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EXISTENCE RESULTS FOR NON-LINEAR SINGULAR INTEGRAL EQUATIONS WITH HILBERT KERNEL IN BANACH SPACES

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Abstract. A class of non-linear singular integral equations with Hilbert kernel and a related class of quasi-linear singular integro-differential equations are investigated by applying Schauder's fixed point theorem in Banach spaces.

Keywords: non-linear singular integral equation, Schauder's fixed point theorem, Banach space

MSC 2010: 45F15, 45G10

1. Introduction

During the last years the theory of non-linear singular integral equations has successfully developed. Many problems of mathematical physics like elasticity, plasticity, viscoelasticity, aerodynamics and fluid mechanics reduced to the solution of non-linear singular integral equations. Non-linear singular integral and integro-differential equations and related Riemann-Hilbert problems have been treated by many authors, see Pogorzelski [16], Guseinov and Mukhtarov [5], Wolfersdorf [17], [18], Junghanns and others [7], [8], Ladopoulos [9]–[12] and many others. The theory of non-linear singular integral and integro-differential equations seems to be a particularly complicated form of the non-linear integral equations. Schauder's fixed point method is one of the basic tools for investigating the existence results of many classes of singular integral and integro-differential equations [1], [2], [4], [6], [7], [17], [18]. We refer to Ladopoulos [9]–[12] for many applications of singular integral and integro-differential equations in engineering and science. In the present paper a class of integral equations with Hilbert kernel and a related class of quasi-linear integro-differential equations are investigated by means of Schauder's fixed point theorem in the Sobolev space W_p^1.

We shall consider the non-linear singular integral equation

$$\varphi(s) + \frac{1}{2\pi}\int_{-\pi}^{\pi} m(s,\sigma,\varphi(\sigma)) \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma = f(s) \tag{1.1}$$

with an additional condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(s)\,\mathrm{d}s = \varepsilon, \tag{1.2}$$

where ε is a positive constant and the functions $\varphi(s)$, $f(s)$, $m(s,\sigma,\varphi(\sigma))$ are continuous positive functions defined on the regions

$$D = \{s\colon\ s \in [-\pi,\pi]\} \quad \text{and} \quad \tilde{D} = \{(s,\sigma,\varphi(\sigma))\colon\ s,\sigma \in [-\pi,\pi],\ -\infty < \varphi < \infty\},$$

respectively.

Throughout the paper $L_p[-\pi,\pi]$ $(1 \leqslant p < \infty)$ means the Banach space of all measurable functions u defined on $[-\pi,\pi]$ with the norm

$$\|u\|_{L_p} = \left[\int_{-\pi}^{\pi} |u(t)|^p\,\mathrm{d}t\right]^{p^{-1}},$$

and $W_p^1[-\pi,\pi]$, $1 \leqslant p < \infty$, means the Sobolev space of all functions $u \in L_p[-\pi,\pi]$ with $u' \in L_p[-\pi,\pi]$.

We will seek the solution of equation (1.1) in the Sobolev space W_p^1.

2. Reduction to fixed point equation

In this section we seek the solution of equation (1.1) and an estimation of the kernel of a fixed point equation. Differentiating both sides of equation (1.1), we have [14]

$$\varphi'(s) + \frac{1}{2\pi}\int_{-\pi}^{\pi} m_\varphi(s,\sigma,\varphi(\sigma))\varphi'(\sigma) \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma = F(s), \tag{2.1}$$

where

$$F(s) = f'(s) - \frac{1}{2\pi}\int_{-\pi}^{\pi} (m_s(s,\sigma,\varphi(\sigma)) + m_\sigma(s,\sigma,\varphi(\sigma))) \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma.$$

Equation (2.1) can be written as the quasi-linear integro-differential equation

$$\varphi'(s) + \frac{1}{2\pi}\int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma))\varphi'(\sigma) \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma = F(s) \tag{2.2}$$

with the additional condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{\varphi}'(s)\,\mathrm{d}s=0,\quad \varphi(0)=\Lambda, \tag{2.3}$$

which is equivalent to equation (1.2), where Λ is a positive constant, $M(s,\sigma,\varphi(\sigma))=m_{\varphi}(s,\sigma,\varphi(\sigma))$ and $M(s,\sigma,\varphi(\sigma))$ satisfy the Hölder-Lipschitz condition as follows:

$$|M(s_2,\sigma_2,\varphi_2(\sigma_2))-M(s_1,\sigma_1,\varphi_1(\sigma_1))|\leqslant\alpha[|s_2-s_1|^{\delta}+|\sigma_2-\sigma_1|^{\delta}+|\varphi_2-\varphi_1|], \tag{2.4}$$

where α is a positive constant and $0<\delta<1$.

From the theory of singular integral equations [3], the solution of equation (2.2) is the solution of the corresponding boundary value problem

$$u(s)+v(s)=c(s) \tag{2.5}$$

with the additional condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}v(s)\,\mathrm{d}s=0,\quad u(s)=\varphi'(s). \tag{2.6}$$

Since the boundary value problem (2.5) has index equal to zero [3], its solution has the form

$$F(z)=\mathrm{e}^{\mathrm{i}\nu(z)}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}M(s,\sigma,\varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)}\frac{\mathrm{e}^{\mathrm{i}\sigma}+z}{\mathrm{e}^{\mathrm{i}\sigma}-z}\,\mathrm{d}\sigma+\mathrm{i}\beta_0\right], \tag{2.7}$$

where $\nu(z)=w(x,y)+\mathrm{i}w_1(x,y)$ and β_0 is an arbitrary constant.

Setting

$$\mathrm{e}^{\mathrm{i}\nu(z)}=\xi+\mathrm{i}\eta,\quad \psi(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}M(s,\sigma,\varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)}\frac{\mathrm{e}^{\mathrm{i}\sigma}+z}{\mathrm{e}^{\mathrm{i}\sigma}-z}\,\mathrm{d}\sigma \tag{2.8}$$

and taking into account the properties of Schwarz kernel, we have

$$\begin{aligned}\operatorname{Re}\psi(t)&=M(s,s,\varphi(s))c(s)\mathrm{e}^{w_1(s)},\\ \operatorname{Im}\psi(t)&=-\frac{1}{2\pi}\int_{-\pi}^{\pi}M(s,\sigma,\varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)}\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma.\end{aligned} \tag{2.9}$$

Substituting (2.8) and (2.9) into equation (2.7), we obtain

$$\begin{aligned} F(z) &= (-\mathrm{i})(\xi + \mathrm{i}\eta)[\psi(z) + \mathrm{i}\beta_0] \\ &= (-\mathrm{i}\xi + \eta)\Big[M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ &\quad - \frac{\mathrm{i}}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma + \mathrm{i}\beta_0 \Big] \\ &= \Big[\xi(s)\beta_0 + \eta(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ &\quad - \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma \Big] \\ &\quad + \mathrm{i}\Big[\eta(s)\beta_0 - \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ &\quad - \frac{\eta(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma \Big]. \end{aligned}$$

Now we have

$$\text{(2.10)} \qquad \begin{cases} u(s) = & \xi(s)\beta_0 + \eta(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ & -\dfrac{\xi(s)}{2\pi} \displaystyle\int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma, \\ v(s) = & \eta(s)\beta_0 - \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ & -\dfrac{\eta(s)}{2\pi} \displaystyle\int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma. \end{cases}$$

In view of condition (2.6), we have

$$\begin{aligned} \int_{-\pi}^{\pi} \Big[&\eta(s)\beta_0 - \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ &- \frac{\eta(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma \Big] = 0. \end{aligned}$$

Hence

$$\text{(2.11)} \qquad \begin{aligned} &\int_{-\pi}^{\pi} \eta(s)\beta_0\,\mathrm{d}s - \int_{-\pi}^{\pi} \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)}\,\mathrm{d}s \\ &\quad - \int_{-\pi}^{\pi} \eta(s)\,\mathrm{d}s \int_{-\pi}^{\pi} \frac{1}{2\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\,\mathrm{d}\sigma = 0, \end{aligned}$$

and by changing the order of integration in the last term of equation (2.11), we obtain

$$(2.12)\quad \beta_0 \int_{-\pi}^{\pi} \eta(\sigma)\,\mathrm{d}\sigma - \int_{-\pi}^{\pi} \xi(\sigma) M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)}\,\mathrm{d}\sigma + \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)}\,\mathrm{d}\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta(s) \cot\frac{s-\sigma}{2}\,\mathrm{d}s = 0.$$

Now by using the Hilbert formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} v(\sigma) \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma = u(s) - u_0 \qquad \text{where } u_0 = u(0,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\sigma)\,\mathrm{d}\sigma$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \eta(s) \cot\frac{s-\sigma}{2}\,\mathrm{d}s = \xi(\sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\sigma)\,\mathrm{d}\sigma.$$

Hence equation (2.11) takes on the form

$$\beta_0 \int_{-\pi}^{\pi} \eta(\sigma)\,\mathrm{d}\sigma - \int_{-\pi}^{\pi} \xi(\sigma) M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)}\,\mathrm{d}\sigma + \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)}\,\mathrm{d}\sigma \left[\xi(\sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\sigma)\,\mathrm{d}\sigma\right] = 0.$$

Therefore

$$\beta_0 \int_{-\pi}^{\pi} \eta(\sigma)\,\mathrm{d}\sigma - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\sigma)\,\mathrm{d}\sigma \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)}\,\mathrm{d}\sigma = 0.$$

In the case of $\beta_0 \int_{-\pi}^{\pi} \eta(\sigma)\,\mathrm{d}\sigma \neq 0$ the corresponding homogeneous equation has no solution. In this case we determine the constant β_0 and consequently the nonhomogeneous equation (2.2) has the unique solution (2.10).

Hence

$$u(s) = \xi(s)\beta_0 + \eta(s) M(s,s,\varphi(s)) c(s) \mathrm{e}^{w_1(s)} - \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)} \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma.$$

From (2.6) we have

$$(2.13)\quad \varphi'(s) = \xi(s)\beta_0 + \eta(s) M(s,s,\varphi(s)) c(s) \mathrm{e}^{w_1(s)} - \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)} \cot\frac{\sigma - s}{2}\,\mathrm{d}\sigma.$$

Taking into account the additional condition (2.3), we obtain an equivalent fixed point equation

$$S\varphi = \varphi, \tag{2.14}$$

where the operator S is defined for any 2π-periodic continuous function $\varphi(s)$ by

$$(S\varphi)(s) = \Lambda + \int_0^s k(s,\sigma,\varphi(\sigma))\,\mathrm{d}\sigma, \tag{2.15}$$

with the kernel function

$$\begin{aligned} k(s,\sigma,\varphi(\sigma)) &= \xi(s)\beta_0 + \eta(s)M(s,s,\varphi(s))c(s)\mathrm{e}^{w_1(s)} \\ &\quad - \frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)}\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma. \end{aligned} \tag{2.16}$$

Lemma 2.1. *The kernel function $k(s,\sigma,\varphi(\sigma))$ given by* (2.16) *is bounded in the space $L_p[-\pi,\pi]$, $p>1$, under the conditions*

$$\begin{cases} |\xi(s_2)-\xi(s_1)| \leqslant \lambda_1|s_2-s_1|^\delta, \\ |\eta(s_2)-\eta(s_1)| \leqslant \lambda_2|s_2-s_1|^\delta, \\ |c(s_2)-c(s_1)| \leqslant \lambda_3|s_2-s_1|^\delta, \ \textit{and} \\ |w_1(s_2)-w_1(s_1)| \leqslant \lambda_4|s_2-s_1|^\delta, \end{cases} \tag{2.17}$$

where λ_i $(i=1,2,3,4)$ are positive constants.

P r o o f. We shall estimate the kernel $k(s,\sigma,\varphi(\sigma))$ for $p>1$ as follows:

$$\|k(s,\sigma,\varphi(\sigma))\|_p \leqslant \|\xi(s)\beta_0\|_p + \|N_1(s)\|_p + \|N_2(s)\|_p \tag{2.18}$$

where

$$N_1(s) = \eta(s)M(s,s,\varphi(s))c(s)\mathrm{e}^{w_1(s)} \tag{2.19}$$

and

$$N_2(s) = \frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)}\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma. \tag{2.20}$$

Now we estimate $\|\xi(s)\beta_0\|_p$.

By using conditions (2.17), we have

$$|\xi(s)| \leqslant n_1 \quad \text{with} \quad n_1 = 2\pi\lambda_1 + |\xi(0)|. \tag{2.21}$$

Therefore

$$\|\xi(s)\beta_0\|_p \leqslant Q_1 \equiv \text{const}, \tag{2.22}$$

where

$$Q_1 = |\beta_0| n_1 (2\pi)^{p^{-1}}.$$

By using conditions (2.4), (2.17) and equality (2.19), from [7], [15] we can estimate $\|N_1(s)\|_p$ as follows:

$$\|N_1(s)\|_p \leqslant \|\eta(s)\|_{p_1} \|c(s)\|_{p_2} \|e^{w_1(s)}\|_{p_3} \|M(s,s,\varphi(s))\|_{p_4}, \tag{2.23}$$

where $p_1^{-1} + p_2^{-1} + p_3^{-1} + p_4^{-1} = p^{-1}$.

To estimate the norm given in (2.23), we proceed in several steps:

(a) Estimation of $\|\eta(s)\|_{p_1}$.

As in (2.21) it is easy to see that

$$|\eta(s)| \leqslant n_2 \quad \text{with} \quad n_2 = 2\pi\lambda_2 + |\eta(0)|. \tag{2.24}$$

Therefore

$$\|\eta(s)\|_{p_1} < \gamma_1, \tag{2.25}$$

where $\gamma_1 = (2\pi)^{p_1^{-1}} n_2$.

Similarly, from (2.24) it is easy to obtain an estimate of $\|c(s)\|_{p_2}$ as follows:

$$|c(s)| \leqslant n_3 \quad \text{with} \quad n_3 = 2\pi\lambda_3 + |c(0)|, \tag{2.26}$$

therefore

$$\|c(s)\|_{p_2} < \gamma_2, \tag{2.27}$$

where $\gamma_2 = (2\pi)^{p_2^{-1}} n_3$.

Also, an estimate of $\|M(s,s,\varphi(s))\|_{p_4}$ follows:

$$|M(s,s,\varphi(s))| \leqslant n_4 \quad \text{with} \quad n_4 = \alpha R + B, \quad B = \max_{s,\sigma\in[-\pi,\pi]} |M(s,s,\varphi(0))|, \tag{2.28}$$

therefore

$$\|M(s,s,\varphi(s))\|_{p_4} < \gamma_3, \tag{2.29}$$

where $\gamma_3 = (2\pi)^{p_4^{-1}} n_4$.

(b) Estimation of $\|e^{w_1(s)}\|_{p_3}$.

It follows from [5] that

$$\|e^{w_1(s)}\|_{p_3} \leqslant \left[1 + \left(\int_{-\pi}^{\pi} (n_5)^{p_3}\,ds\right)^{p_3^{-1}}\right] \exp\left(\left(\int_{-\pi}^{\pi} (n_5)^{p_3}\,ds\right)^{1/p_3}\right) = (1+\gamma_4)e^{\gamma_4},$$

where

$$|w_1(s)| \leqslant n_5, \quad n_5 = (2\pi)\lambda_5 + |w_1(0)|, \quad \gamma_4 = (2\pi)^{p_3^{-1}} n_5. \tag{2.30}$$

Therefore

$$\|e^{w_1(s)}\|_{p_3} \leqslant (1+\gamma_4)e^{\gamma_4}. \tag{2.31}$$

Substituting (2.25), (2.27), (2.29), and (2.31) into (2.23), we obtain

$$\|N_1(s)\|_p \leqslant Q_2, \tag{2.32}$$

where $Q_2 = \gamma_1\gamma_2\gamma_3(1+\gamma_4)e^{\gamma_4}$.

Similarly, we can estimate $\|N_2(s)\|_p$ as follows:

$$\begin{aligned} \|N_2(s)\|_p &= \left\|\frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma))c(\sigma)e^{w_1(\sigma)}\cot\frac{\sigma - s}{2}\,d\sigma\right\| \\ &\leqslant \frac{1}{2\pi}\|\xi(s)\|_{q_1}\left\|\int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma))c(\sigma)e^{w_1(\sigma)}\cot\frac{\sigma - s}{2}\,d\sigma\right\|_{q_2}. \end{aligned} \tag{2.33}$$

Using the well-known inequality, [5],

$$\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\varphi(s)}{s - x}\right\| \leqslant \varrho\|\varphi(s)\|, \tag{2.34}$$

where ϱ is a positive constant, we conclude that

$$\|N_2(s)\|_p \leqslant \varrho\|\xi(s)\|_{q_1}\|M(s,\sigma,\varphi(\sigma))c(\sigma)e^{w_1(\sigma)}\|_{q_2}, \tag{2.35}$$

where $p^{-1} = q_1^{-1} + q_2^{-1}$.

Hence,

$$\|\xi(s)\|_{q_1} \leqslant n_1(2\pi)^{q_1^{-1}}. \tag{2.36}$$

Also,

$$\|c(\sigma)e^{w_1(\sigma)}M(s,\sigma,\varphi(\sigma))\|_{q_2} \leqslant \|c(\sigma)\|_{k_1}\|e^{w_1(\sigma)}\|_{k_2}\|M(s,\sigma,\varphi(\sigma))\|_{k_3}, \tag{2.37}$$

where $q_2^{-1} = k_1^{-1} + k_2^{-1} + k_3^{-1}$, and so

$$\|c(\sigma)\|_{k_1} \leqslant (2\pi)^{k_1^{-1}} n_3, \tag{2.38}$$

$$\|e^{w_1(\sigma)}\|_{k_2} \leqslant (1 + n_5 (2\pi)^{k_2^{-1}}) e^{(2\pi)^{k_2^{-1}} n_5} \tag{2.39}$$

and

$$\|M(s, \sigma, \varphi(\sigma))\|_{k_3} \leqslant (2\pi)^{k_3^{-1}} n_4. \tag{2.40}$$

Substituting (2.38), (2.39), and (2.40) into (2.37), we obtain

$$\|c(\sigma) e^{w_1(\sigma)} M(s, \sigma, \varphi(\sigma))\|_{q_2} \leqslant n_3 n_4 (2\pi)^{k_1^{-1} + k_3^{-1}} (1 + n_5 (2\pi)^{k_2^{-1}}) e^{n_5 (2\pi)^{k_2^{-1}}}; \tag{2.41}$$

using (2.36) and (2.41) in (2.35), we have

$$\|N_2(s)\|_p \leqslant Q_3, \tag{2.42}$$

where

$$Q_3 = \varrho n_1 n_3 n_4 (2\pi)^{q_1^{-1} + k_1^{-1} + k_3^{-1}} (1 + n_5 (2\pi)^{k_2^{-1}}) e^{n_5 (2\pi)^{k_2^{-1}}}.$$

Substituting (2.22), (2.32), and (2.42) into (2.18), we conclude that

$$\|k(s, \sigma, \varphi(\sigma))\| \leqslant Q, \tag{2.43}$$

where

$$Q = Q_1 + Q_2 + Q_3 \equiv \text{const}.$$

□

3. Existence theorem

For non-negative constants R and μ we define the following convex and compact set:

$$A^{\delta}_{R,\mu} = \{\varphi \in W^1_p[-\pi, \pi],\ |\varphi| \leqslant R,\ |\varphi(s_2) - \varphi(s_1)| \leqslant \mu |s_2 - s_1|^{\delta},\ s_1, s_2 \in [-\pi, \pi]\}. \tag{3.1}$$

We are going to prove some assertions about the set $A^{\delta}_{R,\mu}$ and its image $S(A^{\delta}_{R,\mu})$. It is easy to see that the set $A^{\delta}_{R,\mu}$ is a convex set.

Now we shall prove that the operator S defined by (2.15) transforms $A^{\delta}_{R,\mu}$ into itself.

For each $\varphi \in A^{\delta}_{R,\mu}$, we have

$$(S\varphi)(s) = \psi(s), \quad \psi(s) \in W^1_p[-\pi,\pi], \tag{3.2}$$

where $(S\varphi)(s) = \Lambda + \int_0^s k(s,\sigma,\varphi(\sigma))\,\mathrm{d}\sigma$.

Then for any $s \in [-\pi,\pi]$ we have from (2.43) and (3.2) that

$$|\psi(s)| \leqslant \Lambda + \int_{-\pi}^{\pi} |k(s,\sigma,\varphi(\sigma))|\,\mathrm{d}\sigma \leqslant \Lambda + \|k(s,\sigma,\varphi(\sigma))\|_p (2\pi)^{k^{-1}} \leqslant R, \tag{3.3}$$

where

$$R \equiv Q(2\pi)^{k^{-1}} + \Lambda \quad \text{and} \quad p^{-1} + k^{-1} = 1.$$

Now, we evaluate $|\psi(s_2) - \psi(s_1)|$ for $p > 1$, $p^{-1}+k^{-1} = 1$, $k^{-1} = \delta$, $s_1, s_2 \in [-\pi,\pi]$. We have

$$\begin{aligned} |\psi(s_2) - \psi(s_1)| &= \left| \int_0^{s_2} k(s,\sigma,\varphi(\sigma))\,\mathrm{d}\sigma - \int_0^{s_1} k(s,\sigma,\varphi(\sigma))\,\mathrm{d}\sigma \right| \\ &= \left| \int_{s_1}^{s_2} k(s,\sigma,\varphi(\sigma))\,\mathrm{d}\sigma \right| \leqslant \|k(s,\sigma,\varphi(\sigma))\|_p |s_2 - s_1|^{k^{-1}} \\ &\leqslant Q|s_2 - s_1|^{k^{-1}}. \end{aligned} \tag{3.4}$$

If $Q \equiv \mu$, then the operator S maps the space $W^1_p[-\pi,\pi]$ into its convex compact subset $A^{\delta}_{R,\mu}$. In particular, S maps $A^{\delta}_{R,\mu}$ into itself, so that all transformed functions $\psi(s)$ belong to $A^{\delta}_{R,\mu}$.

Hence the following lemma is valid.

Lemma 3.1. *Let functions $\xi(s)$, $\eta(s)$, $c(s)$, $w_1(s)$ and $M(s,\sigma,\varphi(\sigma))$ satisfy the Hilbert-Lipschitz conditions* (2.17). *Then, for arbitrary $\varphi \in A^{\delta}_{R,\mu}$, the transformed points $(S\varphi)(s) = \psi(s)$ belong to the set $A^{\delta}_{R,\mu}$.*

Lemma 3.2. *The operator S defined in* (3.2) *which transforms the set $A^{\delta}_{R,\mu}$ into itself is continuous.*

P r o o f. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of elements of the set $A^{\delta}_{R,\mu}$ converging uniformly to an element $\varphi \in A^{\delta}_{R,\mu}$.

We consider the difference

$$|\psi_n(s) - \psi(s)| \leqslant \int_{-\pi}^{\pi} |k(s,\sigma,\varphi_n(\sigma)) - k(s,\sigma,\varphi(\sigma))|\,\mathrm{d}\sigma,$$

where

$$|k(s,\sigma,\varphi_n(\sigma)) - k(s,\sigma,\varphi(\sigma))| \leqslant |(N_n^1)(s)| + |(N_n^2)(s)|$$

with

$$(N_n^1)(s) = \eta(s)c(s)\mathrm{e}^{w_1(s)}M(s,s,\varphi_n(s)) - \eta(s)c(s)\mathrm{e}^{w_1(s)}M(s,s,\varphi(s))$$

and

$$\begin{aligned}(N_n^2)(s) = {}& \frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} c(\sigma)\mathrm{e}^{w_1(\sigma)}M(s,\sigma,\varphi_n(\sigma))\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma \\ & - \frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} c(\sigma)\mathrm{e}^{w_1(\sigma)}M(s,\sigma,\varphi(\sigma))\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma.\end{aligned}$$

Now, we will show that

$$\lim_{n\to\infty}|\psi_n(s) - \psi(s)| = 0.$$

We have

$$\begin{aligned}\lim_{n\to\infty}|(N_n^1)(s)| &= \lim_{n\to\infty}|\eta(s)c(s)\mathrm{e}^{w_1(s)}M(s,s,\varphi_n(s)) - \eta(s)c(s)\mathrm{e}^{w_1(s)}M(s,s,\varphi(s))| \\ &\leqslant |\eta(s)c(s)\mathrm{e}^{w_1(s)}|\lim_{n\to\infty}|M(s,s,\varphi_n(s)) - M(s,s,\varphi(s))| \\ &\leqslant |\eta(s)c(s)\mathrm{e}^{w_1(s)}|\lim_{n\to\infty}\alpha|\varphi_n(s) - \varphi(s)|,\end{aligned}$$

but

$$\lim_{n\to\infty}|\varphi_n(s) - \varphi(s)| = 0.$$

Hence

$$\lim_{n\to\infty}|(N_n^1)(s)| = 0. \tag{3.5}$$

Also,

$$\begin{aligned}&\lim_{n\to\infty}|(N_n^2)(s)| \\ &\quad = \lim_{n\to\infty}\biggl|\frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} c(\sigma)\mathrm{e}^{w_1(\sigma)}M(s,\sigma,\varphi_n(\sigma))\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma \\ &\qquad - \frac{\xi(s)}{2\pi}\int_{-\pi}^{\pi} c(\sigma)\mathrm{e}^{w_1(\sigma)}M(s,\sigma,\varphi(\sigma))\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma\biggr| \\ &\quad \leqslant |\xi(s)|\lim_{n\to\infty}\biggl|\frac{1}{2\pi}\int_{-\pi}^{\pi} c(\sigma)\mathrm{e}^{w_1(\sigma)}[M(s,\sigma,\varphi_n(\sigma)) - M(s,\sigma,\varphi(\sigma))]\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma\biggr|\end{aligned}$$

and using inequality (2.34), we arrive at

$$\begin{aligned}
&\lim_{n\to\infty} |(N_n^2)(s)| \\
&\quad \leqslant |\xi(s)| \lim_{n\to\infty} \varrho \|c(\sigma)\mathrm{e}^{w_1(\sigma)}[M(s,\sigma,\varphi_n(\sigma)) - M(s,\sigma,\varphi(\sigma))]\| \\
&\quad \leqslant \varrho|\xi(s)||c(\sigma)\mathrm{e}^{w_1(\sigma)}| \lim_{n\to\infty} \alpha|\varphi_n(s)-\varphi(s)| = 0.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} |(N_n^2)(s)| = 0. \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$\lim_{n\to\infty} |\psi_n(s) - \psi(s)| = 0.$$

Consequently, the operator S is continuous. □

By the preceding lemmas and the Arzela-Ascoli theorem [13], [16] the image of $A^{\delta}_{R,\mu}$ is compact. Therefore, we can use Schauder's fixed point theorem. Hence the operator S has at least one fixed point. Thus, we can state the main theorem as follows.

Theorem 3.1. *If the conditions of Lemmas* 3.1 *and* 3.2 *are satisfied, then equation* (1.1) *has at least one solution in the space* $W_p^1[-\pi,\pi]$.

References

[1] *S. M. Amer*: Existence results for a class of non-linear singular integral equations with shift. Il Nuovo Cimento B *120* (2005), 313–333.
[2] *S. M. Amer*: On the solvability of non-linear singular integral equation and integro-differential equations of Cauchy type (II). Proc. Math. Phys. Soc. Egypt No. 78. 2003, pp. 91–110.
[3] *F. D. Gakhov*: Boundary Value Problems. Dover Publ., New York, 1990.
[4] *D. H. Griffel*: Applied Functional Analysis. John Wiley & Sons, New York, 1981.
[5] *A. I. Gusejnov, Kh. Sh. Mukhtarov*: Introduction to the Theory of Nonlinear Singular Integral Equations. Nauka, Moscow, 1980. (In Russian.)
[6] *S. Heikkilä, K. Reffett*: Fixed point theorems and their applications to the theory of Nash equilibria. Nonlinear Anal., Theory, Methods Appl. *64* (2006), 1415–1436.
[7] *P. Junghanns, U. Weber*: On the solvability of nonlinear singular integral equations. Z. Anal. Anwend. *12* (1993), 683–698.
[8] *P. Junghanns, G. Semmler, U. Weber, E. Wegert*: Nonlinear singular integral equations on a finite interval. Math. Methods Appl. Sci. *24* (2001), 1275–1288.
[9] *E. G. Ladopoulos, V. A. Zisis*: Nonlinear finite-part singular integral equations arising in two-dimensional fluid mechanics. Nonlinear Anal., Theory Methods Appl. *42* (2000), 277–290.

[10] *E. G. Ladopoulos, V. A. Zisis*: Existence and uniqueness for nonlinear singular integral equations used in fluid mechanics. Appl. Math. *42* (1997), 345–367.
[11] *E. G. Ladopoulos, V. A. Zisis*: Nonlinear singular integral approximations in Banach spaces. Nonlinear Anal., Theory, Methods Appl. *26* (1996), 1293–1299.
[12] *E. G. Ladopoulos*: Non-linear singular integral equations in real elastodynamics by using Hilbert transformations. Nonlinear Anal., Real World Appl. *6* (2005), 531–536.
[13] *L. A. Lusternik, V. J. Sobolev*: Elements of Functional Analysis. Gordon and Breach Science Publishers, New York, 1986.
[14] *S. G. Mikhlin, S. Prössdorf*: Singular Integral Operators. Akademie-Verlag, Berlin, 1986.
[15] *Nassr-Eddine Tatar*: On the integral inequality with a kernel singular in time and space. J. Inequal. Pure Appl. Math. *4* (2003).
[16] *W. Pogorzelski*: Integral Equations and Their Applications, Vol. 1. Oxford Pergamon Press/PWN, Oxford/Warszawa, 1966.
[17] *L. von Wolfersdorf*: Class of nonlinear singular integral and integro-differential equations with Hilbert kernel. Zeitschrift Anal. Anwend. *8* (1989), 563–570.
[18] *L. von Wolfersdorf*: On the theory of nonlinear singular integral equations of Cauchy type. Math. Methods Appl. Sci. *7* (1985), 493–517.

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