Ningning Yan

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SUPERCONVERGENCE ANALYSIS AND A POSTERIORI ERROR ESTIMATION OF A FINITE ELEMENT METHOD FOR AN OPTIMAL CONTROL PROBLEM GOVERNED BY INTEGRAL EQUATIONS*

Ningning Yan, Beijing

Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. In this paper, we discuss the numerical simulation for a class of constrained optimal control problems governed by integral equations. The Galerkin method is used for the approximation of the problem. A priori error estimates and a superconvergence analysis for the approximation scheme are presented. Based on the results of the superconvergence analysis, a recovery type a posteriori error estimator is provided, which can be used for adaptive mesh refinement.

Keywords: optimal control, integral equation, Galerkin method, superconvergence, a posteriori error estimates

MSC 2010: 65N30, 65R20

1. Introduction

The finite element approximation plays an important role in the numerical treatment of optimal control problems. This approach has been extensively studied in the papers [1], [12], [26]. In particular, a priori error estimates of finite element approximations for optimal control problems governed by linear elliptic equations were established some 30 years ago in, for example, [11], and a posteriori error estimates have been discussed in, e.g., [4] and [21]–[23] in recent years.

Although the finite element method for the optimal control problem governed by partial differential equations has been extensively studied, to the author's knowledge, there are very few similar results for the optimal control problem governed by

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integral equations, even though this kind of problem is also widely used in practical engineering and scientific computations. Recently, an a priori error estimate and a residual type a posteriori error estimate for a class of constrained optimal control problems governed by integral equations have been presented in [6].

In this paper we consider the numerical simulation for a class of constrained optimal control problems governed by integral equations of Fredholm type (see (2.1)–(2.2) for more details). We provide a superconvergence analysis for the Galerkin approximation to these control problems. Based on the results of the superconvergence analysis, a recovery type a posteriori error estimator is established, which can be used for adaptive mesh refinement.

Superconvergence has been investigated for the finite element method of the optimal control problem governed by the partial differential equation, in, e.g., [7], [10], [16], [21], [24], and [28]. It has also been discussed for the Galerkin approximation of integral equations, see, e.g., [5], [17], and [18]. Some techniques used in the above references are utilized in this paper. However, it seems to be not straightforward to extend the existing techniques to the optimal control problems governed by integral equations. Comparing our results with the corresponding ones on constrained optimal control problems governed by partial differential equations, it can be found that there are some significant differences between them. For example, the superconvergence analysis for the constrained optimal control problems governed by integral equations can be applied to general regular meshes, which is in contrast to many well-known superconvergence results where the condition of uniform meshes is required. Moreover, a new recovery operator is introduced in this paper in order to fit the interpolation definition and regular meshes. As a result, the new approach is more suitable for adaptive finite element mesh refinement.

The plan of the paper is as follows. In Section 2 we present the optimal control problem (2.1) governed by integral equations (2.2), and we further provide its Galerkin scheme (finite element method). In Section 3 an a priori error estimate of the Galerkin approximation is derived. In Section 4 we provide the main result of this paper: superconvergence of the Galerkin approximation for the optimal control problem (2.1)–(2.2). On the basis of the superconvergence results established in Section 4 we construct a recovery type a posteriori error estimator in Section 5, which is proved to be asymptotically exact on regular meshes.

2. Model problem and its Galerkin scheme

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and semi-norm $|\cdot|_{m,q,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$. In addition, c and C denote generic positive constants which do not depend on h.

We consider a model problem for the optimal control problem governed by a Fredholm-type integral equation, namely,

$$\min_{u\in K}\left\{\frac{1}{2}\|y-y_0\|_{0,\Omega}^2+\frac{\alpha}{2}\|u\|_{0,\Omega}^2\right\} \tag{2.1}$$

subject to

$$y-\int_\Omega G(s,t)y(s)\,\mathrm{d}s=f+u\quad\text{in }\ \Omega. \tag{2.2}$$

Here $G(\cdot,\cdot)$ is at least in $L^2(\Omega\times\Omega)$ (precise regularity conditions will be given later), $f\in L^2(\Omega)$, and K denotes a convex subset of the space $U=L^2(\Omega)$, the state space is $V=L^2(\Omega)$, $\Omega\subset\mathbb{R}^2$ is a bounded domain. This is a typical optimal control problem governed by integral equations. Its numerical simulation was discussed in, e.g., [6].

It is well known that if 1 is not in the spectrum of the Fredholm integral operator, then the equation

$$y-\int_\Omega G(s,t)y(s)\,\mathrm{d}s=F\quad\text{in }\ \Omega$$

has a unique solution $y\in L^2(\Omega)$ and $\|y\|_{0,\Omega}\leqslant C\|F\|_{0,\Omega}$ for any $F\in L^2(\Omega)$ (compare, for example, Kress [15] or Zabreiko et al. [29]). Moreover, if $G(\cdot,\cdot)$ is smooth enough, we have

$$\|y\|_{m,\Omega}\leqslant C\|F\|_{m,\Omega}\quad\text{for all }\ F\in H^m(\Omega). \tag{2.3}$$

In this paper we always assume that 1 is not in the spectrum of the Fredholm integral operator $G(s,t)$. Let

$$(y,w)_{L^2(\Omega)}:=\int_\Omega y(s)w(s)\,\mathrm{d}s$$

be the $L^2(\Omega)$ scalar product. For simplicity, we denote $(y,w)_{L^2(\Omega)}$ by (y,w) if there is no confusion. Moreover, we let

$$A(w,v):=(w,v)-\int_\Omega\int_\Omega G(s,t)w(s)v(t)\,\mathrm{d}s\,\mathrm{d}t$$

in the sequel. Then the integral equation (2.2) can be rewritten as

$$A(y,w)=(f+u,w)\quad\forall\, w\in L^2(\Omega).$$

We will show that there exists a constant $c > 0$ such that for any $v \in L^2(\Omega)$ there exist a $w \in L^2(\Omega)$, $w \neq 0$, such that

$$c\|v\|_{0,\Omega}\|w\|_{0,\Omega} \leqslant A(v,w) \tag{2.4}$$

and

$$c\|v\|_{0,\Omega}\|w\|_{0,\Omega} \leqslant A(w,v). \tag{2.5}$$

This is true since, for any $v \in L^2(\Omega)$, there is a function $\varphi \in L^2(\Omega)$ such that for all $\psi \in L^2(\Omega)$ we have

$$(\psi,\varphi) - \int_\Omega\int_\Omega G(s,t)\psi(s)\varphi(t)\,\mathrm{d}s\,\mathrm{d}t = \int_\Omega\int_\Omega G(s,t)\psi(s)v(t)\,\mathrm{d}s\,\mathrm{d}t$$

and $\|\varphi\|_{0,\Omega} \leqslant C\|v\|_{0,\Omega}$. Let $w = v + \varphi$. Then

$$\begin{aligned}
A(v,w) &= (v,w) - \int_\Omega\int_\Omega G(s,t)v(s)w(t)\,\mathrm{d}s\,\mathrm{d}t \\
&= (v,v) - \int_\Omega\int_\Omega G(s,t)v(s)v(t)\,\mathrm{d}s\,\mathrm{d}t + (v,\varphi) - \int_\Omega\int_\Omega G(s,t)v(s)\varphi(t)\,\mathrm{d}s\,\mathrm{d}t \\
&= (v,v) = \|v\|_{0,\Omega}^2.
\end{aligned}$$

Therefore, (2.4) follows from

$$\|w\|_{0,\Omega} \leqslant \|v\|_{0,\Omega} + \|\varphi\|_{0,\Omega} \leqslant C\|v\|_{0,\Omega}.$$

Similarly, we can also prove the inequality (2.5). See also [3] and [6] for details.

Using the standard method from [19], it can be proved that the problem (2.1)–(2.2) has a unique solution $(y,u) \in (L^2(\Omega))^2$, and that the pair (y,u) is a solution of (2.1)–(2.2) if and only if there is a co-state $p \in L^2(\Omega)$ such that the triple (y,p,u) satisfies the system

$$A(y,w) = (f+u,w) \quad \forall\, w \in L^2(\Omega), \tag{2.6}$$

$$A(q,p) = (y-y_0,q) \quad \forall\, q \in L^2(\Omega), \tag{2.7}$$

$$(\alpha u + p, v-u) \geqslant 0 \quad \forall\, v \in K. \tag{2.8}$$

In this paper we consider the integral constraint for the control variable

$$K = \left\{ v \in L^2(\Omega_U) \colon \int_\Omega v \geqslant 0 \right\}. \tag{2.9}$$

It can be proved (see, e.g., [13] and [21]) that the solution of (2.6)–(2.8) satisfies

$$u = \frac{1}{\alpha}\left(-p + \max\left\{0, \frac{\int_\Omega p}{\int_\Omega 1}\right\}\right). \tag{2.10}$$

We focus on the optimal control problems with constraints of integral type (2.9). This kind of constrained optimal control problems can be found in many practical and engineering problems, because in many practical problems the control should be constrained in the sense of average or gross on the whole domain. Although optimal control problems governed by PDEs with integral constraints have been extensively investigated (see, e.g., [8], [13] and [21]), to the author's knowledge, this paper is the first to discuss the error analysis of the numerical simulation for the optimal control problems governed by integral equations with integral constraints. Considering the equation (2.10), we have that the regularity of the control u is the same as of the costate p. Moreover, it can be concluded from (2.3) that y, p and u are smooth enough if f, y_0 and the integral kernel $G(\cdot,\cdot)$ are smooth enough. In many practical problems, the integral kernel $G(\cdot,\cdot)$ is not smooth enough. Sometimes, it is even weakly singular. Then the solutions of the control problems may be not so smooth.

Let us consider the finite element approximation of the control problem (2.1)–(2.2). Here we only consider triangular elements, and similar results can be easily obtained for quadrilateral elements and three dimensional problems.

Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into triangles τ so that $\bar{\Omega}^h = \bigcup_{\tau\in T^h} \bar{\tau}$. Let h_τ denote the diameter of the element τ in T^h, and let $h = \max_{\tau\in T^h} h_\tau$. For simplicity, we assume that Ω is a convex polygon so that $\Omega = \Omega^h$.

Associated with T^h we have a finite-dimensional subspace V^h of $L^2(\bar{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of order m ($m \geqslant 0$) for all $\chi \in V^h$ and $\tau \in T^h$. Note that we do not impose a continuity requirement. It is easy to see that $V^h \subset V = U = L^2(\Omega)$. Let $K^h = K \cap V^h$. We have $K^h \subset K$.

The finite element approximation of the control problem (2.1)–(2.2) is defined by

$$\min_{u_h\in K^h}\left\{\frac{1}{2}\|y_h - y_0\|^2_{0,\Omega} + \frac{\alpha}{2}\|u_h\|^2_{0,\Omega}\right\} \tag{2.11}$$

subject to

$$A(y_h, w_h) = (f + u_h, w_h) \quad \forall\, w_h \in V^h \subset L^2(\Omega). \tag{2.12}$$

It follows from the conditions imposed on the kernel $G(\cdot,\cdot)$ of the Fredholm integral operator that for all sufficiently small $h > 0$ the Galerkin equation

$$A(y_h, v_h) = (F, v_h) \quad \forall\, v_h \in V^h$$

has a unique solution for any $F \in L^2(\Omega)$. Similarly to (2.4) and (2.5) it can be shown that there exists a constant $c > 0$ such that for any $v_h \in V^h$ there exist a $w_h \in V^h$, $w_h \neq 0$, such that

$$c\|v_h\|_{0,\Omega}\|w_h\|_{0,\Omega} \leqslant A(v_h, w_h) \tag{2.13}$$

and

$$c\|v_h\|_{0,\Omega}\|w_h\|_{0,\Omega} \leqslant A(w_h, v_h). \tag{2.14}$$

It is well known (see, e.g., [6] and [19]) that the control problem (2.11)–(2.12) has a unique solution (y_h, u_h), and that a pair (y_h, u_h) is a solution of (2.11)–(2.12) if and only if there exists a co-state $p_h \in V^h$ such that the triple (y_h, p_h, u_h) satisfies the following optimality condition:

$$A(y_h, w_h) = (f + u_h, w_h) \quad \forall\, w_h \in V^h \subset L^2(\Omega), \tag{2.15}$$

$$A(q_h, p_h) = (y_h - y_0, q_h) \quad \forall\, q_h \in V^h \subset L^2(\Omega), \tag{2.16}$$

$$(\alpha u_h + p_h, v_h - u_h) \geqslant 0 \quad \forall\, v_h \in K^h. \tag{2.17}$$

3. A priori error estimate

In this section we will provide an error analysis for the optimal control problem (2.6)–(2.8) and its finite element approximation (2.15)–(2.17).

Theorem 3.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of the systems (2.6)–(2.8) and (2.15)–(2.17), respectively. Let V^h be the finite element space of order m defined in the previous section. Assume that $y, p \in H^{m+1}(\Omega)$. Then we have*

$$\|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} \leqslant Ch^{m+1}. \tag{3.1}$$

Proof. Let $(y_h(u), p_h(u))$ be the solution of the system of equations:

$$A(y_h(u), w_h) = (f + u, w_h) \quad \forall\, w_h \in V^h, \tag{3.2}$$

$$A(q_h, p_h(u)) = (y_h(u) - y_0, q_h) \quad \forall\, q_h \in V^h. \tag{3.3}$$

It follows from (3.2)–(3.3) and (2.15)–(2.16) that

$$\begin{aligned} \|y_h(u) - y_h\|_{0,\Omega}^2 &= (y_h(u) - y_h, y_h(u) - y_h) \\ &= A(y_h(u) - y_h, p_h(u) - p_h) = (u - u_h, p_h(u) - p_h). \end{aligned} \tag{3.4}$$

Let $u_I \in V^h$ be the L^2-project of u. Then we have

$$\int_\Omega u_I = \int_\Omega u \geqslant 0,$$

and hence $u_I \in K^h$.

Moreover, note that $u_h \in K$. Thus, (2.8), (2.17), and (3.4) imply that

$$\begin{aligned}
\alpha\|u - u_h\|_{0,\Omega}^2 &+ \|y_h(u) - y_h\|_{0,\Omega}^2 \\
&= \alpha(u - u_h, u - u_h) + (u - u_h, p_h(u) - p_h) \\
&= (\alpha u + p, u - u_h) + (\alpha u_h + p_h, u_h - u_I) \\
&\quad + (\alpha u_h + p_h, u_I - u) + (p_h(u) - p, u - u_h) \\
&\leqslant 0 + 0 + (\alpha u_h + p_h, u_I - u) + (p_h(u) - p, u - u_h) \\
&= (p_h(u) - p, u - u_h) \\
&\leqslant C(\alpha)\|p_h(u) - p\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u - u_h\|_{0,\Omega}^2,
\end{aligned}$$

where we have used the property of the L^2-projection:

$$(u - u_I, w_h) = 0 \quad \forall\, w_h \in V^h.$$

Therefore, we have

$$\|u - u_h\|_{0,\Omega}^2 + \|y_h(u) - y_h\|_{0,\Omega}^2 \leqslant C\|p_h(u) - p\|_{0,\Omega}^2. \tag{3.5}$$

Setting $p_I \in V^h$ to be the L^2-project of p again, it follows from (2.14) that there exists a function $w_h \in V^h$ such that

$$c\|p_h(u) - p_I\|_{0,\Omega}\|w_h\|_{0,\Omega} \leqslant A(w_h, p_h(u) - p_I),$$

where $p_h(u)$ is the solution of the equations (3.2)–(3.3). Then (3.3) and (2.7) imply that

$$\begin{aligned}
c\|p_h(u) - p_I\|_{0,\Omega}\|w_h\|_{0,\Omega} \leqslant A(w_h, p_h(u) - p_I) &= A(w_h, p_h(u) - p) + A(w_h, p - p_I) \\
&= (y_h(u) - y, w_h) + A(w_h, p - p_I) \\
&\leqslant C(\|y_h(u) - y\|_{0,\Omega} + \|p - p_I\|_{0,\Omega})\|w_h\|_{0,\Omega}.
\end{aligned}$$

Therefore, we have

$$\|p_h(u) - p_I\|_{0,\Omega} \leqslant C(\|y_h(u) - y\|_{0,\Omega} + \|p - p_I\|_{0,\Omega}), \tag{3.6}$$

and hence,

$$(3.7)\quad \|p_h(u)-p\|_{0,\Omega} \leqslant \|p_h(u)-p_I\|_{0,\Omega}+\|p_I-p\|_{0,\Omega} \leqslant C(\|y_h(u)-y\|_{0,\Omega}+\|p-p_I\|_{0,\Omega}).$$

Similarly, let $y_I \in V^h$ be the L^2-projection of y. There exists a function $v_h \in V^h$ such that

$$\begin{aligned} c\|y_h(u) - y_I\|_{0,\Omega}\|v_h\|_{0,\Omega} &\leqslant A(y_h(u) - y_I, v_h) \\ &= A(y_h(u) - y, v_h) + A(y - y_I, v_h) \\ &= 0 + A(y - y_I, v_h) \leqslant C\|y - y_I\|_{0,\Omega}\|v_h\|_{0,\Omega}, \end{aligned}$$

and hence,

$$(3.8)\qquad \|y_h(u) - y\|_{0,\Omega} \leqslant \|y_h(u) - y_I\|_{0,\Omega} + \|y_I - y\|_{0,\Omega} \leqslant C\|y - y_I\|_{0,\Omega}.$$

Summing up, it follows from (3.5), (3.7), and (3.8) that

$$(3.9)\qquad \|u - u_h\|_{0,\Omega}^2 + \|y_h(u) - y_h\|_{0,\Omega}^2 \leqslant C(\|y - y_I\|_{0,\Omega} + \|p - p_I\|_{0,\Omega})^2,$$

which, together with (3.8), yields that

$$(3.10)\quad \|y-y_h\|_{0,\Omega} \leqslant \|y-y_h(u)\|_{0,\Omega}+\|y_h(u)-y_h\|_{0,\Omega} \leqslant C(\|y-y_I\|_{0,\Omega}+\|p-p_I\|_{0,\Omega}).$$

Moreover, it can be concluded from (2.14), (2.16), and (3.3) that

$$(3.11)\qquad \begin{aligned} c\|p_h(u) - p_h\|_{0,\Omega}\|w_h\|_{0,\Omega} &\leqslant A(w_h, p_h(u) - p_h) = (y_h(u) - y_h, w_h) \\ &\leqslant \|y_h(u) - y_h\|_{0,\Omega}\|w_h\|_{0,\Omega}. \end{aligned}$$

Thus, (3.7)–(3.9) and (3.11) imply that

$$(3.12)\qquad \begin{aligned} \|p - p_h\|_{0,\Omega} &\leqslant \|p - p_h(u)\|_{0,\Omega} + \|p_h(u) - p_h\|_{0,\Omega} \\ &\leqslant C(\|y - y_I\|_{0,\Omega} + \|p - p_I\|_{0,\Omega}). \end{aligned}$$

Therefore, it follows from (3.9), (3.10), (3.12), and the well-known interpolation error estimate (see, e.g., [9]) that

$$\begin{aligned} \|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} &\leqslant C(\|y - y_I\|_{0,\Omega} + \|p - p_I\|_{0,\Omega}) \\ &\leqslant Ch^{m+1}(\|y\|_{m+1,\Omega} + \|p\|_{m+1,\Omega}), \end{aligned}$$

which proves (3.1). □

R e m a r k 3.1. There is an extensive list of papers devoted to the a priori error analysis for optimal control problems governed by PDEs (see, e.g., [11], [21], and [25]). Moreover, a priori and a posteriori error estimates of the finite element method for optimal control problem governed by integral equations with a different constraint have been investigated in [6]. The proof of Theorem 3.1 uses techniques from the above references.

4. Supconvergence analysis

Next, we will discuss the superconvergence analysis of the optimal control problem (2.6)–(2.8).

Theorem 4.1. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of the systems* (2.6)–(2.8) *and* (2.15)–(2.17)*, respectively. Let* V^h *be the finite element space of order* m *defined in Section* 2*. Assume that* $y,p\in H^{m+1}(\Omega)$ *and* $\partial_t^{m+1}G(s,t)\in L^2(\Omega\times\Omega)$*, where* $\partial_t^{m+1}G(s,t)$ *is the derivative of order* $m+1$ *of the function* $G(s,t)$ *with respect to the variable* t*. Then*

$$\|y_I-y_h\|_{0,\Omega}+\|p_I-p_h\|_{0,\Omega}+\|u_I-u_h\|_{0,\Omega}\leqslant Ch^{2(m+1)}, \tag{4.1}$$

where y_I*,* p_I*,* $u_I\in V^h$ *are the* L^2*-projections of* y*,* p *and* u*, respectively.*

P r o o f. Recalling (3.4), we have

$$\|y_h(u)-y_h\|^2_{0,\Omega}=(u-u_h,p_h(u)-p_h)=(u_I-u_h,p_h(u)-p_h), \tag{4.2}$$

where $(y_h(u),p_h(u))$ is the solution of the equations (3.2)–(3.3). Using (2.8), (2.17) and (4.2), we obtain

$$\begin{aligned}\alpha\|u_I-u_h\|^2_{0,\Omega}&+\|y_h(u)-y_h\|^2_{0,\Omega}\\ &=\alpha(u_I-u_h,u_I-u_h)+(u_I-u_h,p_h(u)-p_h)\\ &=(\alpha u_I+p_h(u),u_I-u_h)+(\alpha u_h+p_h,u_h-u_I)\\ &\leqslant(\alpha u_I+p_h(u),u_I-u_h)+0\\ &=(\alpha u+p,u-u_h)+(\alpha u+p,u_I-u)\\ &\quad+\alpha(u_I-u,u_I-u_h)+(p_h(u)-p,u_I-u_h)\\ &\leqslant 0+(\alpha u+p,u_I-u)+0+(p_h(u)-p,u_I-u_h).\end{aligned} \tag{4.3}$$

Recalling (2.10), we conclude that $\alpha u+p$ is a constant on the whole domain. Thus,

$$(\alpha u+p,u_I-u)=(\alpha u+p)\int_\Omega(u_I-u)=0. \tag{4.4}$$

In addition, it is easy to see that

$$(p_h(u) - p, u_I - u_h) = (p_h(u) - p_I, u_I - u_h) \leqslant C(\alpha)\|p_h(u) - p_I\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u_I - u_h\|_{0,\Omega}^2. \tag{4.5}$$

Then, it follows from (4.3)–(4.5) that

$$\|u_I - u_h\|_{0,\Omega}^2 + \|y_h(u) - y_h\|_{0,\Omega}^2 \leqslant C\|p_h(u) - p_I\|_{0,\Omega}^2. \tag{4.6}$$

By virtue of (2.14), there exists a function $w_h \in V^h$ such that

$$c\|p_h(u) - p_I\|_{0,\Omega}\|w_h\|_{0,\Omega} \leqslant A(w_h, p_h(u) - p_I).$$

Hence, it follows from (2.7) and (3.3) that

$$\begin{aligned} c\|p_h(u) - p_I\|_{0,\Omega}\|w_h\|_{0,\Omega} &\leqslant A(w_h, p_h(u) - p_I) = A(w_h, p - p_I) + A(w_h, p_h(u) - p) \\ &= A(w_h, p - p_I) + (y_h(u) - y, w_h) \\ &= A(w_h, p - p_I) + (y_h(u) - y_I, w_h). \end{aligned} \tag{4.7}$$

Note that

$$\begin{aligned} A(w_h, p - p_I) &= (w_h, p - p_I) - \int_\Omega\int_\Omega G(s,t)w_h(s)(p - p_I)(t)\,\mathrm{d}s\,\mathrm{d}t \\ &= 0 - \int_\Omega w_h(s)\Big(\int_\Omega (G(s,t) - G_I^t(s,t))(p - p_I)(t)\,\mathrm{d}t\Big)\,\mathrm{d}s \\ &\leqslant Ch^{2(m+1)}\int_\Omega\Big(\int_\Omega (\partial_t^{m+1}G(s,t))^2\,\mathrm{d}t\Big)^{1/2}|p|_{m+1,\Omega}w_h(s)\,\mathrm{d}s \\ &\leqslant Ch^{2(m+1)}|p|_{m+1,\Omega}\Big(\int_\Omega\int_\Omega (\partial_t^{m+1}G(s,t))^2\,\mathrm{d}t\,\mathrm{d}s\Big)^{1/2}\Big(\int_\Omega w_h^2(s)\,\mathrm{d}s\Big)^{1/2} \\ &\leqslant Ch^{2(m+1)}\|G\|_{m+1,\Omega\times\Omega}\|p\|_{m+1,\Omega}\|w_h\|_{0,\Omega}, \end{aligned} \tag{4.8}$$

where $G_I^t(s,t)$ is the L^2-projection of $G(s,t)$ defined by

$$\int_\Omega G_I^t(s,t)v_h(t)\,\mathrm{d}t = \int_\Omega G(s,t)v_h(t)\,\mathrm{d}t \quad \forall\, v_h(t) \in V^h$$

and $\partial_t^{m+1}G(s,t)$ is the derivative of order $m+1$ of the function $G(s,t)$ with respect to the variable t. Moreover, it is easy to see that

$$(y_h(u) - y_I, w_h) \leqslant \|y_h(u) - y_I\|_{0,\Omega}\|w_h\|_{0,\Omega}. \tag{4.9}$$

Thus, (4.7)–(4.9) imply

$$\|p_h(u) - p_I\|_{0,\Omega} \leqslant Ch^{2(m+1)} + \|y_h(u) - y_I\|_{0,\Omega}. \tag{4.10}$$

Similarly, it can be deduced that

$$\begin{aligned} c\|y_h(u) - y_I\|_{0,\Omega}\|v_h\|_{0,\Omega} &\leqslant A(y_h(u) - y_I, v_h) \\ &= A(y - y_I, v_h) + A(y_h(u) - y, v_h) \\ &= A(y - y_I, v_h) + 0 \\ &\leqslant Ch^{2(m+1)}\|G\|_{m+1,\Omega\times\Omega}\|y\|_{m+1,\Omega}\|v_h\|_{0,\Omega}, \end{aligned}$$

and hence,

$$\|y_h(u) - y_I\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.11}$$

Summing up, it follows from (4.6), (4.10), and (4.11) that

$$\|u_I - u_h\|_{0,\Omega}^2 + \|y_h(u) - y_h\|_{0,\Omega}^2 \leqslant Ch^{4(m+1)}. \tag{4.12}$$

Moreover, (4.11) and (4.12) imply that

$$\|y_I - y_h\|_{0,\Omega} \leqslant \|y_I - y_h(u)\|_{0,\Omega} + \|y_h(u) - y_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.13}$$

Furthermore, (3.11) and (4.12) imply that

$$\|p_h(u) - p_h\|_{0,\Omega} \leqslant C\|y_h(u) - y_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.14}$$

Then, by means of (4.10), (4.11), and (4.14), we obtain that

$$\|p_I - p_h\|_{0,\Omega} \leqslant \|p_I - p_h(u)\|_{0,\Omega} + \|p_h(u) - p_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.15}$$

Thus, (4.1) is proved by (4.12), (4.13), and (4.15). □

In Theorem 4.1 we have proved the supercloseness property for the finite element approximation of the optimal control problem (2.6)–(2.8). It has been shown that the error between (y_h, p_h, u_h) and (y_I, p_I, u_I) is smaller than the error between (y_h, p_h, u_h) and (y, p, u) (see Theorems 3.1 and 4.1), and the order of error is improved from $m+1$ to $2(m+1)$. In order to obtain global superconvergence by using the supercloseness result provided by Theorem 4.1, we construct a postprocessor as follows.

For any element $\tau_0 \in T^h$, we construct its neighborhood $\Omega_0 = \bigcup_{i=0}^{3} \tau_i \subset \Omega$ as follows: Let l_i, $i = 1, 2, 3$, be the three edges of τ_0.

a) If $l_i \cap \partial\Omega = \emptyset$, $i = 1, 2, 3$, set τ_i to be the element such that l_i is an edge of τ_i.

b) If there is one edge of τ_0 on the boundary $\partial\Omega$, e.g., $l_i \cap \partial\Omega = \emptyset$, $i = 1, 2$, and $l_3 \subset \partial\Omega$, set τ_i to be the element such that l_i is an edge of τ_i for $i = 1, 2$, and set τ_3 to be the element such that the intersection point of l_1 and l_2 is a vertex of τ_3.

c) If there are two edges of τ_0 on the boundary $\partial\Omega$, e.g., $l_1 \cap \partial\Omega = \emptyset$, and $l_i \subset \partial\Omega$, $i = 2, 3$, set τ_1 to be the element such that l_1 is an edge of τ_1, and set τ_i, $i = 2, 3$, to be the element such that two end points of l_1 are vertices of τ_2 and τ_3, respectively.

Then we set $\tilde{\Pi}_{2h} w|_{\Omega_0} \in P_{2m+1}$ such that

$$\int_{\tau_0} (\tilde{\Pi}_{2h} w - w) v = 0 \quad \forall\, v \in P_{m-1}$$

and

$$\int_{\tau_i} (\tilde{\Pi}_{2h} w - w) v = 0 \quad \forall\, v \in P_m,\ \ i = 1, 2, 3,$$

where P_k is the space of polynomials of order k. Note that the number of the degrees of freedom for the interpolation $\tilde{\Pi}_{2h} w$ defined above is

$$\frac{1}{2} m(m+1) + \frac{3}{2}(m+1)(m+2) = \frac{1}{2}(m+1)(m+3m+6) = \frac{1}{2}(2m+2)(2m+3),$$

which is the same as the number of the degrees of freedom for the polynomials of order $2m+1$. Because the above interpolation conditions are independent, it can be proved that the interpolation $\tilde{\Pi}_{2h} w$ exists and is unique for all $w \in L^2(\Omega)$ and fixed $\Omega_0 = \bigcup_{i=0}^{3} \tau_i$. Let $\Pi_{2h} w = \tilde{\Pi}_{2h} w$ on the element τ_0. Then it can be deduced that

$$\|\Pi_{2h} w_h\|_{0,\Omega} \leqslant C \|w_h\|_{0,\Omega} \quad \forall\, w_h \in V^h, \tag{4.16}$$

$$\Pi_{2h} w_I = \Pi_{2h} w \quad \forall\, w \in L^2(\Omega), \tag{4.17}$$

$$\|\Pi_{2h} w - w\|_{0,\Omega} \leqslant C h^{2m+2} \|w\|_{2m+2,\Omega} \quad \forall\, w \in H^{2m+2}(\Omega). \tag{4.18}$$

Using the above properties and Theorem 4.1, we obtain the following global superconvergence result.

Theorem 4.2. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of systems (2.6)–(2.8) and (2.15)–(2.17), respectively. Let V^h be the finite element space of order m defined* *in Section 2. Assume that $u, y, p \in H^{2m+2}(\Omega)$ and $G(s,t) \in H^{m+1}(\Omega \times \Omega)$. Then we have*

$$\|y - \Pi_{2h} y_h\|_{0,\Omega} + \|p - \Pi_{2h} p_h\|_{0,\Omega} + \|u - \Pi_{2h} u_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.19}$$

P r o o f. Note that

$$\|y - \Pi_{2h} y_h\|_{0,\Omega} \leqslant \|y - \Pi_{2h} y\|_{0,\Omega} + \|\Pi_{2h} y - \Pi_{2h} y_I\|_{0,\Omega} + \|\Pi_{2h} y_I - \Pi_{2h} y_h\|_{0,\Omega}. \tag{4.20}$$

It follows from property (4.16) and Theorem 4.1 that

$$\|\Pi_{2h} y_I - \Pi_{2h} y_h\|_{0,\Omega} \leqslant C\|y_I - y_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.21}$$

Then it is easy to see from (4.17), (4.18), (4.20), and (4.21) that

$$\|y - \Pi_{2h} y_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.22}$$

Similarly, we can prove that

$$\|p - \Pi_{2h} p_h\|_{0,\Omega} \leqslant Ch^{2(m+1)} \tag{4.23}$$

and

$$\|u - \Pi_{2h} u_h\|_{0,\Omega} \leqslant Ch^{2(m+1)}. \tag{4.24}$$

Hence (4.19) is the direct result of (4.22)–(4.24). □

R e m a r k 4.1. There are many papers devoted to the superconvergence analysis for integral equations (see, e.g., [5], [17], [18], and [27]) and to the optimal control problems governed by PDEs (see, e.g., [7], [10], [16], [21], [24], and [28]). The proofs of Theorems 4.1 and 4.2 use techniques similar those in references just cited.

Using the global superconvergence result given in Theorem 4.2, we can construct a recovery type a posteriori error estimator as follows.

Set

$$\eta^2 = \|y_h - \Pi_{2h} y_h\|^2_{0,\Omega} + \|p_h - \Pi_{2h} p_h\|^2_{0,\Omega} + \|u_h - \Pi_{2h} u_h\|^2_{0,\Omega}, \tag{5.1}$$

where Π_{2h} was defined in the last section. We have the following asymptotically exact a posteriori error estimate.

Theorem 5.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of the systems (2.6)–(2.8) and (2.15)–(2.17), respectively. Let V^h be the finite element space of order m defined in Section 2. Assume that $u, y, p \in H^{2m+2}(\Omega)$ and $\partial_t^{m+1} G(s,t) \in L^2(\Omega \times \Omega)$. Then we have*

$$\eta^2 = \|y - y_h\|^2_{0,\Omega} + \|p - p_h\|^2_{0,\Omega} + \|u - u_h\|^2_{0,\Omega} + O(h^{4(m+1)}) \tag{5.2}$$

and

$$\lim_{h\to 0} \frac{\eta^2}{\|y - y_h\|^2_{0,\Omega} + \|p - p_h\|^2_{0,\Omega} + \|u - u_h\|^2_{0,\Omega}} = 1 \tag{5.3}$$

if for some $\varepsilon > 0$,

$$\|y - y_h\|^2_{0,\Omega} + \|p - p_h\|^2_{0,\Omega} + \|u - u_h\|^2_{0,\Omega} \geqslant C h^{4(m+1)-\varepsilon},$$

where η is defined by (5.1).

Proof. From Theorem 4.2 we know that

$$\big|\|y_h - \Pi_{2h} y_h\|_{0,\Omega} - \|y - y_h\|_{0,\Omega}\big| \leqslant \|y - \Pi_{2h} y_h\|_{0,\Omega} \leqslant C h^{2(m+1)},$$

which leads to the expression

$$\|y_h - \Pi_{2h} y_h\|_{0,\Omega} = \|y - y_h\|_{0,\Omega} + O(h^{2(m+1)}). \tag{5.4}$$

Similarly, we have

$$\|p_h - \Pi_{2h} p_h\|_{0,\Omega} = \|p - p_h\|_{0,\Omega} + O(h^{2(m+1)}) \tag{5.5}$$

and

$$\|u_h - \Pi_{2h} u_h\|_{0,\Omega} = \|u - u_h\|_{0,\Omega} + O(h^{2(m+1)}). \tag{5.6}$$

Then (5.2) follows from (5.4)–(5.6), and (5.3) is the direct result of (5.2). □

R e m a r k 5.1. In this section we have used the combined recovery techniques of least square fitting (see, e.g., [30]) and post-processing by interpolation (see, e.g., [17]) for constructing a recovery type a posteriori error estimator. Noting that there is no requirement of uniform meshes in Theorem 5.1, which is often required in many general superconvergence analysis and recovery type a posteriori error estimates, the recovery type a posteriori error estimator defined in this section is more suitable for adaptive mesh refinement.

6. Discussion

In this paper we have analyzed the superconvergence of the finite element discretizations of a class of constrained optimal control problems governed by Fredholm-type integral equations. Based on the superconvergence analysis, global superconvergence and a recovery type a posteriori error estimates are provided. There are many important issues that remain to be studied. For example, the focus will be on the full discretization of (2.12) by using appropriate quadrature approximations for the Fredholm integral operator, and on numerical calculation based on applied optimal control problems of the forms (2.1)–(2.2). These include discussions on the discretization of problems governed by integral equations with weakly singular kernels (using suitable graded partitions or adaptive techniques), and optimal control problems with more complicated control constraints, as well as computational issues such as the integration of the numerical simulation into the mathematical programming algorithms that are used to solve the finite-dimensional optimization problems.

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Author's address: *N. Yan*, LSEC, Institute of System Sciences, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China, e-mail: `ynn@amss.ac.cn`.