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ANALYSIS OF A NON-STANDARD MIXED FINITE ELEMENT METHOD WITH APPLICATIONS TO SUPERCONVERGENCE

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Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. We show that a non-standard mixed finite element method proposed by Barrios and Gatica in 2007, is a higher order perturbation of the least-squares mixed finite element method. Therefore, it is also superconvergent whenever the least-squares mixed finite element method is superconvergent. Superconvergence of the latter was earlier investigated by Brandts, Chen and Yang between 2004 and 2006. Since the new method leads to a non-symmetric system matrix, its application seems however more expensive than applying the least-squares mixed finite element method.

Keywords: least-squares mixed finite element method, non-standard mixed finite element method, superconvergence, supercloseness

MSC 2010: 65N30

1. Introduction

In the past decades, many types of finite element methods for the numerical approximation of solutions to partial differential equations have been formulated and investigated. After the standard [14] and mixed [9], [10], [19] finite element methods, the more recent least-squares mixed finite element methods [11], [12], [13], [16], [17], [2] got a fair amount of attention. Similar to the mixed finite element methods, they use the formulation of second order elliptic equations as systems of first order equations. However, instead of applying a Galerkin-type orthogonality approach, they aim to minimize the residual. This avoids the technicalities one encounters in the mixed method that arise as a result of the Babuška-Brezzi [9] conditions and leads to symmetric positive definite systems of linear algebraic equations, instead of merely indefinite symmetric systems. Moreover, the author of this paper and his co-authors have proved in [6], [7], [8] that, using classical examples of standard and mixed finite element spaces, the least-squares mixed finite element approximations are, under some mild conditions, only higher order perturbations of the approximations that one would obtain using the standard and the mixed finite element method. This, for instance, immediately implies superconvergence for the least-squares mixed finite element method whenever the other two methods exhibit superconvergence.

In a recent paper [1], the authors proposed an alternative way to circumvent the Babuška-Brezzi conditions that complicate the mixed finite element method. By adding two terms to the mixed formulation, they arrived at a new, non-standard mixed finite element method. The current paper aims to analyze the approximation porperties of this method in a way different from that in [1]. By comparing the new non-standard approximations to those obtained from the least-squares mixed finite element method directly, the conclusion is the same as for this least-squares mixed method: when classical examples of finite element spaces are used and the model problem at hand is well-behaved, the non-standard method is a higher order perturbation of the standard and the mixed method. Approximation and superconvergence properties are thus inherited by the new method as well. However, its system matrix is non-symmetric. Thus, while obtaining approximations that very much resemble those obtained with the known methods, the costs of applying the method seem to be considerably higher. This, of course, questions the practical use of the new method.

2. Preliminaries

Using an elliptic model problem, we introduce the mixed finite element method and its non-standard variant from [1]. We also recall the least-squares mixed finite element method, since it is our objective to compare it with the non-standard method. Simultaneously, notation concerning Sobolev spaces and norms and inner products will be set, and some minor results given.

2.1. Sobolev spaces and norms, bilinear forms, and a model problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex polytopic domain and let $A\colon \Omega \to \mathbb{R}^{2\times 2}$ be symmetric with Lipschitz continuous coefficients and with eigenvalues in the interval $[\beta^2, \beta^{-2}]$ for some $\beta \in (0,1]$. We use the standard notation for Sobolev spaces and their norms and semi-norms; the L_2-norm and inner product we denote by $|\cdot|_0$ and $(\cdot,\cdot)$. Apart from the usual norms on $\mathbf{H}(\mathrm{div};\Omega)$ and $H_0^1(\Omega)$ we define

$$\|\mathbf{q}\|^2_{\mathrm{div},A} = (\mathbf{q},\mathbf{q})_{\mathrm{div},A} \tag{1}$$

where

$$(\mathbf{r},\mathbf{q})_{\mathrm{div},A} = d(\mathbf{r},\mathbf{q}) + (\mathrm{div}\,\mathbf{r}, \mathrm{div}\,\mathbf{q}) \quad \text{and} \quad d(\mathbf{r},\mathbf{q}) = (A^{-1}\mathbf{r},\mathbf{q}). \tag{2}$$

Furthermore, the energy inner product and energy norm on $H_0^1(\Omega)$ are

$$a(w,v) = (A\nabla w, \nabla v) \quad \text{and} \quad |v|_{1,A}^2 = a(v,v). \tag{3}$$

We equip the product space $H_0^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ with the canonical inner product and norm

$$\begin{aligned}(w,\mathbf{r};v,\mathbf{q})_{1\times\mathrm{div},A} &= a(w,v) + (\mathbf{r},\mathbf{q})_{\mathrm{div},A}, \quad \text{and} \\ \|(v,\mathbf{q})\|_{1\times\mathrm{div},A}^2 &= (v,\mathbf{q};v,\mathbf{q})_{1\times\mathrm{div},A}.\end{aligned} \tag{4}$$

The weighted norms above are equivalent to the usual norms and semi-norms on $H^1(\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$, which we get from the choice $A = I$, the identity matrix. The Poincaré-Friedrichs inequality and the assumption on the eigenvalues of A show that

$$d_A = \sup_{0\neq v\in H_0^1(\Omega)} \frac{|v|_0}{|v|_{1,A}} < \infty. \tag{5}$$

The constant d_A depends only on the diameter of Ω and on β. To conclude, define the bilinear form $b\colon\ H^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ by

$$b(v,\mathbf{q}) = (v, \mathrm{div}\,\mathbf{q}), \tag{6}$$

then with the constant γ defined by

$$0 < \gamma = \sqrt{\frac{d_A^2}{d_A^2+1}} < 1, \tag{7}$$

the following useful lemma was proved in [8].

Lemma 2.1. *For all* $v \in H_0^1(\Omega)$ *and* $\mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega)$,

$$b(v,\mathbf{q}) \leqslant \gamma |v|_{1,A} \|\mathbf{q}\|_{\mathrm{div},A} \leqslant \frac{1}{2}\gamma \|(v,\mathbf{q})\|_{1\times\mathrm{div},A}^2. \tag{8}$$

We are now able to describe the model problem that is used in this paper. Given $f \in H^{-1}(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$-\,\mathrm{div}(A\nabla u) = f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{9}$$

An equivalent first-order system is to find functions $u \in H_0^1(\Omega)$ and $\mathbf{p} \in \mathbf{H}(\mathrm{div};\Omega)$ such that

$$\mathbf{p} = -A\nabla u \ \text{ in } \ \Omega, \quad \mathrm{div}\,\mathbf{p} = f \ \text{ in } \ \Omega. \tag{10}$$

We have now introduced all the necessary notation to describe a number of finite element methods.

2.2. The mixed and the least-squares mixed finite element method

The mixed finite element method [9], [10] is based on the first-order formulation (10). It can be shown that $(u, \mathbf{p})$ is the unique pair in $L^2(\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ such that

$$\forall (w, \mathbf{q}) \in L^2(\Omega) \times \mathbf{H}(\mathrm{div};\Omega), \quad d(\mathbf{p}, \mathbf{q}) - b(u, \mathbf{q}) + b(w, \mathbf{p}) = (f, w). \tag{11}$$

Let $W_h \subset L^2(\Omega)$ and $\mathbf{\Gamma}_h \subset \mathbf{H}(\mathrm{div};\Omega)$ be finite element subspaces indexed by the discretization parameter h that signifies the maximum diameter of the finite element in the partition $\mathcal{T}_h$ of Ω. Assume that $W_h = \mathrm{div}\,\mathbf{\Gamma}_h$. Then the discrete version of (11), which is to find $(u_h^m, \mathbf{p}_h^m) \in W_h \times \mathbf{\Gamma}_h$ such that

$$\forall\, (w_h, \mathbf{q}_h) \in W_h \times \mathbf{\Gamma}_h, \quad d(\mathbf{p}_h^m, \mathbf{q}_h) - b(u_h^m, \mathbf{q}_h) + b(w_h, \mathbf{p}_h^m) = (f, w_h), \tag{12}$$

has a unique solution $(u_h^m, \mathbf{p}_h^m)$. Notice that (12) immediately implies that

$$\forall\, w_h \in W_h, \quad (\mathrm{div}(\mathbf{p} - \mathbf{p}_h^m), w_h) = 0. \tag{13}$$

To derive optimal error bounds for a sequence of approximations, the pair $W_h, \mathbf{\Gamma}_h$ also needs to satisfy Babuška-Brezzi conditions uniformly in the discretization parameter h,

$$\inf_{h>0} \inf_{0 \neq w \in W_h} \sup_{\mathbf{q}_h \in \mathbf{\Gamma}_h} \frac{b(w_h, \mathbf{q}_h)}{|w|_0 \|\mathbf{q}_h\|_{\mathrm{div}}} > 0. \tag{14}$$

In order to circumvent the design of Babuška-Brezzi stable finite element spaces, the least-squares mixed finite element method was developed. See [11], [12], [13], [16], [17], [2] for details on this method. It is based on the minimization of the functional $J\colon H_0^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega) \to \mathbb{R}$ defined by

$$J(v, \mathbf{q}) = (f - \mathrm{div}\,\mathbf{q}, f - \mathrm{div}\,\mathbf{q}) + (\mathbf{q} + A\nabla v, A^{-1}(\mathbf{q} + A\nabla v)). \tag{15}$$

Clearly, $J(v, \mathbf{q}) \geqslant 0$ for all $(v, \mathbf{q}) \in H_0^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$, and the solution $(u, \mathbf{p}) \in H_0^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$ of (10) is a pair for which $J = 0$. Setting the first variation in (15) to zero gives

$$\forall\, (v, \mathbf{q}) \in H_0^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega), \quad B(u, \mathbf{p}|v, \mathbf{q}) = (f, \mathrm{div}\,\mathbf{q}), \tag{16}$$

where the bilinear form B is defined in terms of (4) and (6) by

$$B(w, \mathbf{r}|v, \mathbf{q}) = (w, \mathbf{r}; v, \mathbf{q})_{1\times\mathrm{div},A} - b(w, \mathbf{q}) - b(v, \mathbf{r}). \tag{17}$$

This form is continuous on $H_0^1(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega)$ as well as coercive, since

$$B(v, \mathbf{q}|v, \mathbf{q}) = \|(v, \mathbf{q})\|^2_{1\times\mathrm{div},A} - 2b(v, \mathbf{q}) \geqslant (1-\gamma)\|(v, \mathbf{q})\|^2_{1\times\mathrm{div},A} \tag{18}$$

due to Lemma 2.1, and $\gamma < 1$. Thus, by the Lax-Milgram lemma, $(u, \mathbf{p})$ is the unique solution of (16), and given any Lagrange finite element space $V_h \subset H_0^1(\Omega)$ [14], there also exists a unique pair $(u_h^l, \mathbf{p}_h^l) \in V_h \times \mathbf{\Gamma}_h$, called the least-squares mixed finite element approximation, such that

$$\forall\, (v_h, \mathbf{q}_h) \in V_h \times \mathbf{\Gamma}_h, \quad B(u_h^l, \mathbf{p}_h^l|v_h, \mathbf{q}_h) = (f, \mathrm{div}\, \mathbf{q}_h). \tag{19}$$

In [8], the approximation u_h^l was compared with the standard finite element approximation $u_h^s \in V_h$, satisfying

$$a(u_h^s, v_h) = (f, v_h) \quad \text{for all } \ v_h \in V_h, \tag{20}$$

and $\mathbf{p}_h^l$ was compared with its mixed finite element counterpart given in (11). The main result of that paper is that they are superclose under some relatively mild conditions.

Theorem 2.2. *If $(W_h, \mathbf{\Gamma}_h)$ with $W_h = \mathrm{div}(\mathbf{\Gamma}_h)$ is a Babuška-Brezzi stable pair and W_h contains the piecewise constants, then*

$$\|(u_h^l - u_h^s, \mathbf{p}_h^l - \mathbf{p}_h^m)\|_{1\times\mathrm{div},A} \leqslant Ch\|(u - u_h^s, \mathbf{p} - \mathbf{p}_h^m)\|_{1\times\mathrm{div},A}. \tag{21}$$

This result allows a priori bounds for the least-squares mixed method to be refined, and also for superconvergence results of standard and mixed methods [15], [20], [3], [4], [5] to be transferred towards the least-squares mixed finite element method. See [7] for details.

2.3. A non-standard mixed finite element method

A recent alternative to the least-squares mixed finite element method, also with the goal to avoid the Babuška-Brezzi condition, was given in [1]. In that paper it was proposed to add to the mixed weak formulation (11) the equalities

$$\frac{1}{2}(A\nabla u + \mathbf{p}, A^{-1}(A\nabla v - \mathbf{q})) = 0 \quad \text{and} \quad (\mathrm{div}\, \mathbf{p}, \mathrm{div}\, \mathbf{q}) = (f, \mathrm{div}\, \mathbf{q}),$$

which hold since $\mathbf{p} = -A\nabla u$ and $\operatorname{div}\mathbf{p} = f$. Of course, as a consequence, u must now be sought in $H_0^1(\Omega)$ instead of in $L^2(\Omega)$. Here, however, we propose a minor modification of that idea, which is to include a factor of one half also in the second equality and to add, correspondingly,

$$\frac{1}{2}(\operatorname{div}\mathbf{p}, \operatorname{div}\mathbf{q}) = \frac{1}{2}(f, \operatorname{div}\mathbf{q}). \tag{22}$$

It will turn out that the analysis of this method becomes even easier with this factor included, and we suspect that also its numerical behavior may slightly improve due to this modification.

Remark 2.3. We stress that including the factor of one half does not influence the validity of the results that are proved in this paper. It merely simplifies their proofs.

Green's formula can be used to obtain the second equality in

$$\begin{aligned}(A\nabla u + \mathbf{p}, A^{-1}(A\nabla v - \mathbf{q})) &= (A\nabla u, \nabla v) - (\nabla u, \mathbf{q}) + (\mathbf{p}, \nabla v) - (\mathbf{p}, A^{-1}\mathbf{q}) \\ &= a(u, v) + b(u, \mathbf{q}) - b(v, \mathbf{p}) - d(\mathbf{p}, \mathbf{q}),\end{aligned}$$

hence, the resulting non-standard mixed formulation is to find $(u, \mathbf{p})$ in $H_0^1(\Omega) \times \mathbf{H}(\operatorname{div};\Omega)$ such that for all $(v, \mathbf{q}) \in H_0^1(\Omega) \times \mathbf{H}(\operatorname{div};\Omega)$,

$$\frac{1}{2}a(u,v)+\frac{1}{2}d(\mathbf{p},\mathbf{q})-\frac{1}{2}b(u,\mathbf{q})+\frac{1}{2}b(v,\mathbf{p})+\frac{1}{2}(\operatorname{div}\mathbf{p},\operatorname{div}\mathbf{q}) = (f,v)+\frac{1}{2}(f,\operatorname{div}\mathbf{q}). \tag{23}$$

Therefore, defining the bilinear form

$$C(w,\mathbf{r}|v,\mathbf{q}) = a(w,v) + d(\mathbf{r},\mathbf{q}) - b(w,\mathbf{q}) + b(v,\mathbf{r}) + (\operatorname{div}\mathbf{r},\operatorname{div}\mathbf{q}), \tag{24}$$

the corresponding discrete formulation is to find $(u_h, \mathbf{p}_h) \in V_h \times \mathbf{\Gamma}_h$ such that

$$C(u_h, \mathbf{p}_h | v_h, \mathbf{q}_h) = 2(f, v_h) + (f, \operatorname{div}\mathbf{q}_h). \tag{25}$$

Note that C is a non-symmetric bilinear form. Nonetheless, in [1] it was shown that C is both continuous and coercive, hence the discrete problem is well-posed.

3. Analysis of the non-standard mixed finite element method

Here we provide an analysis of the non-standard mixed finite element method in terms of the least-squares mixed finite element method. As turns out, the similarities between the two allow for a relatively easy comparison.

3.1. A relation between the bilinear forms B and C

Consider the bilinear form B from (17) and the bilinear form C from (24). Once C is rewritten using (2) from (24) into

$$C(w, \mathbf{r}|v, \mathbf{q}) = (w, \mathbf{r}; v, \mathbf{q})_{1\times \mathrm{div},A} - b(w, \mathbf{q}) + b(v, \mathbf{r}) \tag{26}$$

we see immediately that C is coercive, since

$$C(v, \mathbf{q}|v, \mathbf{q}) = \|(v, \mathbf{q})\|^2_{1\times \mathrm{div},A}. \tag{27}$$

In fact, C is a skew-symmetric perturbation of the natural inner product on $H^1_0(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega)$. Notice that this is due to the factor one half in (22). Without it, however, the coercivity would still be trivial. We also immediately see the following relation.

Proposition 3.1. *We have* $C(w, \mathbf{r}|v, \mathbf{q}) = B(w, \mathbf{r}|v, \mathbf{q}) + 2b(v, \mathbf{r})$.

Thus, rather than interpreting the new method as a non-standard mixed finite element method, as was done in [1], it is more natural to see it as a slight variation of the least-squares mixed finite element method, in particular after including the factor one half in (22). Indeed, we simply add together

$$B(u^l_h, \mathbf{p}^l_h|v_h, \mathbf{q}_h) = (f, \mathrm{div}\, \mathbf{q}_h) \quad \text{and} \quad 2b(v_h, \mathbf{p}_h) = 2(f, v_h) \tag{28}$$

and end up with the method proposed in Section 2.3. It is however not immediately clear what its merits are. The coercivity constant is now equal to one and not to $1 - \gamma$ as we saw in (18). This is in principle an advantage. The price to pay is that a linear system needs to be solved that is of the form

$$(M + S)x = b \tag{29}$$

with M symmetric positive definite and S skew-symmetric, or, if the factor one half is not included, of a less special non-symmetric system.

3.2. A supercloseness result

For all $(v_h, \mathbf{q}_h) \in V_h \times \mathbf{\Gamma}_h$ we have

$$\begin{aligned} C(u, \mathbf{p}|v, \mathbf{q}) &= 2(f, v) + (f, \mathrm{div}\, \mathbf{q}) \quad \text{and} \\ C(u_h, \mathbf{p}_h|v_h, \mathbf{q}_h) &= 2(f, v_h) + (f, \mathrm{div}\, \mathbf{q}_h). \end{aligned} \tag{30}$$

Subtraction leads to the error equation

$$C(u-u_h,\mathbf{p}-\mathbf{p}_h|v_h,\mathbf{q}_h)=0. \tag{31}$$

We can now compare the approximations $u_h,\mathbf{p}_h$ with their least-squares mixed finite element counterparts $u_h^l,\mathbf{p}_h^l$ as follows.

Lemma 3.2. *We have* $C(u_h-u_h^l,\mathbf{p}_h-\mathbf{p}_h^l|v_h,\mathbf{q}_h)=2(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^l),v_h)$.

Proof. Using (31) and the relation with the bilinear form B from Lemma 3.1 we find

$$\begin{aligned}C(u_h-u_h^l,\mathbf{p}_h-\mathbf{p}_h^l|v_h,\mathbf{q}_h)&=C(u-u_h^l,\mathbf{p}-\mathbf{p}_h^l|v_h,\mathbf{q}_h)\\&=B(u-u_h^l,\mathbf{p}-\mathbf{p}_h^l|v_h,\mathbf{q}_h)+2(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^l),v_h).\end{aligned}$$

The term with B vanishes, since it is the error equation for the least-squares mixed method. □

To continue, we need the L^2-orthogonal projection P_h on the space of piecewise constants. The following lemma is about the least-squares mixed finite element method. It could have easily been proved in the paper [8] but in that context there was no need for the result. Now there is.

Lemma 3.3. *Assume that W_h contains the piecewise constants. Then for all $v_h\in V_h$ we have*

$$|(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^l),v_h)|\leqslant Ch\|(u-u_h^s,\mathbf{p}-\mathbf{p}_h^m)\|_{1\times\operatorname{div},A}|v_h|_{1,A}. \tag{32}$$

Proof. Using (13) and the fact that W_h contains the piecewise constants, we can rewrite

$$\begin{aligned}(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^l),v_h)&=(\operatorname{div}(\mathbf{p}_h^m-\mathbf{p}_h^l),v_h)+(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^m),v_h)\\&=-(A^{-1}(\mathbf{p}_h^m-\mathbf{p}_h^l),A\nabla v_h)+(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^m),v_h-P_hv_h).\end{aligned} \tag{33}$$

Taking the absolute value and using the Cauchy-Schwarz inequality then yields that

$$|(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^l),v_h)|\leqslant\|\mathbf{p}_h^m-\mathbf{p}_h^l\|_{\operatorname{div},A}|v_h|_{1,A}+|\operatorname{div}(\mathbf{p}-\mathbf{p}_h^m)|_0Ch|v_h|_1. \tag{34}$$

Using Theorem 2.2 and the equivalence of $|\cdot|_{1,A}$ and $|\cdot|_1$ gives

$$|(\operatorname{div}(\mathbf{p}-\mathbf{p}_h^l),v_h)|\leqslant Ch\|(u-u_h^s,\mathbf{p}-\mathbf{p}_h^m)\|_{1\times\operatorname{div},A}|v_h|_{1,A}. \tag{35}$$

This proves the statement. □

We will now prove our main result, showing that the non-standard mixed finite element method is a higher order perturbation of the least-squares mixed finite element method.

Theorem 3.4. *If* $(W_h, \mathbf{\Gamma}_h)$ *with* $W_h = \mathrm{div}(\mathbf{\Gamma}_h)$ *is a Babuška-Brezzi stable pair and* W_h *contains the piecewise constants, then*

$$\|(u_h - u_h^l, \mathbf{p}_h - \mathbf{p}_h^l)\|_{1\times\mathrm{div},A} \leqslant Ch\|(u - u_h^s, \mathbf{p} - \mathbf{p}_h^m)\|_{1\times\mathrm{div},A}. \tag{36}$$

Proof. The coercivity (27) of C and Lemmas 3.2 and 3.3 give

$$\begin{aligned}
&\|(u_h - u_h^l, \mathbf{p}_h - \mathbf{p}_h^l)\|^2_{1\times\mathrm{div},A} \\
&\quad = C(u_h - u_h^l, \mathbf{p}_h - \mathbf{p}_h^l | u_h - u_h^l, \mathbf{p}_h - \mathbf{p}_h^l) \\
&\quad = 2(\mathrm{div}(\mathbf{p} - \mathbf{p}_h^l), u_h - u_h^l) \\
&\quad \leqslant Ch\|(u - u_h^s, \mathbf{p} - \mathbf{p}_h^m)\|_{1\times\mathrm{div},A} |u_h - u_h^l|_{1,A} \\
&\quad \leqslant Ch\|(u - u_h^s, \mathbf{p} - \mathbf{p}_h^m)\|_{1\times\mathrm{div},A} \|(u_h - u_h^l, \mathbf{p}_h - \mathbf{p}_h^l)\|_{1\times\mathrm{div},A}.
\end{aligned}$$

Dividing by $\|(u_h - u_h^l, \mathbf{p}_h - \mathbf{p}_h^l)\|_{1\times\mathrm{div},A}$ gives the statement. □

The conclusion is that the non-standard mixed finite element method inherits not only a priori bounds but also supercloseness and superconvergence properties of the least-squares mixed finite element method.

3.3. Example: lowest order standard and mixed finite element spaces

We will give one example. For more examples, we refer to [7]. Suppose that V_h is the space of continuous piecewise linear functions [14] relative to a uniform partition of the domain $\Omega \subset \mathbb{R}^n$ with $n = 2$. It is well known [4] that the standard method yields u_h^s superclose to the nodal interpolant $L_h u \in V_h$,

$$|L_h u - u_h^s|_1 \leqslant Ch^2|u|_3, \quad \text{whereas } |u - u_h^s| \leqslant Ch|u|_2. \tag{37}$$

Moreover, let $\mathbf{\Gamma}_h \subset \mathbf{H}(\mathrm{div};\Omega)$ be the space of lowest order Raviart-Thomas [10], [19] functions with respect to this partition. For the mixed finite element method with $W_h = \mathrm{div}(\mathbf{\Gamma}_h)$ it is also known [3], though only for $n = 2$, that

$$\|\Pi_h \mathbf{p} - \mathbf{p}_h^m\|_{\mathrm{div}} \leqslant Ch^2|\mathbf{p}|_2 \leqslant Ch^2|u|_3, \quad \text{whereas } \|\mathbf{p} - \mathbf{p}_h^m\| \leqslant Ch|\mathbf{p}|_1, \tag{38}$$

and where Π_h is the Fortin interpolant of $\mathbf{p}$ in $\mathbf{\Gamma}_h$. Now, Theorem 2.2 and the triangle inequality yield that

$$\|(u_h^l - L_h u, \mathbf{p}_h^l - \Pi_h \mathbf{p})\|_{1\times\mathrm{div},A} \leqslant Ch^2|u|_3, \tag{39}$$

as was shown in [7]. With Theorem 3.4 from the present paper we now have the corresponding result also for the non-standard mixed finite element method,

$$\|(u_h - L_h u, \mathbf{p}_h - \Pi_h \mathbf{p})\|_{1\times\mathrm{div},A} \leqslant Ch^2|u|_3. \tag{40}$$

Thus, using the same post-processors as for u_h^s and u_h^l and for $\mathbf{p}_h^m$ and for $\mathbf{p}_h^l$, also the approximation order of u_h and $\mathbf{p}_h$ can be enhanced in an inexpensive fashion.

4. Concluding remarks

We studied a non-standard mixed finite element method proposed in [1]. This method leads to a coercive though non-symmetric bilinear form even for symmetric elliptic problems. By including a factor one half in the formulation of their method, the bilinear form C could be written as a skew-symmetric perturbation of a symmetric positive definite bilinear form, and more easily compared with the bilinear form that results from the least-squares mixed finite element method.

The main result is that the new method is only a higher order perturbation of the least-squares mixed finite element method and thus that the non-symmetry does not spoil the approximation properties of the method. On the other hand, for exactly the same reason, it cannot be better than the least-squares mixed finite element method in a significant way. Thus, we advise against its use for the model problem studied, because solving the non-symmetric linear system would be too expensive compared to the usual symmetric positive definite system that needs to be solved.

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