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SUPERCONVERGENCE ESTIMATES OF FINITE ELEMENT METHODS FOR AMERICAN OPTIONS*

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Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. In this paper we are concerned with finite element approximations to the evaluation of American options. First, following W. Allegretto etc., SIAM J. Numer. Anal. *39* (2001), 834–857, we introduce a novel practical approach to the discussed problem, which involves the exact reformulation of the original problem and the implementation of the numerical solution over a very small region so that this algorithm is very rapid and highly accurate. Secondly by means of a superapproximation and interpolation postprocessing analysis technique, we present sharp L^2-, L^∞-norm error estimates and an H^1-norm superconvergence estimate for this finite element method. As a by-product, the global superconvergence result can be used to generate an efficient a posteriori error estimator.

Keywords: American options, variational inequality, finite element methods, optimal and superconvergent estimates, interpolation postprocessing, a posteriori error estimators

MSC 2010: 90A09, 65K10, 65M12, 65M60

1. Introduction

The option is one of the most important financial derivatives, and a wide variety of options of American style are traded in exchanges. Thus, the problem of pricing American options is clearly important in theory and practice. It has been shown by Black and Scholes [7], McKean [33], and Merton [34] that the valuation of American call or put options can be determined as the solution of a free boundary value problem of degenerate parabolic type. The unknown function in the model equation

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corresponds to the valuation function, and the free boundary represents the time path of critical stock prices beyond which an early exercise is warranted. The free boundary is also known as an optimal exercise curve in the pricing of American options.

The optimal exercise curve must be identified as part of the solution of the modelling equation, which makes it difficult to price American options in theory and applications, and does not lead to explicit, available closed-form formulas associated with the valuation of American options (see, for example, [19]). This makes the valuation of American options quite different from that of European versions. To overcome this difficulty, it is common practice to use appropriate approximation methods to price American options. In the last two decades, the research on this problem has focussed on both analytical and numerical methods, and plenty of literature is now available. For analytical approximations there are some approaches, such as the interpolation method, the compound option approximation method, the quadratic approximation method, etc. See, for example, [24], [6], [32], and the references therein. For numerical approximations, approaches such as the binomial method, the Monte Carlo simulation method, the finite difference method, the finite element method, genetic algorithm approximation, the domain decomposition method are typical. See, for instance, [10], [21], [8], [9], [22], [20], [12], [2], [18], [3], [4], [30], [29], [36], [31], and the references therein. Although convergence analysis is given in some cases (see, for example, [3], [30], and [40] for finite element approximations, and [21], [22] for binomial methods and finite difference methods), to the best of the authors' knowledge there are no known error estimates for most of the numerical methods. There are two intrinsic difficulties to the partial differential equations of pricing American options:

(1) the optimal regularity of the solution $V(\cdot,t)$ to the American option pricing model is of $W^{2,\infty}$, and $V_t(\cdot,t) \in L^\infty$;
(2) the initial data is only in $W^{1,\infty}$.

Mathematically, it is common practice to introduce a change of variables in order to remove the degeneracy from the classical Black-Scholes model. However, at the same time this introduces a new difficulty: the resulting problem needs to be solved over an infinite region in the space variable. In practice, this is dealt with by numerically solving the new problem over a large but finite range (see, for example, [9], [20], [23], [26], [39]). Then, as mentioned in [3], two difficulties arise: (1) the computer simulations must be run over a "large" region and thus are relatively slow; (2) an artificial boundary value must be imposed, which affects the accuracy of the simulation. Especially, when the interest rate is greater than the dividend, the accuracy problem becomes serious because of the specific nature of the convection term in the Black-Scholes's partial differential equation. In [3], a new nonlocal boundary condition is introduced to eliminate the above two difficulties, which is mathematically exact and allows us to reformulate the original problem as a variational inequality on a very narrow region without changing the solution. Numerical results show (see [3] and [4]) that compared with the existing computational methods, this approach provides very rapid option pricing.

In this paper we are concerned with sharp L^2-, L^∞-norm error estimates and an H^1-norm global superconvergence estimate of finite element methods to enhance the finite element approximations given in [3] by means of superapproximation estimates between the finite element solution and an interpolating function of the exact solution, and an interpolation postprocessing technique. As a by-product of the superconvergence property, we illustrate that the approximation of higher accuracy can be used to form an a posteriori error estimator for this finite element method.

This paper is organized in the following way. Following [3], in Section 2 the American option is exactly reformulated as the linear complementarity form of the heat equation with a nonlocal boundary condition on a bounded domain. In addition, we give an equivalent variational inequality to the linear complementarity form and the semidiscrete finite element approximate formula. Notation for function spaces and their norms are provided here, and a stability result is established. In Section 3, sharp L^2- and L^∞-norm error estimates are presented. Also, an H^1-norm superapproximation is discussed in this section. Section 4 is devoted to the study of global superconvergence with the help of an interpolation postprocessing technique. In Section 5, on the basis of the global superconvergence, an efficient a posteriori error estimator is given to assess the accuracy of finite element solutions in applications.

2. The finite element method

In this section we give an equivalent variational inequality for the pricing model of the American option and its semidiscrete finite element scheme, and present a stability result of this scheme.

Let us consider an American call option on a stock with exercise price K, dividend rate d, maturity date T_0, and interest r. As usual, we assume that the capital market is frictionless and arbitrage free with continuous trading possibilities. Let $S = S(t)$ be the underlying asset price and let S follow the lognormal diffusion with constant volatility σ and expected return μ:

$$\mathrm{d}S = \mu\, S\, \mathrm{d}t + \sigma S\, \mathrm{d}Z, \tag{2.1}$$

where $\{Z(t)\colon\ t \geqslant 0\}$ is the standard Brownian motion. It may be assumed that $d > 0$. In fact, if $d = 0$, then the value of the American call options equals that of the corresponding European call options (see, for instance, [23], [26], and [34]). It is well known that the price $V(S,t)$ of the American call option is the solution of the free boundary value problem [26], [33], [34], [23]:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-d)S\frac{\partial V}{\partial S}-rV=0,\quad 0<S<S^*(t),\ 0<t\leqslant T_0, \tag{2.2}$$

$$V(S,t)>(S-K)_+,\quad 0<S<S^*(t),\ 0<t\leqslant T_0, \tag{2.3}$$

$$V(S,t)=(S-K)_+,\quad S\geqslant S^*(t),\ 0\leqslant t\leqslant T_0, \tag{2.4}$$

$$V(S^*(t),t)=(S^*(t)-K)_+,\quad \frac{\partial V}{\partial S}(S^*(t),t)=1,\quad 0<t\leqslant T_0, \tag{2.5}$$

$$V(S,T_0)=(S-K)_+,\quad S\geqslant 0, \tag{2.6}$$

$$V(0,t)=0,\quad 0\leqslant t\leqslant T_0, \tag{2.7}$$

where $S^*(t)$ is the free boundary, which is a non-increasing function, and $z_+=\max(z,0)$. Here, the free boundary $S^*(t)$ also represents the early exercise price: the option should be exercised if the stock price S is greater than or equal to $S^*(t)$ at time t; otherwise, the option should be held.

Let

$$T=\frac{1}{2}\sigma^2T_0,\quad \alpha=\frac{r-d}{\sigma^2}-\frac{1}{2},\quad \beta=\alpha^2+\frac{2r}{\sigma^2}.$$

Then, with the standard transforms,

$$V(S,t)=K\mathrm{e}^{-\alpha x-\beta\tau}\varphi(x,\tau),\quad T_0-t=\frac{2\tau}{\sigma^2},\quad \text{and}\quad S=K\mathrm{e}^x, \tag{2.8}$$

(2.2)–(2.7) becomes (see, also, [3])

$$\frac{\partial\varphi}{\partial\tau}-\frac{\partial^2\varphi}{\partial x^2}=0,\quad \varphi(x,\tau)>g(x,\tau),\quad x<x^*(\tau),\ 0<\tau\leqslant T, \tag{2.9}$$

$$\varphi(x,\tau)=g(x,\tau),\quad x\geqslant x^*(\tau),\ 0\leqslant\tau\leqslant T, \tag{2.10}$$

$$\varphi(x^*(\tau),\tau)=g(x^*(\tau),\tau),\quad \frac{\partial\varphi}{\partial x}(x^*(\tau),\tau)=\frac{\partial g}{\partial x}(x^*(\tau),\tau),\quad 0<\tau\leqslant T, \tag{2.11}$$

$$\varphi(x,0)=g(x,0),\quad -\infty<x<\infty, \tag{2.12}$$

$$\lim_{x\to-\infty}\mathrm{e}^{-\alpha x-\beta\tau}\varphi(x,\tau)=0,\quad 0\leqslant\tau\leqslant T, \tag{2.13}$$

where

$$g(x,\tau)=\mathrm{e}^{\alpha x+\beta\tau}(\mathrm{e}^x-1)_+,\quad x^*(\tau)=\ln(S^*(T_0-2\tau/\sigma^2)/K).$$

Since (see, for example, [23] and [26])

$$S_0\leqslant S^*(t)\leqslant S_\infty,\quad 0\leqslant t\leqslant T_0,$$

we have

$$0 \leqslant x^*(\tau) \leqslant X, \quad 0 \leqslant \tau \leqslant T,$$

where

$$S_0 = S^*(T_0) = \max\Big(\frac{rK}{d}, K\Big), \quad S_\infty = \frac{\sqrt{\beta}-\alpha}{\sqrt{\beta}-\alpha-1}K, \quad X = \ln\Big(\frac{S_\infty}{K}\Big).$$

From [3] we know that X is very small. Obviously, we can see from (2.10) that

$$\varphi(X,\tau) = g(X,\tau), \quad \tau \in [0,T].$$

Thus, in order to solve (2.9)–(2.13) on $[0,X]\times[0,T]$, we need to prescribe a boundary condition at $x=0$. In fact, from [3] we have

$$\varphi_x(0,\tau) = A\varphi(0,\tau), \quad 0 \leqslant \tau \leqslant T, \tag{2.14}$$

where

$$A\varphi(0,t) = \frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t (t-s)^{-1/2}\varphi(0,s)\,\mathrm{d}s\Big). \tag{2.15}$$

Although (2.14) is a complicated and nonlocal condition, it guarantees that we can restrict our consideration to $x \geqslant 0$, since for $x<0$ we have [3]

$$\varphi(x,t) = -\frac{x}{\sqrt{4\pi}}\int_0^t (t-s)^{-3/2}\mathrm{e}^{-x^2/4(t-s)}\varphi(0,s)\,\mathrm{d}s. \tag{2.16}$$

Note that the initial data $g(x,0)$ is only in $W^{1,\infty}(0,X)$, and its derivative is bounded but not continuous. To overcome this difficulty, we let

$$\varphi = u + g,$$

which reduces (2.9)–(2.13) to

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f, \quad u(x,t) > 0, \quad 0 < x < x^*(t),\ 0 < t \leqslant T, \tag{2.17}$$

$$u(x,t) = 0, \quad x^*(t) \leqslant x \leqslant X,\ 0 \leqslant t \leqslant T, \tag{2.18}$$

$$u(x^*(t),t) = 0, \quad u_x(x^*(t),t) = 0, \quad 0 < t \leqslant T, \tag{2.19}$$

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant X, \tag{2.20}$$

$$u_x(0,t) = Au(0,t) - b(t), \quad 0 \leqslant t \leqslant T, \tag{2.21}$$

where

$$f(x,t)=\frac{2}{\sigma^2}(r-d\mathrm{e}^x)\mathrm{e}^{\alpha x+\beta t},\quad b(t)=\mathrm{e}^{\beta t}.$$

Thus, we obtain the following linear complementarity problem, which is equivalent to (2.17)–(2.21):

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\geqslant f,\quad u(x,t)\geqslant 0,\quad 0<x<X,\ 0<t\leqslant T, \tag{2.22}$$

$$(u_t-u_{xx}-f)u=0,\quad 0<x<X,\ 0<t<T, \tag{2.23}$$

$$u(x,0)=0,\quad 0\leqslant x\leqslant X, \tag{2.24}$$

$$u_x(0,t)=Au(0,t)-b(t),\quad 0\leqslant t\leqslant T, \tag{2.25}$$

$$u(X,t)=0,\quad 0\leqslant t\leqslant T. \tag{2.26}$$

Remark 2.1. From (2.16) we know that the solution $V(S,t)$ of the original problem (2.2)–(2.7) is given by $u(x,\tau)$ as

$$V(S,t)=\begin{cases} S-K, & S>S_\infty,\\ K\mathrm{e}^{-\alpha x-\beta\tau}u(x,\tau)+S-K, & K\leqslant S\leqslant S_\infty,\\ -\int_0^\tau E(x,\tau,s)u(0,s)\,\mathrm{d}s, & 0<S<K,\end{cases}$$

for $t\in[0,T_0]$, where $x=\ln(S/K)$, $\tau=\frac{1}{2}\sigma^2(T_0-t)$, and

$$E(x,\tau,s)=\frac{Kx}{\sqrt{4\pi}}(\tau-s)^{-3/2}\mathrm{e}^{-x^2/4(\tau-s)-\alpha x-\beta\tau}.$$

Next we will discuss the semidiscrete finite element method for the problem (2.22)–(2.26). To this end, we will introduce some notation.

For $J=(0,T)$ and a real number m, we denote by $H^m(J)$ and $H^{-m}(J)$ the Sobolev space and its dual space, respectively, and the norm of $H^{-m}(J)$ is given by

$$\|\psi\|_{H^{-m}(J)}=\sup_{\varphi\in H^m(J)}\frac{\langle\psi,\varphi\rangle}{\|\varphi\|_{H^m(J)}},$$

where $\langle\cdot,\cdot\rangle$ represents the dual pairing between $H^{-m}(J)$ and $H^m(J)$. It is also to be used for the inner product on $L^2(J)$.

For $\Omega=(0,X)$, the norm in the Sobolev space $W_q^k(\Omega)$, $k\in\{0,1,\ldots\}$, $q\in[1,\infty]$, is denoted by $\|\cdot\|_{k,q}$. In particular, if $q=2$ then we set $H^2(\Omega)=W_2^k(\Omega)$ and $\|\cdot\|_k=\|\cdot\|_{k,2}$. The symbol $(\cdot,\cdot)$ stands for the usual scalar product in $L^2(\Omega)$. In addition, suppose that $\mathbf{X}$ is a Banach space and $u(t)\colon[0,T]\to\mathbf{X}$ is an $\mathbf{X}$-valued function. Define the space

$$L^p(0,T;\mathbf{X})=\Big\{u(t);\ \|u\|_{L^p(\mathbf{X})}=\Big(\int_0^T\|u(t)\|_{\mathbf{X}}^p\,\mathrm{d}t\Big)^{1/p}<\infty\Big\},\quad 1\leqslant p\leqslant\infty.$$

From [3] we recall the following result.

Lemma 2.1. *Assume that the operator A is given by (2.15) and let $J_t = (0,t)$ for $t \in (0,T)$. Then A is an isomorphism from $H^{1/4}(J_t)$ to $H^{-1/4}(J_t)$, and there exists a positive constant C_0 such that*

$$\langle A\varphi, \varphi\rangle_t \geqslant C_0 \|\varphi\|^2_{H^{1/4}(J_t)} \quad \forall\, \varphi \in H^{1/4}(J_t),$$

where

$$\langle \varphi, \psi\rangle_t = \int_0^t \varphi(s)\psi(s)\,\mathrm{d}s.$$

Let

$$K^* = \{v \in H^1(J; H_E(\Omega))\colon\ v(0,\cdot) \in H^{1/4}(J),\ v(x,t) \geqslant 0 \text{ a.e. on } Q\},$$

where $Q = \Omega \times J$ and

$$H_E(\Omega) = \{v \in H^1(\Omega)\colon\ v(X) = 0\}.$$

It is easy to show that the problem (2.22)–(2.26) is equivalent to the following variational problem: Find the solution $u \in K^*$ such that for any fixed t

$$\begin{aligned} &(u_t, u - v) + a(u, u - v) + Au(0,t)(u(0,t) - v(0)) \\ &\qquad \leqslant (f, u - v) + b(t)\,(u(0,t) - v(0)) \quad \forall\, v \in K^{**}, \end{aligned} \tag{2.27}$$

$$u(x,0) = 0, \quad x \in \Omega, \tag{2.28}$$

where

$$a(u,v) = \int_\Omega u_x v_x\,\mathrm{d}x, \quad K^{**} = \{v \in H_E(\Omega)\colon\ v \geqslant 0 \text{ on } \Omega\}.$$

Obviously, there exist positive constants C_1 and C_2 such that

$$C_1\|u\|_1^2 \leqslant a(u,u) \leqslant C_2\|u\|_1^2 \quad \forall\, u \in H_E(\Omega). \tag{2.29}$$

From Lemma 2.1 and (2.29) one finds that the solution of (2.27)–(2.28) in K^* exists and is unique. We also refer to [5] for the existence and uniqueness of the solution. See [11], [12], [13], [14], [17], and [36] for various numerical approaches to this problem. Here we only consider the finite element method for problem (2.27)–(2.28).

Let $T_h\colon\ 0 = x_0 < x_1 < \ldots < x_M = X$ be a regular partition of Ω, where M is a positive integer, $h_i = x_i - x_{i-1}$, and $h = \max\limits_{1 \leqslant i \leqslant M} h_i$. Let $V_h \subset H_E(\Omega)$ be the space of continuous and piecewise linear finite element functions. In addition, we define the closed convex subset K_h of the space V_h by

$$K_h = \{v \in V_h \colon\ v \geqslant 0\}.$$

Now we are in a position to give the definition of the semidiscrete finite element approximation to problem (2.27)–(2.28): Find $u_h(t) \in K_h$ such that

$$(u_{h,t}, u_h - v) + a(u_h, u_h - v) + Au_h(0,t)(u_h(0,t) - v(0)) \leqslant (f, u_h - v) + b(t)(u_h(0,t) - v(0)) \quad \forall\, v \in K_h, \tag{2.30}$$

$$u_h(x,0) = 0, \quad x \in \Omega. \tag{2.31}$$

R e m a r k 2.2. Recalling Remark 2.1, we can define the approximation of $V(S,t)$ as

$$V_h(S,t) = \begin{cases} S - K, \quad S > S_\infty, \\ K\mathrm{e}^{-\alpha x - \beta\tau} u_h(x,\tau) + S - K, \quad K \leqslant S \leqslant S_\infty, \\ -\displaystyle\int_0^\tau E(x,\tau,s) u_h(0,s)\,\mathrm{d}s, \quad 0 < S < K, \end{cases}$$

for $t \in [0, T_0]$, where $x = \ln(S/K)$, $\tau = \frac{1}{2}\sigma^2(T_0 - t)$, and

$$E(x,\tau,s) = \frac{Kx}{\sqrt{4\pi}}(\tau - s)^{-3/2}\mathrm{e}^{-x^2/4(\tau-s)-\alpha x - \beta\tau}.$$

First of all, we have the following stability result.

Theorem 2.1. *The semidiscrete scheme* (2.30)–(2.31) *has a unique solution* $u_h(t) \in K_h$*, which satisfies*

$$\|u_h\|_0^2 + \|u_h\|^2_{L^2(J_t;H^1(\Omega))} + \|u_h(0,\cdot)\|^2_{1/4,J_t} \leqslant C(\|f\|^2_{L^2(J_t;L^2(\Omega))} + \|b\|^2_{-1/4,J_t}).$$

P r o o f. From Lemma 2.1 and (2.29) we find that the semidiscrete scheme (2.30)–(2.31) has a unique solution $u_h \in K_h$. Moreover, taking $v = 0$ in (2.30), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_0^2 + |u_h|_1^2 + Au_h(0,t)u_h(0,t) \leqslant (f, u_h) + b(t)u_h(0,t).$$

Hence, integrating the above inequality with respect to t and noticing that $u_h(0) = 0$, we have by means of (2.29), Lemma 2.1 and an ε-type inequality that

$$\|u_h\|_0^2 + \|u_h\|^2_{L^2(J_t;H^1(\Omega))} + \|u_h(0,\cdot)\|^2_{1/4,J_t} \leqslant C(\|f\|^2_{L^2(J_t;L^2(\Omega))} + \|u_h\|^2_{L^2(J_t;L^2(\Omega))} + \|b\|^2_{-1/4,J_t}) + \varepsilon\|u_h(0,\cdot)\|^2_{1/4,J_t},$$

from which we conclude by Gronwall's lemma that

$$\|u_h\|_0^2 + \|u_h\|^2_{L^2(J_t;H^1(\Omega))} + \|u_h(0,\cdot)\|^2_{1/4,J_t} \leqslant C(\|f\|^2_{L^2(J_t;L^2(\Omega))} + \|b\|^2_{-1/4,J_t}).$$

□

3. Superapproximation estimates

As mentioned before, the essential difficulty for the error estimates of the valuation of American options is the low regularity of the exact solutions and the initial data. Even for a variational inequality with higher regularity, the error estimate of linear finite element approximation is $O(h)$ under the conditions that the exact solution $u \in W^{2,\infty}(\Omega)$, $u_t \in H^1(\Omega)$, and the initial data $u_0 \in W^{2,\infty}(\Omega)$. See, for instance, [38] and [15].

In this section we will utilize a superapproximation analysis technique to present sharp L^2- and L^∞-norm error estimates for American option pricing problems. In particular, we will establish a superapproximation property in the H^1-norm, which is a key ingredient of the superconvergence analysis. From the mathematical point of view, the superconvergence of finite element methods in the H^1-norm is beyond all doubt very important. Survey paper [25] conveys a good view on this topic. What is more, $\partial V/\partial S$, denoted by "Δ" (Delta) in the financial community, represents the number of shares of the underlying asset that the writer of the option should hold to hedge away the risk arising from selling the option.

First of all, we need the following lemmas.

Lemma 3.1. *Let $u \in H^1(J; H_E(\Omega))$ and u_h be the solutions of the problems (2.27)–(2.28) and (2.30)–(2.31), respectively. Let $i_h u$ be the linear interpolant of u. Then we have*

$$\Big(\frac{\partial(u_h - u)}{\partial t}, u_h - i_h u\Big) + a(u_h - u, u_h - i_h u) + A(u_h - u)(0,t)(u_h(0,t) - i_h u(0,t)) \leqslant Ch^3 \|u_t - u_{xx} - f\|_\infty \|u\|_{2,\infty}.$$

P r o o f. Taking $v_h = i_h u$ in (2.30), we derive the inequality

$$\Big(\frac{\partial u_h}{\partial t}, u_h - i_h u\Big) + a(u_h, u_h - i_h u) + Au_h(0,t)(u_h(0,t) - i_h u(0,t)) \leqslant (f, u_h - i_h u) + b(t)(u_h(0,t) - i_h u(0,t)). \tag{3.1}$$

On the other hand, taking into account the boundary conditions (2.25) and (2.26), we obtain via integration by parts with respect to x that

$$\Big(\frac{\partial u}{\partial t}, u_h - i_h u\Big) + a(u, u_h - i_h u) + Au(0,t)(u_h(0,t) - i_h u(0,t)) = \Big(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}, u_h - i_h u\Big) + b(t)(u_h(0,t) - i_h u(0,t)), \tag{3.2}$$

which, together with (3.1), leads to

$$
\begin{aligned}
(3.3)\quad & \Big(\frac{\partial(u_h-u)}{\partial t}, u_h - i_h u\Big) + a(u_h-u, u_h - i_h u) \\
&\qquad\qquad + A(u_h-u)(0,t)(u_h(0,t) - i_h u(0,t)) \\
&= \Big(\frac{\partial u_h}{\partial t}, u_h - i_h u\Big) + a(u_h, u_h - i_h u) + Au_h(0,t)(u_h(0,t) - i_h u(0,t)) \\
&\quad - \Big\{\Big(\frac{\partial u}{\partial t}, u_h - i_h u\Big) + a(u, u_h - i_h u) + Au(0,t)(u_h(0,t) - i_h u(0,t))\Big\} \\
&\leqslant (f, u_h - i_h u) + b(t)(u_h(0,t) - i_h u(0,t)) \\
&\quad - \Big\{\Big(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}, u_h - i_h u\Big) + b(t)(u_h(0,t) - i_h u(0,t))\Big\} \\
&= -\Big(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f, u_h - i_h u\Big).
\end{aligned}
$$

Set

$$
\begin{aligned}
Lu &= \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f, \\
\Omega^+(t) &= \{x \in \Omega\colon\ u(x,t) > 0\}, \\
\Omega^-(t) &= \{x \in \Omega\colon\ u(x,t) = 0\}.
\end{aligned}
$$

From a property of American options we know that $\Omega^+(t)$ and $\Omega^-(t)$ are separated strictly by the free boundary $x^*(t)$, and

$$
(3.4)\qquad \Omega = \Omega^+(t) \cup \Omega^-(t), \quad Lu|_{\Omega^+(t)} = 0, \quad Lu|_{\Omega^-(t)} \geqslant 0.
$$

In addition, let

$$
\Omega_I^+(t) = \{x \in \Omega\colon\ i_h u(x,t) > 0\}, \quad \Omega_I^-(t) = \{x \in \Omega\colon\ i_h u(x,t) = 0\}.
$$

Thus, it follows from (3.3), (3.4) and the definitions of $\Omega_I^+(t)$ and $\Omega_I^-(t)$ that

$$
\begin{aligned}
&\Big(\frac{\partial(u_h-u)}{\partial t}, u_h - i_h u\Big) + a(u_h-u, u_h - i_h u) + A(u_h-u)(0,t)(u_h(0,t) - i_h u(0,t)) \\
&\qquad = -(Lu, u_h - i_h u) \\
&\qquad = -\int_{\Omega^+(t)} Lu(u_h - i_h u)\,\mathrm{d}x - \int_{\Omega^-(t)} Lu(u_h - i_h u)\,\mathrm{d}x \\
&\qquad = -\int_{\Omega^-(t)} Lu(u_h - i_h u)\,\mathrm{d}x \\
&\qquad = -\int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu(u_h - i_h u)\,\mathrm{d}x - \int_{\Omega^-(t)\cap\Omega_I^-(t)} Lu(u_h - i_h u)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu(u_h - i_h u)\,\mathrm{d}x - \int_{\Omega^-(t)\cap\Omega_I^-(t)} Lu\,u_h\,\mathrm{d}x \\
&\leqslant -\int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu(u_h - i_h u)\,\mathrm{d}x \leqslant \int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu\,i_h u\,\mathrm{d}x,
\end{aligned}
$$

where we have used the fact that

$$
\int_{\Omega^-(t)\cap\Omega_I^-(t)} Lu\,u_h\,\mathrm{d}x \geqslant 0 \quad \text{and} \quad \int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu\,u_h\,\mathrm{d}x \geqslant 0,
$$

because $Lu \geqslant 0$ and $u_h \geqslant 0$. Therefore, from $Lu \cdot u = 0$ in $\Omega \times J$ we further obtain

$$
\begin{aligned}
\Big(\frac{\partial(u_h - u)}{\partial t}, u_h - i_h u\Big) &+ a(u_h - u, u_h - i_h u) \\
&+ A(u_h - u)(0,t)(u_h(0,t) - i_h u(0,t)) \\
&\leqslant \int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu\,i_h u\,\mathrm{d}x \\
&= \int_{\Omega^-(t)\cap\Omega_I^+(t)} Lu\,(i_h u - u)\,\mathrm{d}x \\
&\leqslant \|Lu\|_\infty \|u - i_h u\|_\infty \operatorname{meas}(\Omega^-(t)\cap\Omega_I^+(t)) \leqslant Ch^3 \|Lu\|_\infty \|u\|_{2,\infty},
\end{aligned}
$$

since $\operatorname{meas}(\Omega^-(t)\cap\Omega_I^+(t)) \leqslant h$. □

Lemma 3.2. *Assume that $i_h u \in V_h$ is the piecewise linear interpolant of u. Then we have*

$$
\begin{aligned}
A(u - i_h u)(0,t) &= 0 \quad \forall\, t \in [0,T], \\
a(u - i_h u, v) &= 0 \quad \forall\, v \in V_h.
\end{aligned}
$$

Proof. Since

$$
i_h u(0,t) = u(0,t) \quad \forall\, t \in [0,T],
$$

we have by (2.15) that

$$
A(u - i_h u)(0,t) = 0 \quad \forall\, t \in [0,T].
$$

Next we will prove the second equality in the lemma. For this purpose, we let $e_i = [x_i, x_{i+1}]$ be an arbitrary element of T_h. Then for $v \in V_h$ we have by integration by parts that

$$a(u-i_hu,v)=\int_\Omega (u-i_hu)_x v_x=\sum_{i=0}^{M-1}\int_{e_i}(u-i_hu)_x v_x$$
$$=\sum_{i=0}^{M-1}\{v_x(u_h-i_hu)\}|_{e_i}-\sum_{i=0}^{M-1}\int_{e_i}(u-i_hu)v_{xx}=0,$$

where we have used the fact that $(u-i_hu)|_{e_i}=0$ and $v_{xx}|_{e_i}=0$. □

Lemma 3.3. *Assume that u and u_h are the solutions of the problems (2.27)–(2.28) and (2.30)–(2.31), respectively. Let i_hu be the linear interpolant of u. Then we have for $u_t\in L^\infty(\Omega)\cap H^{1+\alpha}(\Omega^+(t)\setminus\Omega^{++}(t))$ (with $\alpha>0$ sufficiently small) that*

$$(u_t-i_hu_t,u_h-i_hu)=o(h)\|u_h-i_hu\|_1,$$

where $\Omega^+=\{x\in\Omega\colon\ u>0\}$ and $\Omega^{++}(t)=\{\cup e\colon\ \overline{e}\cap\Omega^+(t)\neq\emptyset,\ e\in T_h\}$, and $\lim\limits_{h\to 0}o(h)/h=0$.

Proof. Since $\Omega=\Omega^+(t)\cup\Omega^-(t)$, we have

$$\begin{aligned}(3.5)\quad &(u_t-i_hu_t,u_h-i_hu)\\ &=\int_{\Omega^+(t)}(u_t-i_hu_t)(u_h-i_hu)+\int_{\Omega^-(t)}(u_t-i_hu_t)(u_h-i_hu)\\ &=\sum_{e\subset\Omega^+(t)\setminus\Omega^{++}(t)}\int_e(u_t-i_hu_t)(u_h-i_hu)+\sum_{e\subset\Omega^{++}(t)}\int_e(u_t-i_hu_t)(u_h-i_hu)\\ &=I_1+I_2.\end{aligned}$$

For I_1 we have

$$\begin{aligned}(3.6)\qquad |I_1|&\leqslant\sum_{e\subset\Omega^+(t)\setminus\Omega^{++}(t)}\|u_t-i_hu_t\|_{0,e}\|u_h-i_hu\|_{0,e}\\ &\leqslant\sum_{e\subset\Omega^+(t)\setminus\Omega^{++}(t)}h^{1/2}\|u_t-i_hu_t\|_{0,e}\|u_h-i_hu\|_{\infty,e}\\ &\leqslant\sum_{e\subset\Omega^+(t)\setminus\Omega^{++}(t)}Ch^{3/2+\alpha}\|u_t\|_{1+\alpha,e}\|u_h-i_hu\|_\infty\\ &\leqslant Ch^{1+\alpha}\|u_t\|_{1+\alpha,\Omega^+(t)\setminus\Omega^{++}(t)}\|u_h-i_hu\|_1,\end{aligned}$$

where we have used an estimate of the interpolation error in a fractional order Sobolev spaces (see, for instance, [16] and [35])

$$\|v-i_hv\|_0\leqslant Ch^{1+\alpha}\|v\|_{1+\alpha},$$

and the finite element inverse estimate

$$\|u_h - i_h u\|_\infty \leqslant C\|u_h - i_h u\|_1.$$

It is obvious that there are at most two elements in $\Omega^{++}(t)$: when $x^*(t)$ is a node of the partition T_h, there are two elements belonging to $\Omega^{++}(t)$; when $x^*(t)$ is not a node of T_h, there is only one element contained in $\Omega^{++}(t)$. Without loss of generality, we assume that $x^*(t)$ is just in one element e_i.

From the Luzin Theorem we know that for every $\varepsilon > 0$ there exists a closed subset $e_i^* \subset e_i$, such that $u_t(\cdot,t)|_{e_i^*}$ is continuous and $\mathrm{meas}(e_i \setminus e_i^*) < \varepsilon$. Let u^* be a continuous function on e_i satisfying

$$u^*|_{e_i^*} = u_t(\cdot,t)|_{e_i^*}.$$

That is, u^* is the continuous extension of $u_t(\cdot,t)$ in $e_i^* \subset e_i$ to e_i. In addition, we define

$$i_h u_t|_{e_i} = i_h u^*|_{e_i}.$$

Then, by means of integration by parts we have that

$$
\begin{aligned}
(3.7)\quad I_2 &= \sum_{e\subset\Omega^{++}(t)} \int_e (u_t - i_h u_t)(u_h - i_h u)\\
&= \int_{e_i} (u_t - i_h u_t)(u_h - i_h u)\\
&= \int_{e_i} \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_{x_{i-1}}^{x} (u_t - i_h u_t)\Big)(u_h - i_h u)\\
&= \Big\{(u_h - i_h u)\int_{x_{i-1}}^{x} (u_t - i_h u_t)\Big\}\Big|_{e_i} - \int_{e_i}\Big\{\int_{x_{i-1}}^{x} (u_t - i_h u_t)\Big\}(u_h - i_h u)'\\
&= (u_h(x_i) - i_h u(x_i))\int_{e_i} (u_t - i_h u_t) - \int_{e_i}\Big\{\int_{x_{i-1}}^{x} (u_t - i_h u_t)\Big\}(u_h - i_h u)'\\
&= I_{21} + I_{22}.
\end{aligned}
$$

Next we will deal with I_{21} and I_{22}.

Since for $h \to 0$

$$\|u_t - i_h u_t\|_{\infty, e_i^*} \to 0$$

uniformly, we can obtain from the finite element inverse inequality $\|u_h - i_h u\|_\infty \leqslant C\|u_h - i_h u\|_1$ and choosing $\varepsilon = o(h)$ that

$$
\begin{aligned}
(3.8)\quad |I_{21}| &= \left|(u_h(x_i) - i_h u(x_i)) \int_{e_i} (u_t - i_h u_t)\right| \\
&\leqslant \|u_h - i_h u\|_\infty \left|\int_{e_i} (u_t - i_h u_t)\right| \\
&\leqslant \|u_h - i_h u\|_\infty \left\{\left|\int_{e_i^*} (u_t - i_h u_t)\right| + \left|\int_{e_i \setminus e_i^*} (u_t - i_h u_t)\right|\right\} \\
&\leqslant C\|u_h - i_h u\|_1 (h\|u_t - i_h u_t\|_{\infty, e_i^*} + \|u_t - i_h u_t\|_{\infty, e_i \setminus e_i^*} \operatorname{meas}(e_i \setminus e_i^*)) \\
&\leqslant C\|u_h - i_h u\|_1 (h\, o(1) + \varepsilon \|u_t\|_\infty) \\
&\leqslant o(h)\|u_h - i_h u\|_1.
\end{aligned}
$$

It follows from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
(3.9)\qquad |I_{22}| &= \left|\int_{e_i} \left\{\int_{x_{i-1}}^{x} (u_t - i_h u_t)\right\}(u_h - i_h u)'\right| \\
&\leqslant \int_{e_i} \left\{\int_{x_{i-1}}^{x} |u_t - i_h u_t|\right\}|(u_h - i_h u)'| \\
&\leqslant \left\{\int_{e_i} |u_t - i_h u_t|\right\}\left\{\int_{e_i} |(u_h - i_h u)'|\right\} \\
&\leqslant h\|u_t - i_h u_t\|_{0, e_i} \|u_h - i_h u\|_{1, e_i}.
\end{aligned}
$$

Similarly to (3.8), we can also obtain

$$
\begin{aligned}
(3.10)\quad \|u_t - i_h u_t\|_{0, e_i}^2 &= \int_{e_i} (u_t - i_h u_t)^2 \\
&= \int_{e_i^*} (u_t - i_h u_t)^2 + \int_{e_i \setminus e_i^*} (u_t - i_h u_t)^2 \\
&\leqslant h\|u_t - i_h u_t\|_{\infty, e_i^*}^2 + \|u_t - i_h u_t\|_{\infty, e_i \setminus e_i^*}^2 \operatorname{meas}(e_i \setminus e_i^*) \\
&\leqslant o(h) + \varepsilon \|u_t\|_\infty = o(h),
\end{aligned}
$$

where we have chosen $\varepsilon = o(h)$. Combining (3.9) with (3.10) leads to

$$|I_{22}| \leqslant o(h^{3/2})\|u_h - i_h u\|_1,$$

which, together with (3.8) and (3.7), yields

$$(3.11)\qquad |I_2| \leqslant o(h)\|u_h - i_h u\|_1.$$

Relations (3.11), (3.5), and (3.6) complete the proof of the lemma. □

Lemma 3.4. *We have*

$$i_h u_t = (i_h u)_t,$$

where i_h is the linear interpolation operator.

P r o o f. For any $t \in [0,T]$ and $x \in e_i = [x_{i-1}, x_i]$ we have

$$i_h u(x,t) = u(x_{i-1},t)L_{i-1}(x) + u(x_i,t)L_i(x),$$

where

$$L_{i-1}(x) = \frac{x - x_i}{x_{i-1} - x_i} \quad \text{and} \quad L_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

are the basis functions corresponding to x_{i-1} and x_i, respectively. Thus, we obtain for $x \in e_i$ that

$$\begin{aligned}
(i_h u)_t(x,t) &= \lim_{\Delta t \to 0} \frac{(i_h u)(x,t+\Delta t) - (i_h u)(x,t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{u(x_{i-1},t+\Delta t) - u(x_{i-1},t)}{\Delta t} L_{i-1}(x) \\
&\quad + \lim_{\Delta t \to 0} \frac{u(x_i,t+\Delta t) - u(x_i,t)}{\Delta t} L_i(x) \\
&= u_t(x_{i-1},t)L_{i-1}(x) + u_t(x_i,t)L_i(x) \\
&= i_h u_t(x,t).
\end{aligned}$$

□

Now we are in a position to prove our superapproximation theorem, which is the main result in this section.

Theorem 3.1. *Under the conditions of Lemma 3.3 we have that for all t*

$$\|u - u_h\|_0^2 + \int_0^t \|u_h - i_h u\|_1^2 \,\mathrm{d}s + \|u(0,\cdot) - u_h(0,\cdot)\|_{1/4,J_t}^2 \leqslant o(h^2).$$

P r o o f. Let

$$\theta(x,t) = u_h(x,t) - i_h u(x,t).$$

Then, from Lemmas 3.1, 3.2, 3.3, and 3.4 we derive

$$\begin{aligned}
&(\theta_t,\theta) + a(\theta,\theta) + A\theta(0,t)\theta(0,t) \\
&\quad = \Big(\frac{\partial(u_h - u)}{\partial t}, \theta\Big) + a(u_h - u,\theta) + A(u_h - u)(0,t)\theta(0,t) \\
&\qquad + \Big(\frac{\partial(u - i_h u)}{\partial t}, \theta\Big) + a(u - i_h u,\theta) + A(u - i_h u)(0,t)\theta(0,t) \\
&\quad \leqslant Ch^3 + \Big(\frac{\partial(u - i_h u)}{\partial t}, \theta\Big) \leqslant Ch^3 + o(h)\|\theta\|_1,
\end{aligned}$$

or

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_0^2 + a(\theta,\theta) + A\theta(0,t)\theta(0,t) \leqslant Ch^3 + o(h)\|\theta\|_1.$$

Hence, noticing that $\theta(x,0)=0$ and integrating the above inequality with respect to t yields according to (2.29), Lemma 2.1 and the ε-type inequality that

$$\|\theta\|_0^2 + \int_0^t \|\theta\|_1^2\,\mathrm{d}s + \|\theta(0,\cdot)\|_{1/4,J_t}^2 \leqslant Ch^3 + o(h^2) + \varepsilon\int_0^t \|\theta\|_1^2\,\mathrm{d}s,$$

which implies

$$\|\theta\|_0^2 + \int_0^t \|\theta\|_1^2\,\mathrm{d}s + \|\theta(0,\cdot)\|_{1/4,J_t}^2 \leqslant o(h^2).$$

Thus, Theorem 3.1 follows from the triangle inequality $\|u-u_h\|_0 \leqslant \|u-i_hu\|_0+\|\theta\|_0$ and the linear interpolation approximation as well as the equality $i_hu(0,\cdot)=u(0,\cdot)$. □

Remark 3.1. In [3] the convergence rate

$$\|u-u_h\|_{L^2(J;H^1(\Omega))} = O(h)$$

was obtained, which is a direct consequence of Theorem 3.1:

$$\|u-u_h\|_{L^2(J;H^1(\Omega))} \leqslant \|u-i_hu\|_{L^2(J;H^1(\Omega))} + \|i_hu-u_h\|_{L^2(J;H^1(\Omega))} \leqslant Ch.$$

Remark 3.2. From the previous Theorem 3.1 we know that the convergence rate of $\|u_h-i_hu\|_{L^2(J;H^1(\Omega))}$ is $o(h)$, which is referred to as superapproximation compared with the approximation capability of the linear finite element space V_h.

Remark 3.3. Using Theorem 3.1 we can directly derive the following L^∞-error estimate

$$\|u-u_h\|_{L^2(J;L^\infty(\Omega))} \leqslant \|u-i_hu\|_{L^2(J;L^\infty(\Omega))} + \|i_hu-u_h\|_{L^2(J;L^\infty(\Omega))} \leqslant o(h), \tag{3.12}$$

where we have used the finite element inverse estimate

$$\|i_hu-u_h\|_{L^2(J;L^\infty(\Omega))} \leqslant C\|i_hu-u_h\|_{L^2(J;H^1(\Omega))}.$$

In particular, from (3.12) we know that

$$\int_J |u(0,t)-u_h(0,t)|^2 = o(h^2), \tag{3.13}$$

which improves the result

$$\int_J |u(0,t)-u_h(0,t)|^2 = O(h)$$

obtained in [3]. The theoretical result (3.13) has been verified numerically by computational examples provided in [3].

R e m a r k 3.4. In [3] the convergence rate

$$\|u(0,\cdot) - u_h(0,\cdot)\|_{1/4,J} = O(h^{1/2})$$

was obtained, which is also improved in Theorem 3.1 to

$$\|u(0,\cdot) - u_h(0,\cdot)\|_{1/4,J} = o(h).$$

4. Global superconvergence

In order to improve the approximation accuracy on a global scale, a reasonable postprocessing method is proposed. See, for example, [27] and [28]. For this purpose, we need to define another postprocessing interpolation operator I_{2h}^2 of degree at most 2 in x. Then we assume that T_h has been gained from T_{2h} with mesh size $2h$ by subdividing each element of T_{2h} into two equal elements so that the number of elements N for T_h is an even number. Therefore, for any function u we can define a piecewise polynomial function $I_{2h}^2 u$ of degree at most 2 associated with the mesh T_{2h} according to the conditions

$$\begin{aligned} &I_{2h}^2 u|_{e_i \cup e_{i+1}} \in P_2, \quad i = 0, 2, 4, \ldots, N-2, \\ &I_{2h}^2 u(x_j) = u(x_j), \quad j = i, i+1, i+2, \end{aligned} \tag{4.1}$$

where P_2 stands for the space of (real) polynomials of degree not exceeding 2, and $e_i = [x_i, x_{i+1}]$, $e_{i+1} = [x_{i+1}, x_{i+2}]$. Then it is easy to check that

$$\begin{aligned} I_{2h}^2 i_h &= I_{2h}^2, \\ \|I_{2h}^2 v\|_1 &\leqslant C\|v\|_1 \qquad \forall\, v \in V_h, \\ \|I_{2h}^2 u - u\|_1 &\leqslant Ch^2\|u\|_3 \quad \forall\, u \in H^3(\Omega). \end{aligned} \tag{4.2}$$

Now we are ready to present our global superconvergence estimate.

Theorem 4.1. *Assume that $u(t)$ and $u_h(t)$ are the solutions of* (2.27)–(2.28) *and* (2.30)–(2.31)*, respectively, and $u \in W^{2,\infty}(\Omega) \cap H^{2+\alpha}(\Omega \setminus \Omega^{++}(t))$ with $\alpha > 0$ sufficiently small. Then under the conditions of Theorem* 3.1 *we have that*

$$\int_0^t \|I_{2h}^2 u_h - u\|_1^2 \,\mathrm{d}s \leqslant o(h^2).$$

Proof. It follows from the property of the interpolation operator I_{2h}^2 described in (4.2) that

$$I_{2h}^2 u_h - u = I_{2h}^2(u_h - i_h u) + (I_{2h}^2 u - u). \tag{4.3}$$

Therefore, we know from Theorem 3.1 and (4.2) that

$$\int_0^t \|I_{2h}^2(u_h - i_h u)\|_1^2 \,\mathrm{d}s \leqslant C \int_0^t \|u_h - i_h u\|_1^2 \,\mathrm{d}s \leqslant o(h^2). \tag{4.4}$$

Moreover, we have

$$\begin{aligned}
|I_{2h}^2 u - u|_1^2 &= \sum_{e \subset \Omega \setminus \Omega^{++}(t)} \int_e [(I_{2h}^2 u - u)]_x^2 + \sum_{e \subset \Omega^{++}(t)} \int_e [(I_{2h}^2 u - u)]_x^2 \\
&= \sum_{e \subset \Omega \setminus \Omega^{++}(t)} |I_{2h}^2 u - u|_{1,e}^2 + \sum_{e \subset \Omega^{++}(t)} |I_{2h}^2 u - u|_{1,e}^2 \\
&\leqslant \sum_{e \subset \Omega \setminus \Omega^{++}(t)} C h^{2(1+\alpha)} |u|_{2+\alpha,e}^2 + h |I_{2h}^2 u - u|_{1,\infty,e}^2 \\
&\leqslant C h^{2(1+\alpha)} |u|_{2+\alpha,\Omega \setminus \Omega^{++}(t)}^2 + C h^3 \|u\|_{2,\infty,e}^2 \\
&\leqslant C h^{2(1+\alpha)} (\|u\|_{2+\alpha,\Omega \setminus \Omega^{++}(t)}^2 + \|u\|_{2,\infty}^2),
\end{aligned}$$

from which we can derive

$$\begin{aligned}
\|I_{2h}^2 u - u\|_1^2 &= \|I_{2h}^2 u - u\|_0^2 + |I_{2h}^2 u - u|_1^2 \\
&\leqslant C h^4 \|u\|_2^2 + C h^{2(1+\alpha)} (\|u\|_{2+\alpha,\Omega \setminus \Omega^{++}(t)}^2 + \|u\|_{2,\infty}^2) \\
&\leqslant C h^{2(1+\alpha)} (\|u\|_{2+\alpha,\Omega \setminus \Omega^{++}(t)}^2 + \|u\|_{2,\infty}^2).
\end{aligned} \tag{4.5}$$

Thus, combining (4.4) and (4.5) with (4.3) completes the proof of the theorem. □

5. A posteriori estimates

In this section we develop an a posteriori estimator for the derivative of the finite element solution. It is of great importance for a finite element method to have a computable a posteriori error estimator by which we can assess the accuracy of the finite element solution in applications. From the viewpoint of finance, this kind of indicator is also meaningful. The raw material of banking is not money but risk. As mentioned before, the quantity $\Delta = \partial V / \partial S$ can be used to reduce the sensitivity of a portfolio to the movement of an underlying asset by taking opposite positions in different financial instruments and make the portfolio risk-free, which is called, in the financial term, "Δ-hedging".

One way to construct error estimators is to employ certain superconvergence properties of the finite element solutions. Next we will show how the superconvergent approximation generated above can be naturally applied to produce an efficient a posteriori error estimator.

Theorem 5.1. *Under the conditions of Theorem* 4.1 *we have that*

$$\|u-u_h\|_{L^2(J;H^1(\Omega))} = \|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))} + o(h). \tag{5.1}$$

In addition, if there exists a positive constant C_0 such that

$$\|u-u_h\|_{L^2(J;H^1(\Omega))} \geqslant C_0 h, \tag{5.2}$$

then

$$\lim_{h\to 0} \frac{\|u-u_h\|_{L^2(J;H^1(\Omega))}}{\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))}} = 1. \tag{5.3}$$

Proof. It follows from Theorem 4.1 and

$$u-u_h = (I_{2h}^2 u_h - u_h) + (u - I_{2h}^2 u_h)$$

that

$$\|u-u_h\|_{L^2(J;H^1(\Omega))} = \|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))} + o(h).$$

Thus, by (5.2) we have

$$\frac{\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))}}{\|u-u_h\|_{L^2(J;H^1(\Omega))}} + o(1) \geqslant 1$$

or

$$\varliminf_{h\to 0} \frac{\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))}}{\|u-u_h\|_{L^2(J;H^1(\Omega))}} \geqslant 1. \tag{5.4}$$

Similarly, it follows from (5.2) and

$$\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))} = \|u-u_h\|_{L^2(J;H^1(\Omega))} + o(h)$$

that

$$\varlimsup_{h\to 0} \frac{\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))}}{\|u-u_h\|_{L^2(J;H^1(\Omega))}} \leqslant 1,$$

which, together with (5.4), leads to (5.3). □

R e m a r k 4. We already know from (5.1) that the computable error estimator $\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))}$ is the principal part of finite element error derivative $\|u - u_h\|_{L^2(J;H^1(\Omega))}$, and can be used as an a posteriori error indicator to assess the accuracy of the derivative of a finite element solution. Also, condition (5.2) seems to be a reasonable assumption because $O(h)$ is the optimal convergence rate of the derivative of the linear finite element solution u_h, and from (5.3) we can further see that $\|I_{2h}^2 u_h - u_h\|_{L^2(J;H^1(\Omega))}$ is an asymptotically exact a posteriori error indicator.

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