$$(10) \qquad \begin{cases} (\mathbf{grad}\, p, \mathbf{grad}\, q) = \lambda(\mathbf{u} - \mathbf{curl}\, \omega, \mathbf{grad}\, q) & \forall\, q \in H^1(\Omega), \\ \displaystyle\int_\Omega p \,\mathrm{d}x = 0 \end{cases}$$

with $\mathbf{u} = \mathbf{curl}\, \psi$, where $\omega = -\Delta\psi$ and

$$a(\omega, \theta) = \int_\Omega \omega\theta \quad \forall\, \omega, \theta \in H^1(\Omega),$$

$$b(\omega, \varphi) = -\int_\Omega \mathbf{curl}\, \omega\, \mathbf{curl}\, \varphi \,\mathrm{d}x \,\mathrm{d}y \quad \forall\, \omega \in H^1(\Omega),\ \ \varphi \in H_0^1(\Omega),$$

$$s(\psi, \varphi) = \int_\Omega \mathbf{curl}\, \psi\, \mathbf{curl}\, \varphi \,\mathrm{d}x \,\mathrm{d}y \quad \forall\, \psi \in H_0^1(\omega),\ \ \varphi \in H_0^1(\Omega).$$

Problem (9) has an eigenvalue sequence $\{\lambda_j\}$ ([1], [2]):

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots, \qquad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$(\psi_1, \omega_1), (\psi_2, \omega_2), \ldots, (\psi_k, \omega_k), \ldots,$$

where $(\mathbf{curl}\, \psi_i, \mathbf{curl}\, \psi_j) = \delta_{ij}$, $\omega_k = -\Delta\psi_k$.

The finite element approximation of (9) is to seek $\lambda_h \in \mathbb{R}$, $(\psi_h, \omega_h) \in V_{0h} \times V_h$ or $W_{0h} \times W_h$ such that $s_h(\psi_h, \psi_h) = 1$ and

$$(11) \qquad \begin{cases} a_h(\omega_h, \theta) + b_h(\theta, \psi_h) = 0 & \forall\, \theta \in V_h \text{ or } W_h, \\ b_h(\omega_h, \varphi) = -\lambda_h s_h(\psi_h, \varphi) & \forall\, \varphi \in V_{0h} \text{ or } W_{0h}, \end{cases}$$

and find $p_h \in V_h$ or W_h such that

$$(12) \qquad \begin{cases} (\mathbf{grad}\, p_h, \mathbf{grad}\, q_h)_h = \lambda(\mathbf{u}_h - \mathbf{curl}\, \omega_h, \mathbf{grad}\, q_h)_h & \forall\, q_h \in V_h \text{ or } W_h, \\ \displaystyle\int_\Omega p_h \,\mathrm{d}x = 0 \end{cases}$$

with $\mathbf{u}_h = \mathbf{curl}\, \psi_h$, where

$$a_h(\omega, \theta) = \int_\Omega \omega\theta \,\mathrm{d}x \,\mathrm{d}y \quad \forall\, \omega, \theta \in V_h \text{ or } W_h,$$

$$b_h(\omega, \varphi) = -\sum_{e\in\mathbf{T}_h} \int_e \mathbf{curl}\, \omega\, \mathbf{curl}\, \varphi \,\mathrm{d}x \,\mathrm{d}y \quad \forall\, \omega \in V_h \text{ or } W_h,\ \ \varphi \in V_{0h} \text{ or } W_{0h},$$

$$s_h(\psi, \varphi) = \sum_{e\in\mathbf{T}_h} \int_e \mathbf{curl}\, \psi\, \mathbf{curl}\, \varphi \,\mathrm{d}x \,\mathrm{d}y \quad \forall\, \psi \in V_{0h} \text{ or } W_{0h},\ \ \varphi \in V_{0h} \text{ or } W_{0h}.$$

In order to get the error expansion of the eigenvalue problem we analyze the original problem first.

The original problem is: Find $(\psi,\omega)\in H_0^1(\Omega)\times H^1(\Omega)$ such that

$$\begin{cases} a(\omega,\theta)+b(\theta,\psi)=0 & \forall\,\theta\in H^1(\Omega),\\ b(\omega,\varphi)=-s(g,\varphi) & \forall\,\varphi\in H_0^1(\Omega). \end{cases} \tag{13}$$

The finite element approximation for (13) is: Find $(R_h\psi,R_h\omega)\in V_{0h}\times V_h$ or $(R_h\psi,R_h\omega)\in W_{0h}\times W_h$ such that

$$\begin{cases} a_h(R_h\omega,\theta)+b_h(\theta,R_h\psi)=0 & \forall\,\theta\in V_h \text{ or } W_h,\\ b_h(R_h\omega,\varphi)=-s_h(g,\varphi) & \forall\,\varphi\in V_{0h} \text{ or } W_{0h}. \end{cases} \tag{14}$$

Lemma 3.1. *Assume $\psi\in H^5(\Omega)$, $\mathbf{T}_h$ is a square mesh for $Q_1^{\rm rot}$ and a uniform mesh for $EQ_1^{\rm rot}$. Then*

$$\|\psi_I-R_h\psi\|_{1,h}+\|\omega_I-R_h\omega\|_0\leqslant Ch^2, \tag{15}$$

$$\|\psi-R_h\psi\|_0+h\|\psi-R_h\psi\|_{1,h}+\|\omega-R_h\omega\|_0\leqslant Ch^2. \tag{16}$$

P r o o f. Since $\omega=-\Delta\psi$, we have

$$\begin{aligned}
\|\omega_I-R_h\omega\|_0^2 &= a_h(\omega_I-R_h\omega,\omega_I-R_h\omega)\\
&= a_h(\omega_I-\omega,\omega_I-R_h\omega)+a_h(\omega-R_h\omega,\omega_I-R_h\omega)\\
&= a_h(\omega_I-\omega,\omega_I-R_h\omega)-\sum_{e\in\mathbf{T}_h}\int_e\Delta\psi(\omega_I-R_h\omega)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad -a_h(R_h\omega,\omega_I-R_h\omega)\\
&= a_h(\omega_I-\omega,\omega_I-R_h\omega)+\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\omega_I-R_h\omega)\,\mathrm{d}s\\
&\quad +\sum_{e\in\mathbf{T}_h}\int_e\mathbf{curl}\,\psi\,\mathbf{curl}(\omega_I-R_h\omega)\,\mathrm{d}x\,\mathrm{d}y+b_h(\omega_I-R_h\omega,R_h\psi)\\
&= a_h(\omega_I-\omega,\omega_I-R_h\omega)+\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\omega_I-R_h\omega)\,\mathrm{d}s\\
&\quad -b_h(\omega_I-R_h\omega,\psi-R_h\psi)\\
&= a_h(\omega_I-\omega,\omega_I-R_h\omega)+\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\omega_I-R_h\omega)\,\mathrm{d}s\\
&\quad -b_h(\omega_I-R_h\omega,\psi-\psi_I)-b_h(\omega_I-R_h\omega,\psi_I-R_h\psi).
\end{aligned}$$

Let us define

$$\begin{aligned}
\mathrm{I} := a_h(\omega_I - \omega, \omega_I - R_h\omega) + \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\psi(\omega_I - R_h\omega)\,\mathrm{d}s \\
- b_h(\omega_I - R_h\omega, \psi - \psi_I), \\
\mathrm{II} := -b_h(\omega_I - R_h\omega, \psi_I - R_h\psi).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{II} =& -b_h(\omega_I - \omega, \psi_I - R_h\psi) - b_h(\omega - R_h\omega, \psi_I - R_h\psi) \\
=& -b_h(\omega_I - \omega, \psi_I - R_h\psi) + \sum_{e\in\mathbf{T}_h} \int_e \mathbf{curl}\,\omega\,\mathbf{curl}(\psi_I - R_h\psi)\,\mathrm{d}x\,\mathrm{d}y \\
& + b_h(R_h\omega, \psi_I - R_h\psi) \\
=& -b_h(\omega_I - \omega, \psi_I - R_h\psi) - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\omega(\psi_I - R_h\psi)\,\mathrm{d}s \\
& - \sum_{e\in\mathbf{T}_h} \int_e \Delta\omega(\psi - R_h\psi)\,\mathrm{d}x\,\mathrm{d}y - s_h(g, \psi_I - R_h\psi) \\
=& -b_h(\omega_I - \omega, \psi_I - R_h\psi) - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\omega(\psi_I - R_h\psi)\,\mathrm{d}s \\
& - \sum_{e\in\mathbf{T}_h} \int_e \Delta g(\psi_I - R_h\psi)\,\mathrm{d}x\,\mathrm{d}y - s_h(g, \psi_I - R_h\psi) \\
=& -b_h(\omega_I - \omega, \psi_I - R_h\psi) - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\omega(\psi_I - R_h\psi)\,\mathrm{d}s \\
& + \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,g(\psi_I - R_h\psi)\,\mathrm{d}s + \sum_{e\in\mathbf{T}_h} \int_e \mathbf{curl}\,g\,\mathbf{curl}(\psi_I - R_h\psi)\,\mathrm{d}s \\
& - s_h(g, \psi_I - R_h\psi) \\
=& -b_h(\omega_I - \omega, \psi_I - R_h\psi) - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\omega(\psi_I - R_h\psi)\,\mathrm{d}s \\
& + \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,g(\psi_I - R_h\psi)\,\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(17)\quad & \|\omega_I - R_h\omega\|_0^2 \\
&= a_h(\omega_I - \omega, \omega_I - R_h\omega) + \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\psi(\omega_I - R_h\omega)\,\mathrm{d}s \\
&\quad - b_h(\omega_I - R_h\omega, \psi - \psi_I) - b_h(\omega_I - \omega, \psi_I - R_h\psi) \\
&\quad - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,\omega(\psi_I - R_h\psi)\,\mathrm{d}s + \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n}\times\mathbf{curl}\,g(\psi_I - R_h\psi)\,\mathrm{d}s.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
C\|\psi_I - R_h\psi\|_{1,h}^2 &\leqslant b_h(\psi_I - R_h\psi, \psi_I - R_h\psi) \\
&= b_h(\psi_I - R_h\psi, \psi_I - \psi) + b_h(\psi_I - R_h\psi, \psi - R_h\psi),
\end{aligned}
$$

where

$$
\begin{aligned}
&b_h(\psi_I - R_h\psi, \psi - R_h\psi) \\
&\quad = b_h(\psi_I - R_h\psi, \psi) - b_h(\psi_I - R_h\psi, R_h\psi) \\
&\quad = -\sum_{e\in\mathbf{T}_h}\int_e \mathbf{curl}(\psi_I - R_h\psi)\,\mathbf{curl}\,\psi\,\mathrm{d}x\,\mathrm{d}y + a_h(R_h\omega, \psi_I - R_h\psi) \\
&\quad = \sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\psi_I - R_h\psi)\,\mathrm{d}s + \sum_{e\in\mathbf{T}_h}\int_e \Delta\psi(\psi_I - R_h\psi)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\quad + a_h(R_h\omega, \psi_I - R_h\psi) \\
&\quad = \sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\psi_I - R_h\psi)\,\mathrm{d}s - \sum_{e\in\mathbf{T}_h}\int_e \omega(\psi_I - R_h\psi)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\quad + a_h(R_h\omega, \psi_I - R_h\psi) \\
&\quad = \sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\psi_I - R_h\psi)\,\mathrm{d}s - a_h(\omega - R_h\omega, \psi_I - R_h\psi) \\
&\quad = \sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\psi_I - R_h\psi)\,\mathrm{d}s - a_h(\omega - \omega_I, \psi_I - R_h\psi) \\
&\quad\quad - a_h(\omega_I - R_h\omega, \psi_I - R_h\psi).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\text{(18)}\quad C\|\psi_I - R_h\psi\|_{1,h}^2 &\leqslant b_h(\psi_I - R_h\psi, \psi_I - \psi) \\
&\quad + \sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi(\psi_I - R_h\psi)\,\mathrm{d}s \\
&\quad - a_h(\omega - \omega_I, \psi_I - R_h\psi) - a_h(\omega_I - R_h\omega, \psi_I - R_h\psi).
\end{aligned}
$$

From (17), (18) and Lemmas 2.1–2.3 we can prove the assertion of Lemma 3.1. □

Here we assume that all eigenvalues have ascent and their geometric multiplicity is one. From Lemma 3.1 and the results of [6], [12], [20], and [24] we have the following theorem.

Theorem 3.1. *Under the conditions of Lemma* 3.1 *let us assume that* $(\lambda, \psi, \omega) \in \mathbb{R} \times H_0^1(\Omega) \times H^1(\Omega)$ *is an eigenpair of* (9) *and* $(\lambda_h, \psi_h, \omega_h) \in \mathbb{R} \times V_{0h} \times V_h$ *or* $\mathbb{R} \times W_{0h} \times W_h$ *is an eigenpair of* (11). *Then*

$$|\lambda_h - \lambda| \leqslant Ch^2, \tag{19}$$

$$\|\omega - \omega_h\|_0 \leqslant Ch^2, \tag{20}$$

$$\|\psi - \psi_h\|_0 + h\|\psi - \psi_h\|_{1,h} \leqslant Ch^2. \tag{21}$$

4. Asymptotic eigenvalue error expansions by $Q_1^{\rm rot}$ and $EQ_1^{\rm rot}$

Theorem 4.1. *When we use the finite element spaces* $Q_1^{\rm rot}$ *and* $EQ_1^{\rm rot}$ *and under the conditions of Theorem* 3.1, *we have*

$$\begin{aligned}\lambda_h - \lambda = {} & \lambda_h s_h(\psi - \psi_I, \overline{\psi}_h) + b_h(\overline{\omega}_h, \psi - \psi_I) \\ & - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \omega_h \mathbf{n} \times \mathbf{curl}\, \psi \,\mathrm{d}s + a_h(\omega - \omega_I, \overline{\omega}_h) \\ & + b_h(\omega - \omega_I, \overline{\psi}_h) - \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n} \times \mathbf{curl}\, \omega \overline{\psi}_h \,\mathrm{d}s \\ & + \lambda \sum_{e\in\mathbf{T}_h} \int_{\partial e} \mathbf{n} \times \mathbf{curl}\, \psi \overline{\psi}_h \,\mathrm{d}s,\end{aligned} \tag{22}$$

where

$$\overline{\psi}_h = \frac{\psi_h}{s_h(\psi, \overline{\psi}_h)}, \quad \overline{\omega}_h = \frac{\omega_h}{s_h(\psi, \psi_h)}.$$

Proof. It is obvious that $s_h(\psi, \overline{\psi}_h) = 1$. Then

$$\begin{aligned}\lambda_h &= \lambda_h s_h(\psi, \overline{\psi}_h) \\ &= \lambda_h s_h(R_h\psi, \overline{\psi}_h) + \lambda_h s_h(\psi - R_h\psi, \overline{\psi}_h) \\ &:= \mathrm{I} + \Pi,\end{aligned}$$

where $(R_h\psi, R_h\omega)$ is the solution of (14) with $g = \lambda\psi$. Then

$$\begin{aligned}\mathrm{I} &= \lambda_h s_h(R_h\psi, \overline{\psi}_h) = -b_h(\overline{\omega}_h, R_h\psi) = a_h(R_h\omega, \overline{\omega}_h) = a_h(\overline{\omega}_h, R_h\omega) \\ &= -b_h(R_h\omega, \overline{\psi}_h) = \lambda s_h(\psi, \overline{\psi}_h) = \lambda,\end{aligned}$$

$$
\begin{aligned}
\Pi &= \lambda_h s_h(\psi - R_h\psi, \overline{\psi}_h) \\
&= \lambda_h(\psi - \psi_I, \overline{\psi}_h) + \lambda_h s_h(\psi_I - R_h\psi, \overline{\psi}_h) \\
&= \lambda_h(\psi - \psi_I, \overline{\psi}_h) - b_h(\overline{\omega}_h, \psi_I - R_h\psi) \\
&= \lambda_h(\psi - \psi_I, \overline{\psi}_h) - b_h(\overline{\omega}_h, \psi_I - \psi) - b_h(\overline{\omega}_h, \psi - R_h\psi) \\
&:= \Pi_1 + \Pi_2,
\end{aligned}
$$

where

$$
\Pi_1 = \lambda_h s_h(\psi - \psi_I, \overline{\psi}_h) - b_h(\omega_h, \psi_I - \psi),
$$

and since $\omega = -\Delta\psi$,

$$
\begin{aligned}
\Pi_2 =& -b_h(\overline{\omega}_h, \psi - R_h\psi) \\
=& \sum_{e\in\mathbf{T}_h}\int_e \mathbf{curl}\,\overline{\omega}_h\,\mathbf{curl}\,\psi\,\mathrm{d}x\,\mathrm{d}y + b_h(\overline{\omega}_h, R_h\psi) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + \sum_{e\in\mathbf{T}_h}\int_e \overline{\omega}_h\,\mathrm{curl}(\mathbf{curl}\,\psi)\,\mathrm{d}x\,\mathrm{d}y \\
& - a_h(R_h\omega, \overline{\omega}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s - \sum_{e\in\mathbf{T}_h}\int_e \overline{\omega}_h\Delta\psi\,\mathrm{d}x\,\mathrm{d}y - a_h(R_h\omega, \overline{\omega}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + \sum_{e\in\mathbf{T}_h}\int_e \omega\overline{\omega}_h\,\mathrm{d}x\,\mathrm{d}y - a_h(R_h\omega, \overline{\omega}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega - R_h\omega, \overline{\omega}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega - \omega_I, \overline{\omega}_h) + a_h(\omega_I - R_h\omega, \overline{\omega}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega - \omega_I, \overline{\omega}_h) - b_h(\omega_I - R_h\omega, \overline{\psi}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega - \omega_I, \overline{\omega}_h) \\
& - b_h(\omega_I - \omega, \overline{\psi}_h) - b_h(\omega - R_h\omega, \overline{\psi}_h) \\
=& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega - \omega_I, \overline{\omega}_h) - b_h(\omega_I - \omega, \overline{\psi}_h) \\
& + \sum_{e\in\mathbf{T}_h}\int_e \mathbf{curl}\,\omega\,\mathbf{curl}\,\overline{\psi}_h\,\mathrm{d}x\,\mathrm{d}y + b_h(R_h\omega, \overline{\psi}_h)
\end{aligned}
$$

$$
\begin{aligned}
= & -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega-\omega_I,\overline{\omega}_h) - b_h(\omega_I-\omega,\overline{\psi}_h) \\
& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\omega\overline{\psi}_h\,\mathrm{d}s \\
& +\sum_{e\in\mathbf{T}_h}\int_{e}\mathrm{curl}(\mathbf{curl}\,\omega)\overline{\psi}_h\,\mathrm{d}x\,\mathrm{d}y + b_h(R_h\omega,\overline{\psi}_h) \\
= & -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega-\omega_I,\overline{\omega}_h) - b_h(\omega_I-\omega,\overline{\psi}_h) \\
& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\omega\overline{\psi}_h\,\mathrm{d}s - \sum_{e\in\mathbf{T}_h}\int_{e}\Delta\omega\overline{\psi}_h\,\mathrm{d}x\,\mathrm{d}y - \lambda s_h(\psi,\overline{\psi}_h) \\
= & -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega-\omega_I,\overline{\omega}_h) - b_h(\omega_I-\omega,\overline{\psi}_h) \\
& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\omega\overline{\psi}_h\,\mathrm{d}s \\
& +\lambda\sum_{e\in\mathbf{T}_h}\int_{e}\mathrm{curl}(\mathbf{curl}\,\psi)\overline{\psi}_h\,\mathrm{d}x\,\mathrm{d}y - \lambda s_h(\psi,\overline{\psi}_h) \\
= & -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\overline{\omega}_h\mathbf{n}\times\mathbf{curl}\,\psi\,\mathrm{d}s + a_h(\omega-\omega_I,\overline{\omega}_h) - b_h(\omega_I-\omega,\overline{\psi}_h) \\
& -\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\omega\overline{\psi}_h\,\mathrm{d}s + \lambda\sum_{e\in\mathbf{T}_h}\int_{\partial e}\mathbf{n}\times\mathbf{curl}\,\psi\overline{\psi}_h\,\mathrm{d}s.
\end{aligned}
$$

Thus we get the theorem. □

Theorem 4.2. *If $\psi \in H^5(\Omega)$ and $\omega \in H^5(\Omega)$ then for the finite element Q_1^{rot}, when T_h is a square mesh, we have*

$$
\begin{aligned}
\lambda_h - \lambda = & -\frac{h^2}{6}\int_\Omega \Delta^2\psi\Delta\psi\,\mathrm{d}x\,\mathrm{d}y + \frac{4h^2}{3}\int_\Omega \Delta\psi\psi_{xxyy}\,\mathrm{d}x\,\mathrm{d}y \\
& + \frac{2\lambda h^2}{3}\int_\Omega \psi_{xxyy}\psi\,\mathrm{d}x\,\mathrm{d}y + O(h^4),
\end{aligned} \tag{23}
$$

and for the finite element EQ_1^{rot}, if $\mathbf{T}_h$ is uniform, we have

$$
\begin{aligned}
\lambda_h - \lambda = & -\frac{2(h^2+k^2)}{3}\int_\Omega \psi_{xxyy}\Delta\psi\,\mathrm{d}x\,\mathrm{d}y \\
& + \frac{\lambda(h^2+k^2)}{3}\int_\Omega \psi_{xxyy}\psi\,\mathrm{d}x\,\mathrm{d}y + O(h^4).
\end{aligned} \tag{24}
$$

P r o o f. It is obvious that ([15])

$$\|\overline{\psi}_h - \psi_h\|_0 \leqslant ch^2, \quad \|\overline{\omega}_h - \omega_h\|_0 \leqslant ch^2.$$

Then Lemma 2.1–2.3, Theorem 3.1 and Theorem 4.1 yield the assertion of this theorem. □

5. Extrapolation and an a posteriori error estimate for eigenvalues

In order to use the extrapolation method, we assume that $\mathbf{T}_{h/2}$ has been obtained from $\mathbf{T}_h$ by dividing each element into four congruent rectangles by connecting the midpoints of its edges. Let $(\lambda_{h/2}, \psi_{h/2}, \omega_{h/2})$ be the eigensolution approximation on the mesh $\mathbf{T}_{h/2}$.

Denote by

$$\lambda_h^{\text{extra}} = \frac{4\lambda_{h/2} - \lambda_h}{3} \tag{25}$$

the extrapolation of λ. Then by Theorem 4.2 we get the following error estimate for the extrapolation $\tilde{\lambda}_h$ and an a posteriori error estimate for the eigenvalue.

Theorem 5.1. *Under the conditions of Theorem* 4.2 *we have*

$$\lambda - \lambda_h^{\text{extra}} = O(h^4) \tag{26}$$

and thus,

$$\lambda_{h/2} - \lambda = \frac{\lambda_h - \lambda_{h/2}}{3} + O(h^4) \tag{27}$$

provides an a posteriori error estimate $\frac{1}{3}(\lambda_h - \lambda_{h/2})$ *for* $\lambda_{h/2} - \lambda$.

6. Numerical results

First, we introduce some notation

$$\begin{aligned} \mathrm{err}_h &= \lambda_h - \lambda, \\ \lambda_h^{\mathrm{extra}} &= \frac{1}{3}(4\lambda_{h/2} - \lambda_h), \\ \mathrm{err}_h^{\mathrm{extra}} &= \lambda_h^{\mathrm{extra}} - \lambda, \\ R_h &= \frac{\log\big(|\mathrm{err}_h|/|\mathrm{err}_{h/2}|\big)}{\log(2)}, \\ R_h^{\mathrm{extra}} &= \frac{\log\big(|\mathrm{err}_h^{\mathrm{extra}}|/|\mathrm{err}_{h/2}^{\mathrm{extra}}|\big)}{\log(2)}. \end{aligned}$$

We compute the first eigenvalue and take $\lambda = 52.3446911$ (accurate enough).

$M \times N$	4×4	8×8	16×16	32×32	64×64
λ_h	52.15082488284	52.31809045313	52.34015032048	52.34368098538	52.34444610834
$\lambda_h^{\mathrm{extra}}$	–	52.37384564323	52.34750360960	52.34485787368	52.34470114933
err_h	−0.19386621716	−0.02660064687	−0.00454077952	−0.00101011462	−0.00024499166
$\mathrm{err}_h^{\mathrm{extra}}$	–	0.02915454323	0.00281250960	0.00016677368	0.00001004933
R_h	–	2.86552818903	2.55044943728	2.16842097918	2.04371449127
R_h^{extra}	–	–	3.37379079440	4.07589449578	4.0527200264

Table 1. Computation of the first eigenvalue by Q_1^{rot}.

$M \times N$	4×4	8×8	16×16	32×32
λ_h	47.18962964955835	50.70035642020135	51.90518931765972	52.23288933622509
$\lambda_h^{\mathrm{extra}}$	–	51.87059867708234	52.30680028347918	52.34212267574687
err_h	−5.15506145044165	−1.64433467979865	−0.43950178234028	−0.11180176377491
$\mathrm{err}_h^{\mathrm{extra}}$	–	−0.47409242291766	−0.03789081652081	−0.00256842425313
R_h	–	1.64848565724684	1.90356304681561	1.97492606784368
R_h^{extra}	–	–	3.64524820130709	3.88289279718122

Table 2. Computation of the first eigenvalue by EQ_1^{rot}.

From Tabs. 1 and 2 we can find that with the extrapolation the approximation accuracy can be improved from $O(h^2)$ to $O(h^4)$, which validates the corresponding theoretical result in Theorem 5.1 computationally. The extrapolation of the eigenvalue gives a more efficient approximation.

7. Concluding remarks

The nonconforming finite elements, $Q_1^{\rm rot}$ and $EQ_1^{\rm rot}$, can give optimal error estimates under some conditions for the stream function-vorticity-pressure method of the Stokes eigenvalue problems. The eigenvalue extrapolation can improve the accuracy of the eigenvalue approximations and give an a posteriori error estimate.

We can also apply $Q_1^{\rm rot}$ and $EQ_1^{\rm rot}$ elements to other problems and also use the extrapolation method to improve the accuracy order.

We also need to notice that the extrapolation method may give "good" results even though the true solution does not satisfy regularity assumptions guaranteeing superconvergence theoretically.

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