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ESTIMATING AN EVEN SPHERICAL MEASURE FROM ITS SINE TRANSFORM*

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Abstract. To reconstruct an even Borel measure on the unit sphere from finitely many values of its sine transform a least square estimator is proposed. Applying results by Gardner, Kiderlen and Milanfar we estimate its rate of convergence and prove strong consistency. We close this paper by giving an estimator for the directional distribution of certain three-dimensional stationary Poisson processes of convex cylinders which have applications in material science.

Keywords: Boolean model, convex cylinder, direction distribution, least square estimator, parameter estimation, Poisson process, spherical measure, sine transform

MSC 2010: 62M30, 65D15, 52A22, 60D05, 60G10

1. Introduction

It is a well-known fact in stochastic geometry and spatial statistics that the directional distribution of a stationary line or fiber processes can be estimated from intersections of the particles of the process with hyperplanes. The main tool in this procedure is the estimation of an even measure on the unit sphere from finitely many values of its cosine transform. Furthermore, the optimization problem to which the latter reduces to can be discretized in a loss-free way (see [6] and [7] for details).

In modern material science porous fiber materials have a wide range of applications, e.g. in light polymer-based non-woven materials, fiber-reinforced textiles, filters or fuel cells. It is a popular approach to model the microscopic structure of

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the latter as stationary Poisson processes of convex cylinders or their Boolean models. For concrete examples, see [8] or the references given in the introduction of [12]. Usually, this means that each fiber F is described as a Minkowski (vector) sum of a one-dimensional linear subspace $L \subseteq \mathbb{R}^3$ and a randomly scaled disc $B \subseteq L^\perp$ perpendicular to L, i.e.

$$F = rB + L = \{rb + l\colon\ b \in B,\ l \in L\}.$$

The parameters of such a model are the intensity of the Poisson process, the distribution of the scaling factor and the directional distribution of the linear subspaces. We will see in Section 3 that the sine transform of the direction distribution is closely connected to intersections of the process or the Boolean models with line segments. Therefore, we need an estimator of an even Borel measure on the unit sphere which is based on finitely many values of its sine transform. After introducing some basic notation, we propose such an estimator in Section 2, prove its strong consistency and estimate its rate of convergence. Furthermore, we will show that it can be discretized in such a way that a solution of the discretized problem is close to a solution of the original problem.

All proofs have been moved to Appendix to make this paper more readable.

2. Least square estimator

2.1. Preliminaries and notation

For $d \geqslant 2$, let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, $\|\cdot\|$ being its canonical norm, B^d its unit ball, S^{d-1} its unit sphere, $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$, and $\mathcal{B}(S^{d-1})$ the Borel σ-algebra on S^{d-1}. Additionally, let $\|\cdot\|_1$ denote the 1-norm on $\mathbb{R}^d$, and for $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$, let

$$x \geqslant 0 :\Longleftrightarrow x^{(1)} \geqslant 0, \dots, x^{(d)} \geqslant 0.$$

For $u \in S^{d-1}$, let δ_u be the Dirac measure concentrated at u. For fixed $\alpha > 0$, let $\mathcal{M}_e^\alpha(S^{d-1})$ denote the space of all even Borel measures on the unit sphere with total mass less than or equal to α equipped with the Prohorov metric

$$d_P(\mu_1, \mu_2) := \inf\{\varepsilon > 0\colon\ \mu_1(A) \leqslant \mu_2(A + \varepsilon B^d) + \varepsilon,\ F\,\text{closed},\ F \subseteq S^{d-1}\},$$

$\mu_1, \mu_2 \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$. $(\mathcal{M}_e^{(\alpha)}(S^{d-1}), d_p)$ is compact, complete, and separable (see Theorem A2.3 in [5]). For $\mu_1, \mu_2 \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$, let $|\mu_1 - \mu_2|$ denote the variation of the possibly signed measure $\mu_1 - \mu_2$.

For $t > 0$, let $N(t)$ denote the t-covering number of $(\mathcal{M}_e^{(\alpha)}(S^{d-1}), d_P)$, i.e. the least number of balls of radius t needed to cover $\mathcal{M}_e^{(\alpha)}(S^{d-1})$, and $H(t)$ the t-entropy (or metric entropy) of $(\mathcal{M}_e^{(\alpha)}(S^{d-1}), d_P)$, i.e. $H(t) := \log N(t)$. More generally, for any set S with pseudometric d, let $N(t,(S,d))$ and $H(t,(S,d))$ denote the t-covering number and the t-entropy of S with respect to d, respectively.

Let $\mathcal{C}(S^{d-1})$ denote the set of all continuous functions on the unit sphere. The sine transform $\mathbf{S}$ of $\mu \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$ is defined as the mapping

$$\begin{aligned}\mathbf{S}\colon\ \mathcal{M}_e^{(\alpha)}(S^{d-1}) &\to \mathcal{C}(S^{d-1})\\ \mu &\mapsto \Big(u \mapsto \int_{S^{d-1}} \sqrt{1-\langle u,v\rangle^2}\,\mu(\mathrm{d}v)\Big).\end{aligned}$$

Note that μ is uniquely determined by its sine transform (for example, see Section 5 in [4]).

Lemma 1. *Let $\varepsilon > 0$ and $\mu_1, \mu_2 \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$ with $d_P(\mu_1,\mu_2) \leqslant \varepsilon$. Then*

$$|\mathbf{S}(\mu_1)(u) - \mathbf{S}(\mu_2)(u)| \leqslant \varepsilon$$

for all $u \in S^{d-1}$.

Let $\{u_1,\dots,u_m\} \subseteq S^{d-1}$ be a finite sequence of unit vectors. It is called ε-dense if

$$\max_{u\in S^{d-1}} \min_{1\leqslant i\leqslant m} \|u-u_i\| \leqslant \varepsilon.$$

It is well known (for example, see Proposition 3.1 in [2]) that for each $d \geqslant 2$ there is a constant $c(d)$ depending only on d such that for all $\varepsilon > 0$ there exists an ε-dense set on S^{d-1} of at most $c(d)\varepsilon^{-(d-1)}$ points. For $i = 1,\dots,m$ we define the spherical Voronoi cell C_i containing u_i as

$$C_i := \{u \in S^{d-1}\colon\ \|u-u_i\| \leqslant \|u-u_j\| \text{ for all } j \in \{1,\dots,m\}\}.$$

Let $k \in \mathbb{N}$, $X_1, X_2, \dots$ be a sequence of independent uniformly sub-Gaussian random variables, i.e. there exist constants A and τ such that

$$A^2(\mathbb{E}\mathrm{e}^{|X_i|^2/A^2} - 1) \leqslant \tau^2, \quad i = 1,2,\dots, \tag{2.1}$$

$\mu_0 \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$, $v_1, v_2, \dots \in S^{d-1}$ and

$$y_i = \mathbf{S}(\mu_0)(v_i) + X_i, \quad i = 1,2,\dots. \tag{2.2}$$

Any measure $\hat{\mu}_k \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$ that is a solution of

$$\min_{\mu\in\mathcal{M}_e^{(\alpha)}(S^{d-1})} \sum_{i=1}^{k} (y_i - \mathbf{S}(\mu)(v_i))^2 \tag{2.3}$$

is called a least square estimator for μ_0 with respect to $\mathcal{M}_e^{(\alpha)}(S^{d-1})$ based on measurements at $v_1, \ldots, v_k$. Note that by Lemma 1 and compactness of $(\mathcal{M}_e^{(\alpha)}(S^{d-1}), d_P)$ such an estimator $\hat{\mu}_k$ always exists. Also, any sequence of independent identically distributed Gaussian random variables with mean zero and variance σ satisfies (2.1) with $A = \tau = 2\sigma$. In this case (2.3) coincides with the maximum likelihood estimator.

2.2. Rate of convergence

Convergence of such least square estimators has been studied extensively in [2]. To apply those results we have to introduce a pseudometric d_k on $\mathbf{S}(\mathcal{M}_e^{(\alpha)}(S^{d-1}))$ by

$$d_k(\mathbf{S}(\mu_1), \mathbf{S}(\mu_2)) := \sqrt{\frac{1}{k}\sum_{i=1}^{k}(\mathbf{S}(\mu_1)(v_i) - \mathbf{S}(\mu_2)(v_i))^2},$$

$\mu_1, \mu_2 \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$. The following holds:

Corollary 1. *Let $\hat{\mu}_k$ be a least square estimator for μ_0 with respect to $\mathcal{M}_e^{(\alpha)}(S^{d-1})$ based on measurements at $v_1, \ldots, v_k$. Then, almost surely, there exist constants $C(A, \tau, d, \alpha)$ and $N(A, \tau, d, \alpha)$ depending only on A, τ, d, and α such that*

$$d_k(\mu_0, \hat{\mu}_k) \leqslant C(A, \tau, d, \alpha) k^{-(2(d+1))^{-1}}$$

for all $k \geqslant N(A, \tau, d, \alpha)$.

Before we prove this result we have to introduce some more notation: For $\mu_0 \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$, let

$$\mathcal{G}_k(\varepsilon, \mu_0) := \{\mathbf{S}(\mu)\colon\ \mu \in \mathcal{M}_e^{(\alpha)}(S^{d-1}),\ |\mathbf{S}(\mu) - \mathbf{S}(\mu_0)|_k \leqslant \varepsilon\}.$$

It was shown in [2] that estimates for the rate of convergence of $\hat{\mu}_k$ can be derived by bounding the t-entropy of $(\mathcal{G}_k(\varepsilon, \mu_0), d_k)$ from above. By Lemma 1 any t-covering of $(\mathcal{M}_e^{(\alpha)}(S^{d-1}), d_P)$ can be used to construct a t-covering of $(\mathcal{G}_k(\varepsilon, \mu_0), d_k)$, and thus,

$$H(t, (\mathcal{G}_k(\varepsilon, \mu_0), d_k)) \leqslant H(t). \tag{2.4}$$

Therefore, the following lemma combined with (2.4) and Corollary 4.2 in [2] immediately yields Corollary 1.

Lemma 2. *There exists a constant $M(d)$ depending only on d such that*

$$H(t) \leqslant M(d)t^{-(d+1)}$$

for all $0 < t \leqslant 1$.

2.3. Strong consistency

Under some mild assumptions on the sequence $v_1, v_2, \ldots$ we can use Corollary 1 to prove consistency of the estimator $\hat{\mu}_k$. As in [6] we call $v_1, v_2, \ldots$ asymptotically smooth if there exists a strictly positive function $h\colon S^{d-1} \to (0,\infty)$ such that

$$d_P\left(\frac{1}{k}\sum_{i=1}^{k}\delta_{v_i}, \frac{1}{d\kappa_d}\int_{(\cdot)} h(v)\,\omega_{d-1}(\mathrm{d}v)\right) \to 0 \quad \text{for } k \to \infty$$

where κ_d denotes the volume of the d-dimensional unit ball and ω_{d-1} denotes the spherical Lebesgue measure on S^{d-1}. The following holds:

Corollary 2. *Let $v_1, v_2, \ldots$ be asymptotically smooth and let $\hat{\mu}_k$ be a least square estimator for μ_0 with respect to $\mathcal{M}_e^{(\alpha)}(S^{d-1})$ based on measurements at $v_1, \ldots, v_k$. Then, almost surely,*

$$d_P(\mu_0, \hat{\mu}_k) \to 0 \quad \textit{for } k \to \infty,$$

i.e. $\hat{\mu}_k$ is a strongly consistent estimator of μ.

2.4. Discretization

In general, we cannot expect to be able to solve (2.3) analytically or even numerically, but in applications it is usually sufficient to find a good enough approximation of $\hat{\mu}_k$. This can be achieved by discretizing the problem as follows: Let $0 < \varepsilon \leqslant 1$ and $\{-u_1, \ldots, -u_m, u_1, \ldots, u_m\}$ be an ε-dense set on S^{d-1} (recall that we can assume $m \leqslant c(d)\varepsilon^{d-1}$). Furthermore, let $S^+ := \{p \in \mathbb{R}^m\colon\ p \geqslant 0,\ \|p\|_1 \leqslant \alpha\}$. Each vector $p = (p^{(1)}, \ldots, p^{(m)}) \in S^+$ can be identified with the measure μ_p on S^{d-1} by defining

$$\mu_p := \sum_{i=1}^{m} \frac{p^{(i)}}{2}(\delta_{u_i} + \delta_{-u_i}).$$

Therefore, we introduce the set

$$\mathcal{M}_e^{(m,\alpha)}(S^{d-1}) := \{\mu_p\colon\ p \in S^+\},$$

and denote by $\hat{\mu}_{k,m}$ the least square estimator for μ_0 with respect to $\mathcal{M}_e^{(m,\alpha)}(S^{d-1})$ based on measurements at $v_1, \ldots, v_k$. It is a short calculation to see that $\hat{\mu}_{k,m}$ can be determined by solving the quadratic program

$$\begin{aligned} \min \quad & \sum_{i=1}^{k}\left(y_i - \sum_{j=1}^{m}\sqrt{1-\langle v_i, u_j\rangle^2}p^{(j)}\right)^2 \\ \text{s.t.} \quad & p^{(1)} + \ldots + p^{(m)} \leqslant \alpha, \\ & p^{(1)}, \ldots, p^{(m)} \in [0, \infty). \end{aligned}$$

The following holds:

Lemma 3. *Let $0 < \varepsilon \leqslant 1$ and let $\{-u_1, \ldots, -u_m, u_1, \ldots, u_m\}$ be an ε-dense set on S^{d-1}. Furthermore, let $\hat{\mu}_{k,m}$ and $\hat{\mu}_k$ be least square estimators for μ_0 with respect to $\mathcal{M}_e^{(m,\alpha)}(S^{d-1})$ and $\mathcal{M}_e^{(\alpha)}(S^{d-1})$, respectively, based on measurements at $v_1, \ldots, v_k$. Then*

$$|d_k(\mu_0, \hat{\mu}_k) - d_k(\mu_0, \hat{\mu}_{k,m})| \leqslant \varepsilon.$$

2.5. Probability measures

In applications the previous results mainly concern the situation when the total mass of the measure to be estimated can only be bounded from above. In case we have a good estimate for the latter we can slightly modify our problem and thus improve the upper bound for the rate of convergence.

Let $\mathcal{P}_e(S^{d-1})$ be the set of all even probability measures on S^{d-1}. $(\mathcal{P}_e(S^{d-1}), d_P)$ is also compact, complete, and separable (cf. [1]). Furthermore, let $k \in \mathbb{N}$, $\mu_0 \in \mathcal{P}_e(S^{d-1})$, $X_1, X_2, \ldots$, $v_1, v_2, \ldots$ and $y_1, y_2, \ldots$ as before. Again, we call any measure $\hat{\mu}_k \in \mathcal{P}_e(S^{d-1})$ which is a solution of

$$\min_{\mu \in \mathcal{P}_e(S^{d-1})} \sum_{i=1}^{k}(y_i - \mathbf{S}(\mu)(v_i))^2$$

a least square estimator for μ_0 with respect to $\mathcal{P}_e(S^{d-1})$ based on measurements at $v_1, \ldots, v_k$. It follows from the proof of Lemma 2 that $H(t, (\mathcal{P}_e(S^{d-1}), d_P)) \leqslant \overline{M}(d)t^{-d}$ for some constant $\overline{M}(d)$ depending only on d. As before, Corollary 4.2 in [2] yields:

Corollary 3. *Let $\hat{\mu}_k$ be a least square estimator for μ_0 with respect to $\mathcal{P}_e(S^{d-1})$ based on measurements at $v_1, \dots, v_k$. Then, almost surely, there exist constants $\overline{C}(A, \tau, d)$ and $\overline{N}(A, \tau, d)$ depending only on A, τ and d such that*

$$d_k(\mu_0, \hat{\mu}_k) \leqslant \begin{cases} \overline{C}(A, \tau, d) k^{-1/4} \log k & \text{if } d = 2, \\ \overline{C}(A, \tau, d) k^{-1/(2d)} & \text{if } d \geqslant 3 \end{cases}$$

for all $k \geqslant \overline{N}(A, \tau, d)$.

A result analogous to Corollary 2 holds in this case as well. Again, let $0 < \varepsilon \leqslant 1$, let $\{-u_1, \dots, -u_m, u_1, \dots, u_m\}$ be an ε-dense set on S^{d-1} and

$$\mathcal{P}_e^{(m)}(S^{d-1}) := \{\mu_p \colon p \in \mathbb{R}^m,\ p \geqslant 0,\ \|p\|_1 = 1\}.$$

Furthermore, let $\hat{\mu}_{k,m}$ be a least square estimator for μ_0 with respect to $\mathcal{P}_e^{(m)}(S^{d-1})$ based on measurements at $v_1, \dots, v_k$. As before $\hat{\mu}_{k,m}$ can be determined by solving a quadratic program which in this case is given by

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} \left(y_i - \sum_{j=1}^{m} \sqrt{1 - \langle v_i, u_j \rangle^2}\, p^{(j)} \right)^2 \\ \text{s.t.} \quad & p^{(1)} + \ldots + p^{(m)} = 1, \\ & p^{(1)}, \dots, p^{(m)} \in [0, \infty) \end{aligned}$$

and Lemma 3 holds in a similar manner.

3. Parameter estimation for processes of convex cylinders

In this section we will show how the least square estimator considered before can be used as an estimator for the directional distribution of certain Poisson processes of convex cylinders. We refer readers unfamiliar with stochastic geometry to [11] or [13]. We will also use basic notions from convex and integral geometry details of which can be found in [9] and [10], respectively.

3.1. Preliminaries and notation

For finite unions of convex sets, let V_j, $j = 0, \dots, 3$, denote the additive extension of the jth intrinsic volume. Furthermore, let $\mathcal{F}'$ denote the set of all non-empty closed subsets of $\mathbb{R}^3$ equipped with the Fell topology and let $\mathcal{B}(\mathcal{F}')$ be the respective σ-algebra. For $u \in S^{d-1}$, let $L(u) := \{\alpha u \colon \alpha \in \mathbb{R}\}$ and $B(u) \subseteq L^\perp$ be the unit ball in $L(u)^\perp$. Additionally, let X be a Poisson process on $\mathcal{F}'$ with intensity measure Θ given by

$$\Theta(\mathcal{A}) = \gamma \int_{S^2} \int_{\mathbb{R}_+} \int_{L(u)^\perp} 1_{\mathcal{A}}(rB(u) + L(u) + x) \lambda_{L(u)^\perp}(\mathrm{d}x) \mathbb{P}(\mathrm{d}r) \mu(\mathrm{d}u), \quad \mathcal{A} \in \mathcal{B}(\mathcal{F}'),$$

where $\gamma \in \mathbb{R}_+ := (0, \infty)$, $\lambda_{L^\perp}$ denotes the Lebesgue measure on $L^\perp$, and $\mathbb{P}$ and μ are probability measures on $\mathbb{R}_+$ and S^2, respectively. Finally, let $R^{(1)}, R^{(2)} \in \mathbb{R}_+$ denote the first and second moments of $\mathbb{P}$, respectively, and let

$$Z := \bigcup_{F \in X} F$$

be the Boolean model induced by X. In this case, the parameters are the intensity γ of the Poisson process, the distribution $\mathbb{P}$ of r and the direction distribution μ of the linear subspaces.

It was mentioned in the introduction that it is not uncommon to use either X or Z as a model for a porous fiber medium. If single fibers can still be distinguished in digital images of the structure it is more convenient to use X; otherwise, Z is a more natural choice.

3.2. Parameter estimation

We can use the following results to estimate some of the parameters of the model:

Theorem 1. *Let $K_0 \subseteq \mathbb{R}^3$ be a compact and convex set with non-empty interior. Then*

$$\begin{aligned} \mathbb{E}V_3(Z \cap K_0) &= (1 - \mathrm{e}^{-\pi\gamma R^{(2)}})V_3(K_0), \\ \mathbb{E}V_2(Z \cap K_0) &= (1 - \mathrm{e}^{-\pi\gamma R^{(2)}})V_2(K_0) + \pi\gamma R^{(1)}\mathrm{e}^{-\pi\gamma R^{(2)}}V_3(K_0). \end{aligned} \tag{3.1}$$

For $u \in S^2$ and $l \in \mathbb{R}_+$,

$$\mathbb{E}V_0(Z \cap l[0,u]) = 1 - \mathrm{e}^{-\gamma\pi R^{(2)}} + 4l\gamma R^{(1)}\mathrm{e}^{-\gamma\pi R^2} \int_{S^2} \sqrt{1 - \langle u, v\rangle^2}\mu(\mathrm{d}v) \tag{3.2}$$

where $[0,u]$ denotes the line segment between the origin and u. Furthermore,

$$\begin{aligned} \mathbb{E}\sum_{C \in X} V_3(C \cap K_0) &= \pi\gamma R^{(2)}V_3(K_0), \\ \mathbb{E}\sum_{C \in X} V_2(C \cap K_0) &= \pi\gamma R^{(2)}V_2(K_0) + \pi\gamma R^{(1)}V_3(K_0), \end{aligned} \tag{3.3}$$

and for $u \in S^2$ and $l \in \mathbb{R}_+$,

$$\mathbb{E}\sum_{C \in X} V_0(C \cap l[0,u]) = \gamma\pi R^{(2)} + 4l\gamma R^{(1)} \int_{S^2} \sqrt{1 - \langle u, v\rangle^2}\mu(\mathrm{d}v). \tag{3.4}$$

Note that $V_3(Z \cap K_0)$ is the volume of the set $Z \cap K_0$, $V_2(Z \cap K_0)$ is half the surface area of the set $Z \cap K_0$, and $V_0(Z \cap l[0,u])$ is the number of line segments Z

cuts out of $l[0,u]$. Analogously, $V_3(C \cap K_0)$, $V_2(C \cap K_0)$ are given by the volume and half the surface area of the respective sets. $V_0(C \cap l[0,u])$ is one if $C \cap l[0,u] \neq \emptyset$ and zero otherwise. Therefore, the left-hand sides of the equations (3.1), (3.2), (3.3) and (3.4) can all be estimated from digital images of the microstructure by standard procedures from digital image analysis.

Since $V_3(K_0)$ and $V_2(K_0)$ are usually known (3.1) or (3.3) can be used to estimate γ, $R^{(1)}$ and $R^{(2)}$. Furthermore, inserting these estimates into (3.2) or (3.4), respectively, yields an estimator for $\mathbf{S}(\mu)(u)$.

More precisely, let $k \in \mathbb{N}$, $v_1, \dots, v_k \in S^2$, and for $i = 1, \dots, k$, let $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{\xi}_{v_i}$ be estimators of $\mathbb{E}V_3(Z \cap K_0)$, $\mathbb{E}V_2(Z \cap K_0)$ and $\mathbb{E}V_0(Z \cap [0, v_i])$, respectively. Let $\alpha = 4l\gamma R^{(1)} \mathrm{e}^{-\pi\gamma R^{(2)}}$. Then

$$\hat{y}_i := \hat{\xi}_{v_i} - \frac{\hat{\xi}_1}{V_3(K_0)}$$

is an estimator for $\mathbf{S}(\alpha\mu)(v_i)$ for $i = 1, \dots, k$ and the total mass of $\alpha\mu$ can be estimated by $\hat{\alpha} := 4l\hat{\xi}_2/V_3(K_0) - \hat{\xi}_1 V_2(K_0)/V_3(K_0)$.

Let $\varepsilon > 0$ and $-u_1, \dots, -u_m, u_1, \dots, u_m \in S^2$. An estimator $\hat{\mu}_{k,m}$ for $\alpha\mu$ can be obtained as a solution of the following quadratic program:

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} \Bigl(\hat{y}_i - \sum_{j=1}^{m} \sqrt{1 - \langle v_i, u_j \rangle^2}\, p^{(j)} \Bigr)^2 \\ \text{s.t.} \quad & p^{(1)} + \ldots + p^{(m)} = \hat{\alpha}, \\ & p^{(1)}, \dots, p^{(m)} \in [0, \infty). \end{aligned}$$

If a large enough number images of the structure are available to obtain good and independent estimators for $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{\xi}_{v_i}$, $i = 1, \dots, k$, we can assume that $\hat{y}_i$, $i = 1, \dots, k$, satisfy (2.2), and thus, all results from the last section hold. The formulas for X in Theorem 1 can be used in an analogous way to construct an estimator for $\alpha\mu$.

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Appendix. Proofs

Proof of Lemma 1.

$$\begin{aligned} |\mathbf{S}(\mu_1)(u) - \mathbf{S}(\mu_2)(u)| &= \Bigl| \int_{S^{d-1}} \sqrt{1 - \langle u, v \rangle^2} (\mu_1 - \mu_2)(\mathrm{d}v) \Bigr| \\ &\leqslant \int_{S^{d-1}} \sqrt{1 - \langle u, v \rangle^2} |\mu_1 - \mu_2|(\mathrm{d}v) \leqslant |\mu_1 - \mu_2|(S^{d-1}) \leqslant \varepsilon. \end{aligned}$$

□

P r o o f of Lemma 2. Let $0 < \varepsilon \leqslant 1$ and let $\{-u_1, \dots, -u_m, u_1, \dots, u_m\}$ be an ε-dense set on S^{d-1}; remember that $m \leqslant c(d)\varepsilon^{-(d-1)}$. Furthermore, for $i = 1, \dots, m$, let C_i denote the spherical Voronoi cell of u_i with respect to the above net, and for any $\mu \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$, let

$$\mu_\varepsilon := \sum_{i=1}^m \mu(C_i)(\delta_{u_i} + \delta_{-u_i}).$$

Obviously, $d_P(\mu, \mu_\varepsilon) \leqslant \varepsilon$. Additionally, let $p_1, \dots, p_k \in \mathbb{R}^m$ be such that $\|p_j\|_1 \leqslant \alpha$, $p_j \geqslant 0$ for $j = 1, \dots, k$, and let

$$S^+ := \{p \in \mathbb{R}^m \colon\ p \geqslant 0,\ \|p\|_1 \leqslant \alpha\} \subseteq \bigcup_{j=1}^k (p_j + \varepsilon B_1^d)$$

where B_1^d denotes the unit ball with respect to $\|\cdot\|_1$. Note that $k \leqslant \alpha^m \varepsilon^{-m}$. Each vector $p = (p^{(1)}, \dots, p^{(m)}) \in S^+$ can be identified with a measure μ_p on S^{d-1} by defining

$$\mu_p := \sum_{i=1}^m \frac{p^{(i)}}{2}(\delta_{u_i} + \delta_{-u_i}).$$

For $p, p' \in S^+$, $\|p - p'\|_1 \leqslant \varepsilon$ implies that $d_P(\mu_p, \mu_{p'}) \leqslant \varepsilon$. Hence, for all $\mu \in \mathcal{M}_e^{(\alpha)}(S^{d-1})$, there exists a $j \in \{1, \dots, k\}$ such that

$$d_P(\mu, \mu_{p_j}) \leqslant d_P(\mu, \mu_\varepsilon) + d(\mu_\varepsilon, \mu_{p_j}) \leqslant 2\varepsilon.$$

Since $k \leqslant \alpha^m \varepsilon^{-m}$ and $m \leqslant c(d)\varepsilon^{-(d-1)}$, the latter implies that there exists a constant $M(d, \alpha)$ depending only on d and α such that

$$\log N(t) = H(t) \leqslant M(d, \alpha) t^{-d} \log \frac{1}{t} \leqslant M(d, \alpha) t^{-(d+1)}$$

for all $0 < t \leqslant 1$. $\square$

P r o o f of Corollary 2. By Corollary 1 we have that, almost surely, $d_k(\mu_0, \hat{\mu}_k) \to 0$ for $k \to \infty$. Let $\tilde{\mu}$ be an accumulation point of the sequence $(\hat{\mu}_k)_{k \in \mathbb{N}}$ which by compactness of $(\mathcal{M}_e^{(\alpha)}(S^{d-1}), d_P)$ always exists. Obviously, almost surely,

$$d_k(\mu_0, \tilde{\mu}) \to 0 \quad \text{for } k \to \infty.$$

Because both $\mathbf{S}(\mu_0)$ and $\mathbf{S}(\tilde{\mu})$ are continuous functions the assumption on $v_1, v_2, \dots$ yields that, almost surely,

$$d_k(\mu_0, \tilde{\mu}) \to \left(\frac{1}{d\kappa_d} \int_{S^{d-1}} (\mathbf{S}(\mu_0)(v) - \mathbf{S}(\tilde{\mu})(v))^2 h(v) \omega_{d-1}(\mathrm{d}v) \right)^{1/2}.$$

Since h is strictly positive we obtain that, almost surely,

$$\mathbf{S}(\mu_0) = \mathbf{S}(\tilde{\mu}).$$

Hence, almost surely, each accumulation point of $(\hat{\mu}_k)_{k\in\mathbb{N}}$ is equal to μ_0. This yields the assertion. $\square$

Proof of Lemma 3.

$$d_k(\mu_0, \hat{\mu}_{k,m}) \leqslant d_k(\mu_0, \hat{\mu}_k) + \min_{\mu \in \mathcal{P}_e^{(m)}(S^{d-1})} d_P(\hat{\mu}_k, \mu) \leqslant d_k(\mu_0, \hat{\mu}_k) + \varepsilon,$$

and since $d_k(\mu_0, \hat{\mu}_k) \leqslant d_k(\mu_0, \hat{\mu}_{k,m})$ this implies the assertion. $\square$

Proof of Theorem 1. By Theorems 1, 2, and 5 as well as Corollary 1 in [3] we have for $j \in \{0, \ldots, 3\}$ and any non-empty compact and convex set K_0 and $l \in \mathbb{R}_+$:

$$\begin{aligned}
\mathbb{E}V_j(Z \cap lK_0) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int_{S^2} \int_{\mathbb{R}_+} \cdots \int_{S^2} \int_{\mathbb{R}_+} \int_{L(u_1)^\perp} \cdots \int_{L(u_k)^\perp} \\
&\quad V_j\big(lK_0 \cap (r_1 B(u_1) + L(u_1) + x_1) \cap \ldots \cap (r_k B(u_k) + L(u_k) + x_k)\big) \\
&\quad \times \lambda_{L(u_1)^\perp}(\mathrm{d}x_1) \ldots \lambda_{L(u_k)^\perp}(\mathrm{d}x_k) \mathbb{P}(\mathrm{d}r_1)\mu(\mathrm{d}u_1) \ldots \mathbb{P}(\mathrm{d}r_k)\mu(\mathrm{d}u_k) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int_{S^2} \int_{\mathbb{R}_+} \cdots \int_{S^2} \int_{\mathbb{R}_+} \sum_{\substack{m_0, \ldots, m_k = j \\ m_0 + \ldots + m_k = 3k+j}}^{3} l^{m_0} r_1^{m_1} \ldots r_k^{m_k} \\
&\quad \times \Phi^{(j)}_{m_0, \{0\}; m_1, L_{u_1}; \ldots, m_k; L_{u_k}}(K_0, B(u_1), \ldots, B(u_k); \mathbb{R}^3 \times \ldots \times \mathbb{R}^3) \\
&\quad \times \mathbb{P}(\mathrm{d}r_1)\mu(\mathrm{d}u_1) \ldots \mathbb{P}(\mathrm{d}r_k)\mu(\mathrm{d}u_k).
\end{aligned}$$

Let $l = 1$ and let K_0 have non-empty interior. For $j = 3$, the above calculation together with Theorem 2 in [3] yields

$$\begin{aligned}
\mathbb{E}V_3(Z \cap K_0) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \left(\int_{\mathbb{R}_+} r^2 \mathbb{P}(\mathrm{d}r) \right)^k \left(\int_{S^2} V_2(K(u))\mu(\mathrm{d}u) \right)^k V_3(K_0) \\
&= (1 - \mathrm{e}^{\pi\gamma R^{(2)}}) V_3(K_0).
\end{aligned}$$

Analogously, we obtain

$$\mathbb{E}V_2(Z \cap K_0) = (1 - \mathrm{e}^{\pi\gamma R^{(2)}}) V_2(K_0) + \pi\gamma R^{(1)} \mathrm{e}^{\pi\gamma R^{(2)}} V_3(K_0)$$

and

$$\begin{aligned}
\mathbb{E}V_0(Z \cap l[0,u]) &= (1 - \mathrm{e}^{\pi\gamma R^{(2)}}) \\
&\quad + l\gamma R^{(1)} \mathrm{e}^{\pi\gamma R^{(2)}} \int_{S^2} \int_{S^2 \cap L(v)^\perp} |\langle u, w \rangle| \, \omega_{L(v)^\perp}(\mathrm{d}w)\mu(\mathrm{d}v).
\end{aligned}$$

Here, $\omega_{L(v)^\perp}$ denotes the spherical Lebesgue measure on $S^2 \cap L(v)^\perp$. Let $\tilde{u}(v)$ denote the orthogonal projection of u on $L(v)^\perp$. Then $\|\tilde{u}(v)\| = \sqrt{1 - \langle u, v\rangle^2}$, and since

$$\int_{S^2 \cap L(v)^\perp} |\langle u_0, w\rangle| \, \omega_{L(v)^\perp}(\mathrm{d}w) = 4$$

for all $u_0 \in S^2 \cap L(v)^\perp$ (see (7.37) in [11]) this yields

$$\mathbb{E}V_0(Z \cap l[0,u]) = (1 - \mathrm{e}^{\pi\gamma R^{(2)}}) + 4l\gamma R^{(1)} \mathrm{e}^{\pi\gamma R^{(2)}} \int_{S^2} \sqrt{1 - \langle u, v\rangle^2} \mu(\mathrm{d}v).$$

The rest of the theorem can be proved in a similar manner. $\square$

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