Applications of Mathematics

Hossein Azari; Shu Hua Zhang

Global superconvergence of finite element methods for parabolic inverse problems

Applications of Mathematics, Vol. 54 (2009), No. 3, 285--294

Persistent URL: `http://dml.cz/dmlcz/140365`

GLOBAL SUPERCONVERGENCE OF FINITE ELEMENT METHODS FOR PARABOLIC INVERSE PROBLEMS*

Hossein Azari, Teheran, Shuhua Zhang, Tianjin

Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. In this article we transform a large class of parabolic inverse problems into a nonclassical parabolic equation whose coefficients consist of trace type functionals of the solution and its derivatives subject to some initial and boundary conditions. For this nonclassical problem, we study finite element methods and present an immediate analysis for global superconvergence for these problems, on basis of which we obtain a posteriori error estimators.

Keywords: inverse problem, global superconvergence, finite element method

MSC 2010: 35R30, 65M32, 76R50, 65M06

1. Introduction

In the present work we study the numerical solutions of some inverse problems, i.e., the determination of some unknown function $p(t)$ in a parabolic equation. The classical example is that one needs to find the temperature distribution $u(x,t)$ as well as the thermal coefficient $a(t)$ which simultaneously satisfy

$$u_t = a(t)u_{xx}, \quad 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1, \tag{1.2}$$

$$u(0,t) = f_1(t), \quad 0 \leqslant t \leqslant T, \tag{1.3}$$

$$u(1,t) = f_2(t), \quad 0 \leqslant t \leqslant T, \tag{1.4}$$

* This research was supported in part by the Shahid Beheshti University, the National Basic Research Program of China (2007CB814906), the National Natural Science Foundation of China (10471103 and 10771158), Social Science Foundation of the Ministry of Education of China (Numerical methods for convertible bonds, 06JA630047), Tianjin Natural Science Foundation (07JCYBJC14300).

and the over-specification

$$a(t)u_x(0,t) = g(t), \quad 0 \leqslant t \leqslant T. \tag{1.5}$$

The well-posedness of the problem is studied in [9]. Moreover, in [8] the numerical solution by the finite difference method is also discussed. However, we investigate the problem (1.1)–(1.5) from a rather different point of view. If we solve equation (1.5) for $a(t)$ and substitute it into the equation (1.1), then the equation (1.1) becomes

$$u_t = \frac{g(t)}{u_x(0,t)} u_{xx}$$

which is nonclassical since a functional of the derivative of the solution with respect to the variable x is involved in the equation. This example motivates us to consider the following general nonclassical problem:

$$u_t = a(x,t,u,u_x,u_x(x_0,t))u_{xx} + b(x,t,u,u_x,u_x(x_0,t)), \tag{1.6}$$

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1, \tag{1.7}$$

$$u(0,t) = f_1(t), \quad 0 \leqslant t \leqslant T, \tag{1.8}$$

$$u(1,t) = f_2(t), \quad 0 \leqslant t \leqslant T, \tag{1.9}$$

where $x_0 \in [0,1]$ is a fixed point.

The problem (1.6)–(1.9) represents a large class of parabolic inverse problems in which an unknown function $p(t)$ as well as the solution itself are to be determined. As another example of the motivation of our study, let us consider the problem of finding $(u(x,t), a(t))$ such that

$$u_t = u_{xx} + a(t)u_x, \quad 0 < x < 1, \ 0 < t < T, \tag{1.10}$$

subject to the initial-boundary conditions (1.2)–(1.4) and an additional condition

$$\int_0^1 u(x,t)\,\mathrm{d}x = g(t), \quad 0 \leqslant t \leqslant T.$$

If one differentiates with respect to the variable t in the above equation and uses the resulting equation, then one obtains

$$a(t) = \frac{g'(t) - [u_x(1,t) - u_x(0,t)]}{f_2(t) - f_1(t)}$$

provided that $f_1(t) \neq f_2(t)$. Consequently, the above inverse problem is of the form of our problem (1.6)–(1.9) with the specific equation

$$u_t = u_{xx} + \frac{g'(t) - [u_x(1,t) - u_x(0,t)]}{f_2(t) - f_1(t)}\, u_x.$$

When an unknown function $a(t)$ appears as the coefficient of a lower order term in a parabolic equation, it is not difficult to see that one is able to transform the problem into our nonclassical form under reasonable conditions. The reader can find many more such examples in [5] and [2].

Recently, the study of parabolic inverse problems has received much attention. For example, finite element methods and finite difference methods have been investigated in [10], [11], and [19], respectively. Here and throughout this paper "parabolic inverse problem" means that an unknown coefficient that is assumed to be a function of only the time variable and the solution of a parabolic equation subject to suitable initial-boundary conditions is to be determined. For the existence and uniqueness of solutions of such problems, the reader can refer to [1], [7], [15], [4], [16], [17], [8], [9], [3], [18], [14] etc.

However, the theory of numerical solution of these problems is far from satisfactory. For the nonclassical problem (1.6)–(1.9), the paper [5] established the global solvability as well as the continuous dependence of the solution upon the data. In the present paper we continue the work of [5], [6] and investigate the numerical calculation for the solution of our problem. We will introduce a new function and transform the problem into a variational form. The classical Galerkin procedure is then applied to our problem, and the global superconvergence of finite element methods is derived in this article. In addition, as a by-product of the superconvergence, a posteriori error estimates are obtained.

2. Superclose estimates

For the sake of simplicity, we assume that $f_1(t) = f_2(t) = 0$ and $x_0 = 0$ in problem (1.6)–(1.9). We shall use the following notation throughout the paper.

Set $I = (0,1)$. For $u, v \in L^2(I)$ we define

$$\langle u, v\rangle = \int_0^1 u(x)v(x)\,\mathrm{d}x,$$

$$\|u\|_0^2 = \langle u, u\rangle = \int_0^1 u^2(x)\,\mathrm{d}x,$$

$$\tilde{H}_0^1(I) = \Big\{ u \in H^1(I),\ \int_0^1 u(x)\,\mathrm{d}x = 0 \Big\},$$

and

$$|u|_1 = \|u\|_{H_0^1(I)} = \|u_x\|_0.$$

Evidently, $\tilde{H}_0^1(I)$ is a Banach space with the $H_0^1(I)$-norm. To transform the problem (1.6)–(1.9) into a variational form, we introduce a new function

$$v(x,t) = u_x(x,t), \quad (x,t) \in \overline{Q}_T,$$

where $u(x,t)$ is the solution of problem (1.6)–(1.9) and $\overline{Q}_T = I \times [0,T]$. Then $v(x,t) \in \tilde{H}_0^1(I)$ and

$$u(x,t) = \int_0^x v(y,t)\,\mathrm{d}y, \quad (x,t) \in \overline{Q}_T.$$

Furthermore, using (1.6) we see that $v(x,t)$ satisfies

$$\left(\int_0^x v(y,t)\,\mathrm{d}y\right)_t = av_x + b \quad \text{in } Q_T, \tag{2.1}$$

and

$$u_t(0,t) = u_t(1,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.2}$$

Let $w \in \tilde{H}_0^1(I)$ be arbitrary. We multiply the equation (2.1) by $w_x(x)$ and integrate it over $(0,1)$ with respect to x. Integrating by parts the term on the left-hand side of the equation (2.1) and using the boundary condition (2.2), we obtain

$$\langle v_t, w\rangle + \langle av_x + b, w_x\rangle = 0, \qquad t \in (0,T], \tag{2.3}$$

$$\langle v, w\rangle = \langle u_0', w\rangle, \quad t = 0, \tag{2.4}$$

where

$$a = a\left(x,t,\int_0^x v(y,t)\,\mathrm{d}y, v(x,t), v(0,t)\right), \quad (x,t) \in \overline{Q}_T, \tag{2.5}$$

$$b = b\left(x,t,\int_0^x v(y,t)\,\mathrm{d}y, v(x,t), v(0,t)\right), \quad (x,t) \in \overline{Q}_T. \tag{2.6}$$

It is easy to see that if $v(x,t)$ is a variational solution of (2.3)–(2.6) and sufficiently smooth, then $u(x,t) = \int_0^x v(y,t)\,\mathrm{d}y$ is a solution of the problem (1.6)–(1.9). Hence if one can obtain an approximate solution for the variational problem (2.3)–(2.6), one also has an approximate solution for the original problem (1.6)–(1.9). Therefore, we shall concentrate on the approximation of the solution of the variational problem (2.3)–(2.6) in the rest of this paper.

For convenience, the following conditions are assumed in the sequel:

(H1) The functions $a(x,t,u,p,q)$ and $b(x,t,u,p,q)$ are twice differentiable with respect to all their arguments.

(H2) (i) $a(x,t,u,p,q) \geqslant a_0 > 0$;

(ii) $|\nabla_{u,p,q} a(\ldots)| + |\nabla_{u,p,q} b(\ldots)| \leqslant A_0$ for $(x,t,u,p,q) \in \overline{Q}_T \times \mathbb{R}^3$, where $\nabla_{u,p,q} a = \bigl(\partial a/\partial u, \partial a/\partial p, \partial a/\partial q\bigr)$.

Next, we will approximate the problem (2.3)–(2.4) by the finite element method. To this purpose, we let $T_h\colon\ 0 \leqslant x_0 < x_1 < \ldots < x_N \leqslant 1$ be a division of the interval $[0,1]$, and $e_i := [x_i, x_{i+1}]$ $(i = 0, 1, \ldots, N-1)$ represents the $(i+1)$st element.

Furthermore, let $V_h \subset \tilde{H}_0^1(I)$ consist of piecewise linear polynomials with respect to the partition T_h. Thus, the semi-discrete finite element scheme of (2.3)–(2.4) is defined as follows: Find $U(x,t) \in V_h$ such that

$$\langle U_t, W\rangle + \langle AU_x + B, W_x\rangle = 0, \quad t \in (0,T],\ W \in V_h, \tag{2.7}$$

$$U(x,0) = i_h u_0'(x), \tag{2.8}$$

where i_h is the piecewise linear interpolation operator, and

$$A = a\biggl(x, t, \int_0^x U(y,t)\,\mathrm{d}y, U(x,t), U(0,t)\biggr), \quad (x,t) \in \overline{Q}_T, \tag{2.9}$$

$$B = b\biggl(x, t, \int_0^x U(y,t)\,\mathrm{d}y, U(x,t), U(0,t)\biggr), \quad (x,t) \in \overline{Q}_T. \tag{2.10}$$

From (2.3) and (2.7) we can derive the following error equation:

$$\langle v_t - U_t, W\rangle + \langle av_x - AU_x + b - B, W_x\rangle = 0, \quad W \in V_h. \tag{2.11}$$

First of all, from [12] we recall the following lemma.

Lemma 2.1. *For any $W \in V_h$ we have*

$$|\langle A(v - i_h v)_x, W_x\rangle| \leqslant Ch^2 \|v\|_2 |W|_1,$$

where A is given by (2.9).

Theorem 2.2. *Under the conditions* (H1) *and* (H2) *we have the supercloseness estimate*

$$\|U - i_h v\|_0 + \|U - i_h v\|_{L^2(0,T;H^1(I))} \leqslant Ch^2.$$

P r o o f. Let

$$\theta(x,t) = U(x,t) - i_h v(x,t).$$

Then from (2.11) we find for any $W \in V_h$ that

$$\begin{aligned}&\langle \theta_t, W\rangle + \langle A\theta_x + B, W_x\rangle \\ &\quad = \langle (a-A)v_x + b, W_x\rangle + \langle v_t - i_h v_t, W\rangle + \langle A(v - i_h v)_x, W_x\rangle\end{aligned}$$

or

$$\begin{aligned}&\langle \theta_t, W\rangle + \langle A\theta_x, W_x\rangle \\ &\quad = \langle (a-A)v_x + b - B, W_x\rangle + \langle v_t - i_h v_t, W\rangle + \langle A(v - i_h v)_x, W_x\rangle \\ &\quad := I + II, \quad W \in V_h.\end{aligned} \tag{2.12}$$

From Lemma 2.1 we obtain

$$\begin{aligned}|II| &= |\langle v_t - i_h v_t, W\rangle + \langle A(v - i_h v)_x, W_x\rangle| \\ &\leqslant Ch^2(\|v_t\|_2 + \|v\|_2)|W|_1.\end{aligned} \tag{2.13}$$

Since

$$\begin{aligned}&(a-A)v_x + b - B \\ &\quad = \Big[a\Big(x,t,\int_0^t v(y,t)\,\mathrm{d}y, v(x,t), v(0,t)\Big) \\ &\qquad - a\Big(x,t,\int_0^t U(y,t)\,\mathrm{d}y, U(x,t), U(0,t)\Big)\Big]v_x \\ &\qquad + b\Big(x,t,\int_0^t v(y,t)\,\mathrm{d}y, v(x,t), v(0,t)\Big) \\ &\qquad - b\Big(x,t,\int_0^t U(y,t)\,\mathrm{d}y, U(x,t), U(0,t)\Big) \\ &\quad = b_1 Z + c_1 Z(0,t) + d_1\int_0^x Z(y,t)\,\mathrm{d}y,\end{aligned}$$

where

$$
\begin{aligned}
b_1 &= \left[\int_0^1 a_p(x,t,\alpha(\tau),\beta(\tau),\gamma(\tau))\,\mathrm{d}\tau\right] v_x + \int_0^1 b_p(x,t,\alpha(\tau),\beta(\tau),\gamma(\tau))\,\mathrm{d}\tau,\\
c_1 &= \left[\int_0^1 a_q(x,t,\alpha(\tau),\beta(\tau),\gamma(\tau))\,\mathrm{d}\tau\right] v_x + \int_0^1 b_q(x,t,\alpha(\tau),\beta(\tau),\gamma(\tau))\,\mathrm{d}\tau,\\
d_1 &= \left[\int_0^1 a_u(x,t,\alpha(\tau),\beta(\tau),\gamma(\tau))\,\mathrm{d}\tau\right] v_x + \int_0^1 b_u(x,t,\alpha(\tau),\beta(\tau),\gamma(\tau))\,\mathrm{d}\tau,
\end{aligned}
$$

$$
\alpha(\tau) = \tau\left(\int_0^\tau U(y,t)\,\mathrm{d}y\right) + (1-\tau)\left(\int_0^\tau v(y,t)\,\mathrm{d}y\right),
$$

$$
\beta(\tau) = \tau U + (1-\tau)v,
$$

$$
\gamma(\tau) = \tau U(0,t) + (1-\tau)v(0,t),
$$

$$
Z(x,t) = v(x,t) - U(x,t),
$$

we have

$$
\begin{aligned}
|I| &= |\langle (a-A)v_x + b - B, W_x\rangle|\\
&= \left|\left\langle b_1 Z + c_1 Z(0,t) + d_1\int_0^x Z(y,t)\,\mathrm{d}y, W_x\right\rangle\right|\\
&\leqslant \left|\left\langle b_1(v-i_hv) + c_1(v-i_hv)(0,t) + d_1\int_0^x (v-i_hv)(y,t)\,\mathrm{d}y, W_x\right\rangle\right|\\
&\quad + \left|\left\langle b_1\theta + c_1\theta(0,t) + d_1\int_0^x \theta(y,t)\,\mathrm{d}y, W_x\right\rangle\right|,
\end{aligned}
$$

which, together with Lemma 2.1, leads to

$$
\begin{aligned}
(2.14)\qquad |I| &\leqslant Ch^2\|v\|_2|W|_1 + \left|\left\langle b_1\theta + c_1\theta(0,t) + d_1\int_0^x \theta(y,t)\,\mathrm{d}y, W_x\right\rangle\right|\\
&\leqslant Ch^2\|v\|_2|W|_1 + C(\|\theta\|_0 + |\theta(0,t)|)|W|_1.
\end{aligned}
$$

Thus, from (2.12), (2.13) and (2.14) we have

$$
(2.15)\quad \langle\theta_t, W\rangle + \langle A\theta_x, W_x\rangle \leqslant Ch^2(\|v_t\|_2 + \|v\|_2)|W|_1 + C(\|\theta\|_0 + |\theta(0,t)|)|W|_1.
$$

It follows from taking $W = \theta$ in (2.15) and applying the inequality (see Cannon and Yin [6])

$$
|\theta(0,t)| \leqslant \varepsilon|\theta|_1 + C\|\theta\|_0
$$

and the ε-inequality that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_0^2 + \langle A\theta_x, \theta_x\rangle \leqslant Ch^4(\|v_t\|_2 + \|v\|_2)^2 + \varepsilon|\theta|_1^2 + C\|\theta\|_0^2,
$$

or by condition (H2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_0^2 + |\theta|_1^2 \leqslant Ch^4(\|v_t\|_2 + \|v\|_2)^2 + C\|\theta\|_0^2.$$

Since $\|\theta(0)\|_0 = 0$, we obtain by integrating the above inequality from 0 to t that

$$\|\theta\|_0^2 + \int_0^t |\theta|_1^2 \,\mathrm{d}\tau \leqslant Ch^4 \int_0^t (\|v_t\|_2^2 + \|v\|_2^2)\,\mathrm{d}\tau + C\int_0^t \|\theta\|_0^2\,\mathrm{d}\tau,$$

which, together with the Gronwall lemma, implies

$$\|\theta\|_0^2 + \int_0^t |\theta|_1^2 \,\mathrm{d}\tau \leqslant Ch^4 \int_0^t (\|v_t\|_2^2 + \|v\|_2^2)\,\mathrm{d}\tau.$$

This completes the proof of the theorem. □

R e m a r k 2.3. From Theorem 2.2 we know that $\|U - i_h v\|_{L^2(0,T;H^1(I))}$ converges at a rate of $O(h^2)$, which is referred to as the finite element solution being superclose to the nodal interpolant.

3. Global superconvergence estimates

By means of the interpolation postprocessing technique [12], on the basis of Theorem 2.2 we can obtain the following global superconvergence estimate.

Theorem 3.1. *Assume that $v \in H^3(I)$. Then we have under the conditions of Theorem* 2.2 *that*

$$\|I_{2h}^2 U - v\|_{L^2(0,T;H^1(I))} \leqslant Ch^2,$$

where I_{2h}^2 is a piecewise polynomial interpolation operator of degree at most 2 *associated with the mesh T_{2h} of mesh size $2h$, from which T_h is assumed to be gained by subdividing each element of T_{2h} into two equal elements so that the number of elements M for T_h is an even number.*

As a by-product of Theorem 3.1 we can immediately obtain the following result [12].

Theorem 3.2. *Under the assumptions of Theorem* 3.1 *we have*

$$\|v-U\|_{L^2(0,T;H^1(I))}=\|U-I_{2h}^2U\|_{L^2(0,T;H^1(I))}+O(h^2).$$

In addition, if there exist positive constants C_0 *and* $\varepsilon\in(0,1)$ *such that*

$$\|v-U\|_{L^2(0,T;H^1(I))}\geqslant C_0h^{2-\varepsilon},$$

then

$$\lim_{h\to 0}\frac{\|v-U\|_{L^2(0,T;H^1(I))}}{\|I_{2h}^2U-U\|_{L^2(0,T;H^1(I))}}=1.$$

Acknowledgements. The authors would like to thank anonymous referees for their valuable suggestions which significantly improved the presentation of this paper.

References

[1] *C. Alvarez, C. Conca, L. Friz, O. Kavian, J. H. Ortega*: Identification of immersed obstacles via boundary measurements. Inverse Probl. *21* (2005), 1531–1552.
[2] *H. Azari, W. Allegretto, Y. Lin, S. Zhang*: Numerical procedures for recovering a time dependent coefficient in a parabolic differential equation. Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms *11* (2004), 181–199.
[3] *H. Azari, Ch. Li, Y. Nie, S. Zhang*: Determination of an unknown coefficient in a parabolic inverse problem. Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. *11* (2004), 665–674.
[4] *H. Azari, S. Zhang*: Identifying a time dependent unknown coefficient in a parabolic inverse problem. Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms. Suppl. *12b* (2005), 32–43.
[5] *J. R. Cannon, H.-M. Yin*: A class of nonlinear non-classical parabolic equations. J. Differ. Equations *79* (1989), 266–288.
[6] *J. R. Cannon, H.-M. Yin*: Numerical solutions of some parabolic inverse problems. Numer. Methods Partial Differ. Equations *6* (1990), 177–191.
[7] *B. Canuto, O. Kavian*: Determining coefficients in a class of heat equations via boundary measurements. SIAM J. Math. Anal. *32* (2001), 963–986 (electronic).
[8] *J. Douglas, Jr., B. F. Jones, Jr.*: The determination of a coefficient in a parabolic differential equation. II. Numerical approximation. J. Math. Mech. *11* (1962), 919–926.
[9] *B. F. Jones, Jr.*: The determination of a coefficient in a parabolic differential equation. I. Existence and uniqueness. J. Math. Mech. *11* (1962), 907–918.
[10] *Y. L. Keung, J. Zou*: Numerical identification of parameters in parabolic systems. Inverse Probl. *14* (1998), 83–100.
[11] *R. A. Khachfe, Y. Jarny*: Numerical solution of 2-D nonlinear inverse heat conduction problems using finite-element techniques. Numer. Heat Transfer, Part B: Fundamentals *37* (2000), 45–67.
[12] *Q. Lin, N. Yan*: The Construction and Analysis of High Efficiency Finite Element Methods. Hebei University Publishers, Baoding, 1996. (In Chinese.)
[13] *Q. Lin, Q. Zhu*: The Preprocessing and Postprocessing for the Finite Element Method. Shanghai Scientific & Technical Publishers, Shanghai, 1994. (In Chinese.)

[14] *A. I. Prilepko, D. G. Orlovskii*: Determination of the parameter of an evolution equation and inverse problems of mathematical physics I. Differ. Equations *21* (1985), 96–104.
[15] *A. G. Ramm*: An inverse problem for the heat equation. J. Math. Anal. Appl. *264* (2001), 691–697.
[16] *A. G. Ramm*: A non-overdetermined inverse problem of finding the potential from the spectral function. Int. J. Differ. Equ. Appl. *3* (2001), 15–29.
[17] *A. G. Ramm*: Inverse problems for parabolic equations applications. Aust. J. Math. Anal. Appl. *2* (2005, Art. 10 (electronic)).
[18] *A. G. Ramm, S. V. Koshkin*: An inverse problem for an abstract evolution equation. Appl. Anal. *79* (2001), 475–482.
[19] *X. T. Xiong, C. L. Fu, H. F. Li*: Central difference schemes in time and error estimate on a non-standard inverse heat conduction problem. Appl. Math. Comput. *157* (2004), 77–91.

Authors' addresses: *H. Azari*, Department of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran, e-mail: `h_azari@sbu.ac.ir`; *S. Zhang* (corresponding author), Department of Mathematics, Tianjin University of Finance and Economics, Tianjin, P. R. China, e-mail: `szhang@tjufe.edu.cn`.