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*Applications of Mathematics*, Vol. 54 (2009), No. 5, 381--390

Persistent URL: [http://dml.cz/dmlcz/140373](http://dml.cz/dmlcz/140373)

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A PARAMETRIZED NEWTON METHOD FOR NONSMOOTH EQUATIONS WITH FINITELY MANY MAXIMUM FUNCTIONS*

SHOU-QIANG DU, Shanghai and Qingdao, YAN GAO, Shanghai

(Received July 13, 2007, in revised version January 28, 2008)

Abstract. In this paper we propose a parametrized Newton method for nonsmooth equations with finitely many maximum functions. The convergence result of this method is proved and numerical experiments are listed.

Keywords: nonsmooth equations, Newton method, convergence

MSC 2010: 65H10, 90C30

1. Introduction

A lot of optimization problems can be transformed to a system of nonsmooth equations

\( F(x) = 0, \)

where \( F: \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitz, for instance, nonlinear complementarity problems, variational inequality problems and \( LC^1 \) optimization problems ([1], [3], [5], [7]–[10]).

In recent years, much attention has been devoted to various forms of methods for solving (1). Most popular methods are the Newton-type ones, which include Newton methods, inexact Newton methods and quasi-Newton methods. They are based on the Clarke-type subdifferentials in each iteration step. For instance, the Newton

*This work was supported by National Science foundation of China (under grant 10671126), key project for fundamental research of STCSM (under grant 06JC14057) and Shanghai leading academic discipline project (S30501) and JWCXSL0801 of Shanghai.
method is given as follows:

\[ x_{k+1} = x_k - V_k^{-1} F(x_k), \]

where \( V_k \) is an element of Clarke generalized Jacobian [7], an element of B-differential [8], and an element of b-differential [11] of \( F \) an \( x_k \). Under the assumption that all elements of \( \partial_{C^1} F(x^*) \), of \( \partial_B F(x^*) \), or of \( \partial_b F(x^*) \), where \( x^* \) is the solution of (2), are nonsingular, the locally super-linear convergence properties were obtained. In [1], Chen and Qi presented a parametrized modification of the generalized Jacobian Newton-like method

\[ x_{k+1} = x_k - \alpha_k (V_k + \lambda_k I)^{-1} F(x_k), \]

where \( V_k \in \partial_B F(x_k) \), \( I \) is the \( n \times n \) identity matrix, the parameters \( \alpha_k \) and \( \lambda_k \) are chosen to ensure convergence and \( V_k + \lambda_k I \) is invertible.

In this paper we study a new method for the equations with max-type functions proposed by Gao in [3]

\[ \max_{j \in J_i} f_{ij}(x) = 0, \]

\[ \vdots \]

\[ \max_{j \in J_n} f_{nj}(x) = 0, \]

where \( f_{ij} : \mathbb{R}^n \to \mathbb{R} \) for \( j \in J_i, i = 1, \ldots, n \), are continuously differentiable, \( J_i \) for \( i = 1, \ldots, n \) are finite index sets. Denote

\[ f_i(x) = \max_{j \in J_i} f_{ij}(x), \quad x \in \mathbb{R}^n, \quad i = 1, \ldots, n, \]

\[ F(x) = (f_1(x), \ldots, f_n(x))^T, \quad x \in \mathbb{R}^n; \]

\[ J_i(x) = \{ j_i \in N : f_{ij}(x) = f_i(x) \}, \quad x \in \mathbb{R}^n, \quad i = 1, \ldots, n; \]

then the equation (3) can be rewritten as (1). Define a new kind of the differential for \( F(x) \) by

\[ \partial_* F(x) = \{ (\nabla f_{j_1}, \ldots, \nabla f_{j_n})^T : j_1 \in J_1(x), \ldots, j_n \in J_n(x) \}, \quad x \in \mathbb{R}^n. \]

Gao [3] gave the super convergence result. Based on [3], Śmietański constructed a new version of finite difference approximation of the generalized Jacobian for a finite maximum function in [9]. In paper [10], Śmietański also proposed a new class of the parametrized Newton-like method for semismooth equations.
Based on [10], we consider the max-type equations proposed in [3] and present a new parametrized Newton method for the system of nonsmooth equations with finite maximum functions. The well-known results on the generalized Jacobian and the semismoothness will be recalled in Section 2. In Section 3, we will give the new method for (3) and study the convergence of this method. Finally, we give a numerical example.

2. Preliminaries

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a locally Lipschitzian function. According to Rademacher’s Theorem, the local Lipschitz continuity of \( F(x) \) implies that \( F(x) \) is differentiable almost everywhere. Let \( D_F \) be the set where \( F(x) \) is differentiable. Then

\[
\partial_B F(x) = \{ \lim_{x_i \to x} JF(x_i), \ x_i \in D_F \}
\]

is called the B-differential of \( F(x) \) at \( x \) ([8]), where \( JF(x_k) \) denotes the usual Jacobian matrix of partial derivatives of \( F(x) \) at \( x_k \). The generalized Jacobian ([2]) of \( F : \mathbb{R}^n \to \mathbb{R}^n \) at \( x \) in the sense of Clark is

\[
\partial F(x) = \text{conv} \partial_B F(x).
\]

From [2] we know \( \partial F(x) \) is nonempty, convex and compact and \( \partial F \) is upper semi-continuous at \( x \). By the definition in [7], a locally Lipschitzian function \( F : \mathbb{R}^n \to \mathbb{R}^n \) is said to be semismooth at \( x \) provided that

\[
\lim_{V \in \partial F(x+th'), h' \to h, t \downarrow 0} \frac{Vh'}{t}
\]

exists for any \( h \in \mathbb{R}^n \).

The semismoothness was originally introduced for functions by Mifflin in [4]. Semismooth functions have many important properties, which are very important in convergence analysis of methods in nonsmooth optimization. We will give some properties for our discussion. If \( F \) is semismooth, let \( F'(x; h) \) denote the classic directional derivative of \( F \) at \( x \) in the direction \( h \), i.e.,

\[
F'(x; h) = \lim_{t \downarrow 0} \frac{F(x + th) - F(x)}{t},
\]

and

\[
F'(x; h) = \lim_{V \in \partial F(x+th'), h' \to h, t \downarrow 0} Vh', \ h \in \mathbb{R}^n.
\]
Lemma 1. Suppose that $F: \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitzian and semismooth at $x$. Then

1. $Vh - F'(x; h) = O(\|h\|)$, \quad $\forall V \in \partial F(x + h)$, $h \in \mathbb{R}^n$,
2. $F(x + h) - F(x) - F'(x; h) = O(\|h\|)$, $h \in \mathbb{R}^n$.

If for any $V \in \partial F(x + h)$, $h \to 0$, we have $Vh - F'(x; h) = O(\|h\|^{1+p})$, where $0 < p \leq 1$, then $F$ is $p$-order semismooth at $x$. When $p = 1$, the function $F(x)$ is called strongly semismooth ([6]).

Lemma 2 ([3]). Suppose that $F(x)$ and $\partial_\star F(x)$ are defined by (3) and by (4), and all $V \in \partial_\star F(x)$ are nonsingular. Then there exists a scalar $\beta > 0$ such that

$$\|V^{-1}\| \leq \beta, \quad \forall V \in \partial_\star F(x).$$

Furthermore, there exists a neighborhood $N(x)$ of $x$ such that

$$\|V^{-1}\| \leq \frac{10}{9} \beta, \quad \forall V \in \partial_\star F(y), \quad y \in N(x).$$

Scalar products and sums of semismooth functions are still semismooth functions (see [4]). Moreover, the equations in (3) are also semismooth.

Lemma 3. Equations of max-type functions (3) form a system of semi-smooth equations.

The terminology of the convergence rate, which is also used in this paper, refers to the following. Let $\{x_k\} \subset \mathbb{R}^n$ be a sequence of vectors tending to the limit $x^* \neq x_k$ for all $k$. The convergence rate is said to be

(a) Q-linear if

$$\limsup_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} < \infty;$$

(b) Q-superlinear if

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0;$$

(c) Q-quadratic if

$$\limsup_{k \to \infty} \frac{\|x_{k+1} - x^*\|^2}{\|x_k - x^*\|^2} < \infty;$$

(d) quadratic if

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|^2}{\|x_k - x^*\|^2} = 0.$$

In these cases, we say that $\{x_k\}$ converges to $x^*$ Q-linearly, Q-superlinearly, Q-quadratically, and quadratically, respectively.
3. A parametrized Newton method

The classical generalized Jacobian method was introduced by Qi and Sun [7] in the form (2). The iteration is locally superlinearly convergent for the semismooth equations. Our parametrized Newton method would be considered as a new approach based on parametrization of the method (2) for solving (3). On the other hand, we extend the method of Gao [3] and Śmietański [10] to a system of nonsmooth equations with finite maximum functions. We give the new parametrization Newton method for (3) in the form

\[ x_{k+1} = x_k - \text{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k)^{-1} F(x_k), \]

where \( V_k \in \partial x F(x_k) \) and \( \lambda_i^{(k)} \in \mathbb{R}, 0 < |\lambda_i^{(k)}| < +\infty \) for \( i = 1, \ldots, n \) and \( k = 0, 1, 2, \ldots \) is a parameter chosen such that the matrix \( \text{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k \) is nonsingular.

**Lemma 4.** Suppose \( x^* \) is a solution of (3). Then

\[ \|\text{diag}(\lambda_i^{(k)} f_i(x_k))\| \leq M \leq \frac{1}{\beta} \]

for all \( x \) in some neighborhood of \( x^* \) and \( M + \varepsilon < 1/\beta \), where \( \lambda_i^{(k)} \in \mathbb{R} \) and \( 0 < |\lambda_i^{(k)}| < +\infty \) for \( i = 1, \ldots, n \) and \( k = 0, 1, 2, \ldots \)

Since each \( f_{ij} \) of (3) is continuous, we get the lemma immediately.

**Theorem.** Suppose that \( x^* \) is a solution of (3), and all \( V \in \partial x F(x^*) \) are nonsingular. Then the iteration method defined by (6) for solving (3) is well-defined and superlinearly convergent to \( x^* \) in a neighborhood of \( x^* \).

**Proof.** Denote by \( N_{x^*} \), a neighborhood of \( x^* \) for any \( x \in N_{x^*} \). For \( V_x \in \partial x F(x) \), Lemma 1 implies that

\[ \|V_x(x - x^*) - F'(x^*; x - x^*)\| = o(\|x - x^*\|), \]

\[ \|F(x) - F(x^*) - F'(x^*; x - x^*)\| = o(\|x - x^*\|). \]

By virtue of Lemma 2 and (7), we get

\[ \|[\text{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1}\| \leq \frac{\beta}{1 - \beta(\varepsilon + M)}. \]

So (6) is well-defined for \( x_k \in N_{x^*} \).
By (8), (10) and the fact of Lipschitz continuity of $F(x)$, we have

$$
\|x_{k+1} - x^*\| = \|x_k - x^* - [\text{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1} F(x_k)\| \\
\leq \|[\text{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1}\| \\
\times ([\text{diag}(\lambda_i^{(k)} f_i(x_k))(x_k - x^*)] + \|F(x_k) - F(x^*) - F'(x^*; x_k - x^*)\| \\
+ \|V_k(x_k - x^*) - F'(x^*; x_k - x^*)\|) \\
= o(\|x_k - x^*\|).
$$

Therefore, the sequence $\{x_k\}$ converges to $x^*$ superlinearly. Thus, we have completed the proof of the theorem. □

**Corollary.** Under the condition of the theorem, if the iteration does not terminate after a finite number of steps, then

$$
\lim_{k \to \infty} \frac{\|F(x_{k+1})\|}{\|F(x_k)\|} = 0
$$

holds in a neighborhood of $x^*$.

**Remark.** If $f_{ij}$ in the equation system (3) are $C^2$ functions, then $F(x)$ is a strongly semismooth function. Since

$$
Vh - F'(x; h) = o(\|h\|^2),
$$

$$
F(x + h) - F(x) - F'(x; h) = o(\|h\|^2),
$$

we obtain that

$$
\|V_x(x - x^*) - F'(x^*; x - x^*)\| = o(\|x - x^*\|^2),
$$

$$
\|F(x) - F(x^*) - F'(x^*; x - x^*)\| = o(\|x - x^*\|^2),
$$

$$
\|\text{diag}(\lambda_i^{(k)} f_i(x_k))(x - x^*)\| = o(\|x - x^*\|^2).
$$

So the sequence $\{x_k\}$ converges to $x^*$ quadratically.
4. Numerical test

The system of equations of max-type functions have concrete background, for instance, complementarity problems, variational inequality problems, Karush-Kuhn-Tucker systems of nonlinear programs and other mechanics and engineering problems. In order to show the performance of the parametrized Newton method (6), in this section, we present some numerical results of our method. All the experiments were run on a Pentium IV 1.8GHz using Matlab 7.0.

Example 1.

\[
\begin{align*}
\max \{ f_{11}(x_1, x_2), f_{12}(x_1, x_2) \} &= 0, \\
\max \{ f_{21}(x_1, x_2), f_{22}(x_1, x_2) \} &= 0,
\end{align*}
\]

where

\[
\begin{align*}
f_{11} &= \frac{1}{3} x_1^2, & f_{12} &= x_1^2, & f_{21} &= \frac{1}{3} x_1^2, & f_{22} &= x_1^2.
\end{align*}
\]

From (3) we know

\[
F(x) = (f_1(x), f_2(x))^T,
\]

where \( f_1(x) = x_1^2, f_2(x) = x_1^2, x \in \mathbb{R}^2. \)

\[\begin{array}{ccc|c}
\hline
x_0 &=& (1, 10)^T & \lambda_1 = 0.01 & \lambda_2 = 10 \\
\hline
\text{Step} & x & F(x) & \\
\hline
1 & (1.0000, 10.0000)^T & (1.0000, 1.0000)^T & \\
2 & (0.5025, 9.9995)^T & (0.2525, 0.2525)^T & \\
3 & (0.2519, 9.9993)^T & (0.0634, 0.0634)^T & \\
4 & (0.1261, 9.9991)^T & (0.0159, 0.0159)^T & \\
5 & (0.0631, 9.99906)^T & (0.00398, 0.00398)^T & \\
6 & (0.031554, 9.99903)^T & 1.0e - 003 \ast (0.99563, 0.99563)^T & \\
7 & (0.01578, 9.99902)^T & 1.0e - 003 \ast (0.248986, 0.248986)^T & \\
8 & (0.00789, 9.99901)^T & 1.0e - 004 \ast (0.62256, 0.62256)^T & \\
9 & (0.003945, 9.99900)^T & 1.0e - 004 \ast (0.155653, 0.155653)^T & \\
10 & (0.001973, 9.99900)^T & 1.0e - 005 \ast (0.38915, 0.38915)^T & \\
11 & (0.000986, 9.99900)^T & 1.0e - 006 \ast (0.972888, 0.972888)^T & \\
12 & (0.000493, 9.99900)^T & 1.0e - 006 \ast (0.243225, 0.243225)^T & \\
13 & (0.000247, 9.99900)^T & 1.0e - 007 \ast (0.608064, 0.608064)^T & \\
14 & (0.000123, 9.99900)^T & 1.0e - 007 \ast (0.152016, 0.152016)^T & \\
15 & (0.0000616, 9.9990001)^T & 1.0e - 008 \ast (0.3800416, 0.3800416)^T & \\
16 & (0.0000308, 9.9990000)^T & 1.0e - 009 \ast (0.9501046, 0.9501046)^T & \\
17 & (0.0000154, 9.9990000)^T & 1.0e - 009 \ast (0.2375262, 0.2375262)^T & \\
18 & (0.0000077, 9.9990000)^T & 1.0e - 010 \ast (0.5938156, 0.5938156)^T & \\
19 & (0.0000039, 9.9990000)^T & 1.0e - 010 \ast (0.1484539, 0.1484539)^T & \\
20 & (0.0000019, 9.9990000)^T & 1.0e - 011 \ast (0.3711348, 0.3711348)^T & \\
21 & (0.000000963, 9.999000001)^T & 1.0e - 012 \ast (0.9278370, 0.9278370)^T & \\
\hline
\end{array}
\]

Table 1. Results for Example 1 with initial point \( x_0 = (1, 10)^T \) and \( \lambda_1 = 0.01, \lambda_2 = 10. \)
According to (6), we have
\[
\begin{pmatrix}
  x_1^{(k+1)} \\
  x_2^{(k+1)}
\end{pmatrix}
= \begin{pmatrix}
  x_1^{(k)} \\
  x_2^{(k)}
\end{pmatrix} - \begin{pmatrix}
  \lambda_1 x_1^{(k)} * x_1^{(k)} + 2x_1^{(k)} \\
  2x_1^{(k)}
\end{pmatrix} \cdot \begin{pmatrix}
  0 \\
  \lambda_2 x_1^{(k)} * x_1^{(k)}
\end{pmatrix}^{-1}
\begin{pmatrix}
  x_1^{(k)} * x_1^{(k)} \\
  x_1^{(k)} * x_1^{(k)}
\end{pmatrix}.
\]
To test the points, we use
\[
| x^{(k+1)} - x^k | < \varepsilon.
\]
In this example, we let \( \varepsilon = 10^{-6} \).

<table>
<thead>
<tr>
<th>Step</th>
<th>( x_0 = (1000, 1000)^T )</th>
<th>( \lambda_1 = 0.0001 )</th>
<th>( \lambda_2 = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1000, 1000)^T</td>
<td>(1000000, 1000000)^T</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.0e + 002 * (5.2381, 9.9998)^T</td>
<td>1.0e + 005 * (2.74376, 2.74376)^T</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.0e + 002 * (2.68589, 9.99963)^T</td>
<td>1.0e + 004 * (7.21401, 7.21401)^T</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.0e + 002 * (1.36074, 9.99957)^T</td>
<td>1.0e + 004 * (1.85162, 1.85162)^T</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.0e + 002 * (0.68497, 9.99953)^T</td>
<td>1.0e + 003 * (4.69182, 4.69182)^T</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.0e + 002 * (0.34365, 9.99952)^T</td>
<td>1.0e + 003 * (1.18098, 1.18098)^T</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.0e + 002 * (0.17212, 9.99951)^T</td>
<td>1.0e + 002 * (2.96258, 2.96258)^T</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.0e + 002 * (0.08613, 9.99950)^T</td>
<td>(74.19182, 74.19182)^T</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.0e + 002 * (0.04309, 9.99950)^T</td>
<td>(18.56393, 18.56393)^T</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.0e + 002 * (0.02155, 9.99950)^T</td>
<td>(4.64298, 4.64298)^T</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.0e + 002 * (0.01077, 9.99950)^T</td>
<td>(1.16010, 1.16010)^T</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1.0e + 002 * (0.00539, 9.99950)^T</td>
<td>(0.29028, 0.29028)^T</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1.0e + 002 * (0.00269, 9.99950)^T</td>
<td>(0.07257, 0.07257)^T</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>1.0e + 002 * (0.00135, 9.99950)^T</td>
<td>(0.01814, 0.01814)^T</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>1.0e + 002 * (0.00067, 9.99950)^T</td>
<td>(0.00454, 0.00454)^T</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1.0e + 002 * (0.00034, 9.99950)^T</td>
<td>(0.00113, 0.00113)^T</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>1.0e + 002 * (0.00017, 9.99950)^T</td>
<td>1.0e - 003 * (0.28351, 0.28351)^T</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>1.0e + 002 * (0.000084, 9.99950)^T</td>
<td>1.0e - 004 * (0.70877, 0.70877)^T</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>1.0e + 002 * (0.000042, 9.99950)^T</td>
<td>1.0e - 004 * (0.17719, 0.17719)^T</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1.0e + 002 * (0.000021, 9.99950)^T</td>
<td>1.0e - 005 * (0.44298, 0.44298)^T</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>1.0e + 002 * (0.000011, 9.99950)^T</td>
<td>1.0e - 005 * (0.11074, 0.11074)^T</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>1.0e + 002 * (0.000005, 9.99950)^T</td>
<td>1.0e - 006 * (0.27686, 0.27686)^T</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>1.0e + 002 * (0.000003, 9.99950)^T</td>
<td>1.0e - 007 * (0.69216, 0.69216)^T</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>1.0e + 002 * (0.000001, 9.99950)^T</td>
<td>1.0e - 007 * (0.17304, 0.17304)^T</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>1.0e + 002 * (0.00000066, 9.99950000)^T</td>
<td>1.0e - 008 * (0.432598, 0.432598)^T</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>1.0e + 002 * (0.00000033, 9.99950000)^T</td>
<td>1.0e - 008 * (0.108149, 0.108149)^T</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>1.0e + 002 * (0.00000016, 9.99950000)^T</td>
<td>1.0e - 009 * (0.270374, 0.270374)^T</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>1.0e + 002 * (0.00000008, 9.99950000)^T</td>
<td>1.0e - 010 * (0.675934, 0.675934)^T</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>1.0e + 002 * (0.00000004, 9.99950000)^T</td>
<td>1.0e - 010 * (0.168983, 0.168983)^T</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>1.0e + 002 * (0.00000002, 9.99950000)^T</td>
<td>1.0e - 011 * (0.422459, 0.422459)^T</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>1.0e + 002 * (0.00000001, 9.99950000)^T</td>
<td>1.0e - 011 * (0.105615, 0.105615)^T</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>1.0e + 002 * (0.00000005, 9.99950000)^T</td>
<td>1.0e - 012 * (0.264037, 0.264037)^T</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Results for Example 1 with initial point \( x_0 = (1000, 1000)^T \) and \( \lambda_1 = 0.0001 \), \( \lambda_2 = 2 \).
From (3) we know

Example 2.

\[ \begin{align*}
  x_0 &= (10, 10)^T \\
  \lambda_1 &= 0.003 \\
  \lambda_2 &= 0.002 \\
  F(x) &= (f_1(x), f_2(x))^T \\
\end{align*} \]

<table>
<thead>
<tr>
<th>Step</th>
<th>( x )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (10.0000, 10.0000)^T )</td>
<td>( (150.0000, 100.0000)^T )</td>
</tr>
<tr>
<td>2</td>
<td>( (5.0491, 5.0868)^T )</td>
<td>( (38.6227, 25.4937)^T )</td>
</tr>
<tr>
<td>3</td>
<td>( (2.5373, 2.5657)^T )</td>
<td>( (9.8018, 6.4379)^T )</td>
</tr>
<tr>
<td>4</td>
<td>( (1.2719, 1.2885)^T )</td>
<td>( (2.4691, 1.6177)^T )</td>
</tr>
<tr>
<td>5</td>
<td>( (0.6368, 0.6457)^T )</td>
<td>( (0.6196, 0.4055)^T )</td>
</tr>
<tr>
<td>6</td>
<td>( (0.3186, 0.3232)^T )</td>
<td>( (0.1552, 0.1015)^T )</td>
</tr>
<tr>
<td>7</td>
<td>( (0.1593, 0.1617)^T )</td>
<td>( (0.0388, 0.0254)^T )</td>
</tr>
<tr>
<td>8</td>
<td>( (0.0797, 0.0809)^T )</td>
<td>( (0.0097, 0.0063)^T )</td>
</tr>
<tr>
<td>9</td>
<td>( (0.0398, 0.0404)^T )</td>
<td>( (0.0024, 0.0016)^T )</td>
</tr>
<tr>
<td>10</td>
<td>( (0.0199, 0.0202)^T )</td>
<td>( 1.0e-003 \ast (0.6073, 0.3970)^T )</td>
</tr>
<tr>
<td>11</td>
<td>( (0.00996, 0.01011)^T )</td>
<td>( 1.0e-003 \ast (0.1518, 0.0992)^T )</td>
</tr>
<tr>
<td>12</td>
<td>( (0.00498, 0.00506)^T )</td>
<td>( 1.0e-004 \ast (0.3796, 0.2481)^T )</td>
</tr>
<tr>
<td>13</td>
<td>( (0.00249, 0.00253)^T )</td>
<td>( 1.0e-005 \ast (0.9491, 0.6203)^T )</td>
</tr>
<tr>
<td>14</td>
<td>( (0.00125, 0.00126)^T )</td>
<td>( 1.0e-005 \ast (0.2373, 0.1551)^T )</td>
</tr>
<tr>
<td>15</td>
<td>( 1.0e-003 \ast (0.6226, 0.6319)^T )</td>
<td>( 1.0e-006 \ast (0.5932, 0.3877)^T )</td>
</tr>
<tr>
<td>16</td>
<td>( 1.0e-003 \ast (0.31132, 0.31597)^T )</td>
<td>( 1.0e-006 \ast (0.1483, 0.0969)^T )</td>
</tr>
<tr>
<td>17</td>
<td>( 1.0e-003 \ast (0.15566, 0.15798)^T )</td>
<td>( 1.0e-007 \ast (0.3707, 0.2423)^T )</td>
</tr>
<tr>
<td>18</td>
<td>( 1.0e-004 \ast (0.77830, 0.78992)^T )</td>
<td>( 1.0e-008 \ast (0.9268, 0.6058)^T )</td>
</tr>
<tr>
<td>19</td>
<td>( 1.0e-004 \ast (0.38915, 0.39496)^T )</td>
<td>( 1.0e-008 \ast (0.2317, 0.1514)^T )</td>
</tr>
<tr>
<td>20</td>
<td>( 1.0e-004 \ast (0.19458, 0.19748)^T )</td>
<td>( 1.0e-009 \ast (0.5793, 0.3786)^T )</td>
</tr>
<tr>
<td>21</td>
<td>( 1.0e-005 \ast (0.97288, 0.98740)^T )</td>
<td>( 1.0e-009 \ast (0.1448, 0.0946)^T )</td>
</tr>
<tr>
<td>22</td>
<td>( 1.0e-005 \ast (0.48644, 0.49370)^T )</td>
<td>( 1.0e-010 \ast (0.3620, 0.2366)^T )</td>
</tr>
<tr>
<td>23</td>
<td>( 1.0e-005 \ast (0.24322, 0.24685)^T )</td>
<td>( 1.0e-011 \ast (0.9051, 0.5916)^T )</td>
</tr>
<tr>
<td>24</td>
<td>( 1.0e-005 \ast (0.12161, 0.12342)^T )</td>
<td>( 1.0e-011 \ast (0.2263, 0.1479)^T )</td>
</tr>
<tr>
<td>25</td>
<td>( 1.0e-006 \ast (0.60805, 0.61712)^T )</td>
<td>( 1.0e-012 \ast (0.5657, 0.3697)^T )</td>
</tr>
</tbody>
</table>

Table 3. Results for Example 2 with initial point \( x_0 = (10, 10)^T \) and \( \lambda_1 = 0.003, \lambda_2 = 0.002 \).

Example 2.

\[
\begin{align*}
  \max \{ f_{11}(x_1, x_2), f_{12}(x_1, x_2) \} &= 0, \\
  \max \{ f_{21}(x_1, x_2), f_{22}(x_1, x_2) \} &= 0,
\end{align*}
\]

where

\[
\begin{align*}
  f_{11} &= \frac{1}{2} x_1^2 + \frac{1}{3} x_2^2, \\
  f_{12} &= \frac{1}{2} x_1^2 + x_2^2, \\
  f_{21} &= \frac{1}{4} x_1^2, \\
  f_{22} &= x_1^2.
\end{align*}
\]

From (3) we know

\[
F(x) = (f_1(x), f_2(x))^T.
\]

According to (6), we also have

\[
\begin{pmatrix}
  x_1^{(k+1)} \\
  x_2^{(k+1)}
\end{pmatrix} =
\begin{pmatrix}
  x_1^{(k)} \\
  x_2^{(k)}
\end{pmatrix} -
\begin{pmatrix}
  \lambda_1 \ast (f_1(x^{(k)})) + x_1^{(k)} \\
  2 \ast x_1^{(k)}
\end{pmatrix}
\begin{pmatrix}
  2 \ast x_2^{(k)} \\
  \lambda_2 \ast (f_2(x^{(k)}))
\end{pmatrix}^{-1}
\begin{pmatrix}
  f_1(x^{(k)}) \\
  f_2(x^{(k)})
\end{pmatrix}
\]

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where
\[ f_1(x) = \frac{1}{2}x_1^2 + x_2^2, \quad f_2(x) = x_1^2, \quad x \in \mathbb{R}^2. \]

To test the points, we use
\[ |x^{(k+1)} - x^k| < \varepsilon. \]

In this example, we also let \( \varepsilon = 10^{-6} \).

We also can test the example with other initial points and \( \lambda \).

**Acknowledgement.** The authors are grateful to the anonymous referee for valuable suggestions and comments.

**References**


**Authors’ addresses:** Shou-qiang Du, School of Management, University of Shanghai for Science and Technology, Shanghai, 200093, P. R. China, e-mail: dsq@9330163.com, and College of Mathematics, Qingdao University, Qingdao, 266071, P. R. China; Yan Gao, School of Management, University of Shanghai for Science and Technology, Shanghai, 200093, P. R. China, e-mail: gaoyan1962@263.net.