Applications of Mathematics

Nesrine Fterich

Global solution to a generalized nonisothermal Ginzburg-Landau system

Applications of Mathematics, Vol. 55 (2010), No. 1, 1–46

Persistent URL: http://dml.cz/dmlcz/140384

GLOBAL SOLUTION TO A GENERALIZED NONISOTHERMAL GINZBURG-LANDAU SYSTEM

Nesrine Fterich, Poitiers

(Received November 8, 2006, in revised version March 10, 2009)

Abstract. The article deals with a nonlinear generalized Ginzburg-Landau (Allen-Cahn) system of PDEs accounting for nonisothermal phase transition phenomena which was recently derived by A. Miranville and G. Schimperna: Nonisothermal phase separation based on a microforce balance, Discrete Contin. Dyn. Syst., Ser. B, *5* (2005), 753–768. The existence of solutions to a related Neumann-Robin problem is established in an $N \leqslant 3$-dimensional space setting. A fixed point procedure guarantees the existence of solutions locally in time. Next, Sobolev embeddings, interpolation inequalities, Moser iterations estimates and results on renormalized solutions for a parabolic equation with L^1 data are used to handle a suitable a priori estimate which allows to extend our local solutions to the whole time interval. The uniqueness result is justified by proper contracting estimates.

Keywords: nonisothermal Ginzburg-Landau (Allen-Cahn) system, microforce balance, existence and uniqueness results, renormalized solutions, Moser iterations

MSC 2010: 35Q56

1. Introduction

We are concerned with a Neumann-Robin problem related to a system of nonlinear PDE, namely, the generalized Ginzburg-Landau (Allen-Cahn) equations which model nonisothermal phase transition phenomena. More precisely, we investigate the following equations:

$$(\Phi(\theta))_t - \Delta\theta = \varrho_t^2 + \theta\varrho\varrho_t \quad \text{in } Q_T = \Omega \times]0,T[, \tag{1}$$

$$\varrho_t - \Delta\varrho + f'(\varrho) = -\varrho\,(\theta - \theta_c) \quad \text{in } Q_T, \tag{2}$$

$$\partial_n\varrho = \partial_n\theta + n_0(\theta - \theta_\Gamma) = 0 \quad \text{on } \Gamma \times (0,T), \tag{3}$$

$$\theta(\cdot,0) = \theta_0(x), \quad \varrho(\cdot,0) = \varrho_0(x) \quad \text{in } \Omega, \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \leqslant 3$, with smooth boundary $\partial\Omega = \Gamma$, $T > 0$ is a fixed time, $(\cdot)_t = \partial \cdot /\partial t$, Δ denotes the Laplacian, $\partial_n = \partial/\partial n$ the outer normal derivative on Γ, f a double-well potential, $\theta_c > 0$ the critical temperature at which the transition takes place, $\theta_\Gamma > 0$ represents the exterior (absolute) temperature on the boundary, n_0 is a positive proportionality parameter, θ_0 (assumed a.e. greater than a positive constant $\underline{\theta}$) and ϱ_0 are initial (given) values of the unknown fields which are here the absolute temperature θ and the order parameter or phase field ϱ. In the above equation (1), $\Phi\colon \mathbb{R} \to \mathbb{R}$ is a C^1, increasing function such that $\Phi(0) = 0$ and there exists a positive real number p such that

$$\begin{cases} p \geqslant 2 & \text{if } N \leqslant 2, \\ 2 \leqslant p < 3 & \text{if } N = 3, \end{cases} \tag{5}$$

and two positive constants c, c' satisfying

$$cr^p \leqslant \Phi(r) \leqslant c'r^p \quad \forall\, r \in \mathbb{R}^+. \tag{6}$$

We note that we have got rid of the precise value of some physical parameters in the above formulation (1)–(2), since they do not affect our analysis.

Our aim in this article is to establish a rigorous mathematical analysis for our generalized problem (1)–(4). More precisely, we are concerned with the existence and uniqueness results to the generalized Ginzburg-Landau system (1)–(4) under suitable hypotheses on the data. To solve equation (1), we introduce a strictly monotone function denoted by $\gamma\colon \mathbb{R} \to \mathbb{R}$ which coincides with Φ on $\mathbb{R}^+$ and satisfies the necessary conditions (C1)–(C2), mentioned in the next section.

It is clear that this problem looks difficult to deal with, due to the term $\Phi(\theta)$ and to the presence of strong nonlinearities, especially, the term $\theta\varrho\varrho_t$. The boundedness of ϱ, the positivity of θ and the existence of a lower bound θ^* are the key points to prove the local existence in time, the global existence and the uniqueness of solutions (θ, u, ϱ) to the system (1)–(4), where

$$u = \gamma(\theta) \tag{7}$$

represents the third unknown function introduced to overcome the difficulties. The case $p = 2$ and $N = 3$ was recently treated by Miranville and Schimperna in [37], where they showed similar results of global existence and uniqueness of solutions. Thus, we adapt here the techniques of [37] to prove our results.

Schauder's fixed-point theorem is exploited twice to prove the local existence of solutions to the problem (1)–(4). The global existence result follows from a simple combination of the uniform a priori estimates and Theorem 4.1 on local existence.

The main mathematical difficulty, in proving global existence results, comes from establishing the regularity

$$\varrho_t^2 + \theta\varrho\varrho_t \in L^2(Q_t). \tag{8}$$

To have (8) in three-space dimension, we use Moser iterations procedure, Agmon-Douglis-Nirenberg estimates ([28]) and renormalized solutions estimates of parabolic PDE, with initial data in L^1 (see [8], [9], and [10]). The concept of renormalized solutions has been introduced by R. J. DiPerna and J.-L. Lions in [24] and [25] to study Boltzmann equations and first-order equations.

To prove (8) in two-space dimension, we have to discuss it according to the values of p. In fact, by noting the same procedure as in three dimensional case, we realize that (8), for technical reasons, is valid only for $2 \leqslant p < 5$. On the other hand, making use of the Ladyzhenskaya inequality (see [28, Chapter II, (3.1)]) and the continuous embedding $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$ we can establish (8) for all $p > 2$.

Since the renormalized solution estimates are valid only in two and three-space dimensions, in order to improve (8) in one-space dimension we use the continuous embedding $V \hookrightarrow L^\infty(\Omega)$, the Gagliardo-Nirenberg inequality ([15, p. 194]) and the Agmon inequality ([41]).

2. Justification of the model

In a recent paper [36], Miranville and Schimperna introduced thermodynamically consistent models of nonisothermal phase transitions based on a balance law for internal microforces proposed by M. Gurtin in [27]. These models belong to a new family of systems of equations of Ginzburg-Landau (Allen-Cahn) type. We give here an idea on the physical derivation of these models. It turns out in [36] that, owing to the two laws of thermodynamics and the following internal microforce balance, first proposed by M. Gurtin in [27]:

$$\operatorname{div}\zeta + \pi = 0, \tag{9}$$

where ζ (a vector) corresponds to the microstress and π (a scalar) corresponds to the internal microforces (i.e., forces which arise from the interactions between atoms); this yields the constitutive relations between the order parameter ϱ and the temperature θ

$$\frac{1}{\theta}(\pi + \partial_\varrho\psi)\varrho_t = -\beta\varrho_t - a \cdot \nabla\frac{1}{\theta}, \tag{10}$$

$$q = b\varrho_t + B\nabla\frac{1}{\theta}, \tag{11}$$

where $\beta > 0$ is a scalar, a, b are two vectors and B is, in some sense, a positive semi-definite matrix. Then the relation

$$\zeta = \partial_{\nabla \varrho} \psi, \tag{12}$$

the first law of thermodynamics (energy equation) and (10)–(11) give the system of equations

$$\beta \varrho_t + a \cdot \nabla \frac{1}{\theta} + \frac{1}{\theta} \partial_\varrho \psi - \frac{1}{\theta} \operatorname{div}(\partial_{\nabla \varrho} \psi) = 0, \tag{13}$$

$$e_t = -\operatorname{div}\Big(b \varrho_t + B \nabla \frac{1}{\theta} - \varrho_t \partial_{\nabla \varrho} \psi \Big), \tag{14}$$

where e is the internal energy (see [36]). We emphasize that the system in this family exhibits many similarities with the so-called "models of phase transition with micro-movements" proposed by M. Frémond and coauthors in [11] (see also the recent monograph [26]) and mathematically analyzed in a series of articles, among which we quote [32]–[35].

Our system (1)–(2) follows from (13)–(14) by assuming that $\beta\theta$ is a positive constant (which we still denote by β), that $a = b = 0$ and that $B = \theta^2 I$ (I being the identity matrix). Moreover, we take the free energy of the form

$$\psi = -c_V \frac{c_p}{p-1} \theta^p + c \varrho^2 (\theta - \theta_c) + f(\varrho) + \frac{\alpha}{2} |\nabla \varrho|^2, \tag{15}$$

where $c_V > 0$, $c > 0$, $\alpha > 0$ (see [4]), $p \in [1, +\infty[$ and $c_p = c(p) > 0$. Knowing that

$$e = \partial_{1/\theta} \frac{\psi}{\theta} = \psi - \theta \partial_\theta \psi, \tag{16}$$

it is then easy to see that (1)–(2) can be recovered (by normalizing some of the constants). We note that the "entropic" contribution $E(\theta) = -c_V (c_p/(p-1)) \theta^p$ in the expression of the free energy (15) and $\Phi(\theta)$ in (1) fulfil the relation

$$E(\theta) - \theta E'(\theta) = c_V c_p \theta^p = c_V \Phi(\theta). \tag{17}$$

In general, any concave function E with $E(0) = 0$ might be physically admissible, in the sense that such assumptions are sufficient to ensure the thermodynamic consistency of the model. From a mathematical point of view, the choice of $E(\theta)$ (and more precisely, the corresponding term $(\Phi(\theta))_t$ in (1)) turns out to ensure the existence of global in time solutions, since it provides a priori information on the large value of θ.

Let us mention other related works. In a recent paper [37], under certain assumptions, Miranville and Schimperna treated the case $p = 2$ and $c_p = 1$ (i.e. $E(\theta) = -c_V\theta^2$) and proved the existence and uniqueness of global solutions in $\Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^3$. If $E(\theta) = -c_V\theta\ln\theta$ (see the quoted works on Frémond type models), one should probably only expect local in time existence result, at least in three-space dimension (see [35]), due to the lower growth rate at $+\infty$.

Our work is organized as follows. The next section is devoted to notation, assumptions and statements of the main results. Section 4 is concerned with the local in time existence result, performed by means of a fixed point procedure. Sections 5 and 6 treat the global existence and properties of solutions. This follows from a simple combination of uniform a priori estimates and Theorem 4.1 of local existence. Finally, in Section 7, we establish the uniqueness result and, more precisely, the continuous dependence estimates.

3. Notation, assumptions and basic theorem

Throughout the paper, let $H = L^2(\Omega)$, $V = H^1(\Omega)$ and $W = H^2(\Omega)$. Identifying, as usual, H with its dual H', we recall that $W \hookrightarrow V \hookrightarrow H \hookrightarrow V'$ with dense and compact injections. We denote by $(\cdot,\cdot)$ the inner product in H and by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V. The norm in H or in H^N is simply indicated by $|\cdot|$ and the norm in V by $\|\cdot\|$. Moreover, we denote by A the Riesz isomorphism of V onto V' and set

$$J\colon\ V \to V';\quad \langle Jv, w\rangle = \int_\Omega \nabla v \nabla w \,\mathrm{d}x + n_0 \int_\Gamma vw \,\mathrm{d}\sigma, \tag{18}$$

where v, w are elements of V. The norm $\|\cdot\|_J = \langle J\cdot,\cdot\rangle^{1/2}$ is of course equivalent to $\|\cdot\|$ and we will use it whenever necessary. We define the scalar product in V' by

$$((w_1, w_2))_* = (w_1, J^{-1}w_2). \tag{19}$$

The norm in the generic Banach space X will be generally denoted by $\|\cdot\|_X$. Sometimes, for $X = L^p(\Omega)$, we will write $|\cdot|_p$ instead of $\|\cdot\|_{L^p(\Omega)}$, for brevity. For $v \in \mathbb{R}$, we make use of the quantities $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$ so that $v = v^+ - v^-$ and $|v| = v^+ + v^-$.

Now, we are ready to state our mathematical problem and the related results properly. We note that, in what follows, the following assumptions are assumed to hold true.

First, let $\gamma\colon\ \mathbb{R} \to \mathbb{R}$ be a function satisfying the following two properties:

(C1) γ has the form

$$\gamma(r) = \begin{cases} \Phi(r) = rG(r) & \text{if } r > 0, \\ -r^2 & \text{if } r \leqslant 0, \end{cases}$$

where $G\colon \mathbb{R}^+ \to \mathbb{R}$ is C^2 with $G(0) = 0$, and there exists a positive real number p depending only on the dimension N such that

$$\begin{cases} p \geqslant 2 & \text{if } N \leqslant 2, \\ 2 \leqslant p < 3 & \text{if } N = 3, \end{cases}$$

and two positive constants c_1, c_2 such that

$$c_1 r^{p-2} \leqslant G'(r) \leqslant c_2 r^{p-2}, \quad \forall\, r \in \mathbb{R}^+.$$

We denote by α the inverse of γ. We shall need the sequences of functions

$$\gamma_\varepsilon(r) = \varepsilon r + \eta_\varepsilon(r), \tag{20}$$

where

$$\eta_\varepsilon(r) = \begin{cases} \gamma(r) & \text{if } |r| \leqslant \dfrac{1}{\varepsilon}, \\ rG\Big(\dfrac{1}{\varepsilon}\Big) & \text{if } r \geqslant \dfrac{1}{\varepsilon}, \\ \dfrac{r}{\varepsilon} & \text{if } r \leqslant -\dfrac{1}{\varepsilon} \end{cases} \tag{21}$$

for $\varepsilon \in (0,1)$ and $r \in \mathbb{R}$. Next, we set $\alpha_\varepsilon = \gamma_\varepsilon^{-1}$. So we have

$$\alpha_\varepsilon \to \alpha \quad \text{and} \quad \gamma_\varepsilon \to \gamma \tag{22}$$

in the sense of graphs (also called "G-convergence", see [5]). Finally, we let $\hat{\alpha}_\varepsilon$ and $\hat{\alpha}$ be respectively the antiderivatives of α_ε and α which vanish at 0.

(C2) There exist positive constants $c_{\alpha,i}$, $i = 1,2,3,4$, independent of ε such that

$$\hat{\alpha}_\varepsilon(r) \geqslant c_{\alpha,1}\alpha_\varepsilon^2(r) - c_{\alpha,2} \geqslant c_{\alpha,3}|r| - c_{\alpha,4} \quad \forall\, \varepsilon \in (0,1)\ \forall\, r \in \mathbb{R}.$$

Secondly, referring to the above notation, we introduce the functions

$$g = n_0\theta_\Gamma, \quad F'(r) = f'(r) - (1+\theta_c)r, \ \ r \in \mathbb{R},$$

and introduce the following assumptions on the data:
(H1) $\theta_0 \in V,\ u_0 = \gamma(\theta_0),\ \exists\, \underline{\theta} > 0;\ \theta_0 \geqslant \underline{\theta} \geqslant 0$ a.e. in Ω,
(H2) $g \in H^1(0, +\infty; L^2(\Gamma))$,
(H3) $\exists\, \underline{g} > 0;\ g \geqslant \underline{g}$ a.e. in $\Gamma \times (0, \infty)$,
(H4) $F \in C^2(\mathbb{R}),\ F'(0) = 0$,
(H5) $\exists\, \underline{\varrho} < 0,\ \overline{\varrho} > 0;\ F'(r) < 0,\ \forall\, r > \overline{\varrho},\ F'(r) < 0,\ \forall\, r < \underline{\varrho}$,
(H6) $\varrho_0 \in W^{2-1/p, 2p}(\Omega),\ \underline{\varrho} \leqslant \varrho_0 \leqslant \overline{\varrho}$ a.e. in Ω.

We set

$$\langle k, v\rangle = \int_\Gamma gv, \quad \forall\, v \in V. \tag{23}$$

Now, we are ready to state the main result of the paper.

Theorem 3.1. *Let $T > 0$ be an arbitrary final time. Assume that assumptions* (C1)–(C2) *and* (H1)–(H6) *hold true. Then the problem*

$$u_t + J\theta = \varrho_t^2 + \theta\varrho\varrho_t + k \qquad \text{in } Q_T, \tag{24}$$

$$\varrho_t + A\varrho + F'(\varrho) = -\theta\varrho \qquad \text{in } Q_T, \tag{25}$$

$$\partial_n \varrho = \partial_n\theta + n_0(\theta - \theta_\Gamma) = 0 \qquad \text{on } \Gamma \times (0, T), \tag{26}$$

$$\theta(\cdot, 0) = \theta_0(x),\ \ u(\cdot, 0) = u_0(x),\ \ \varrho(\cdot, 0) = \varrho_0(x) \ \text{ a.e. in } \Omega \tag{27}$$

admits a unique solution (θ, u, ϱ),

$$\theta \in H^1(0, T; H) \cap L^\infty(0, T; V), \tag{28}$$

$$u \in H^1(0, T; V') \cap L^\infty(0, T; H), \tag{29}$$

$$\varrho \in L^2(0, T; W) \cap H^1(0, T; H) \cap L^\infty(0, T; V), \quad \varrho_t, A\varrho \in L^{2p}(Q_T). \tag{30}$$

Moreover, we have

$$\underline{\varrho} \leqslant \varrho \leqslant \overline{\varrho} \quad \text{a.e. in } Q_T, \tag{31}$$

and there exist two constants $a, b > 0$ depending only on Ω, Γ, $\underline{g}$, $\underline{\theta}$, $\underline{\varrho}$, $\overline{\varrho}$ and n_0 such that

$$\theta(x, t) \geqslant a\mathrm{e}^{-bT} \quad \text{a.e. in } Q_T. \tag{32}$$

The proof of these results will be carried out throughout the remainder of the paper. We note that we will omit the proof of several results, in the sequel, since they are detailed in [37].

4. Local existence

We start by presenting our local existence theorem.

Theorem 4.1. *Under assumptions* (C1)–(C2) *and* (H1)–(H6), *there exists a positive constant* $\hat{T} \in\,]0,T]$ *such that problem* (24)–(30) *admits at least a solution defined on* $Q_{\hat{T}}$. *Moreover, we have*

$$\underline{\varrho} \leqslant \varrho \leqslant \overline{\varrho} \quad \text{a.e. in } Q_{\hat{T}}, \tag{33}$$

and there exist two constants $a, b > 0$ *depending only on* Ω, Γ, $\underline{g}$, $\underline{\theta}$, $\underline{\varrho}$, $\overline{\theta}$ *and* n_0 *such that*

$$\theta(x,t) \geqslant a\mathrm{e}^{-b\hat{T}} \quad \text{a.e. in } Q_{\hat{T}}. \tag{34}$$

P r o o f. We warn that, in what follows, we employ the same letter c for different constants, even in the same formula. We assume that c depends only on p and the data specified in (H1)–(H6). In particular, this generic constant will not be allowed to depend on $\hat{T}$. The constants depending on further parameters (e.g., on $\hat{T}$) not included in the above list will be denoted, e.g., by $c(\hat{T})$. A notation like c_i, $i \in \mathbb{N}$ (or $c_i(\hat{T})$, $i \in \mathbb{N}$) will be used to indicate specific constants whose precise value is needed in the course of the procedure. Also, we denote by m_i, $i \in \mathbb{N}$, some continuous and nonnegative functions defined on $[0, +\infty)$.

Now, we detail the local existence result. To this aim, we apply the Schauder fixed-point theorem to a suitable operator $\mathcal{T}$ constructed as will be specified in a while. For $R > 0$, let us consider the space for the fixed-point argument

$$\Theta_p(\hat{T}, R) = \{w \in L^{2p}(Q_{\hat{T}});\ w \geqslant 0 \text{ a.e. in } Q_{\hat{T}},\ \|w\|_{L^{2p}(Q_{\hat{T}})} \leqslant R\}, \tag{35}$$

where $\hat{T} \in\,]0,T]$ will be determined later in such a way that $\mathcal{T}\colon\ \Theta_p(\hat{T}, R) \to \Theta_p(\hat{T}, R)$ is a compact and continuous operator. The space $\Theta_p(\hat{T}, R)$ is endowed with the natural L^{2p}-norm. Now, we consider the following auxiliary problems for ϱ and θ, whose well-posedness is guaranteed by standard arguments.

P r o b l e m 1. Given $R, \hat{T} > 0$ and $\tilde{\theta} \in \Theta_p(\hat{T}, R)$, find a function $\varrho = \mathcal{T}_1(\tilde{\theta})\colon\ Q_{\hat{T}} \to \mathbb{R}$ satisfying

$$\varrho \in L^2(0, \hat{T}; W) \cap H^1(0, \hat{T}; H) \cap L^\infty(0, \hat{T}; V), \tag{36}$$

$$\varrho_t + A\varrho + F'(\varrho) = -\tilde{\theta}\varrho \quad \text{in } V',\ \text{a.e. in } (0, \hat{T}), \tag{37}$$

$$\varrho(\cdot, 0) = \varrho_0(x) \quad \text{a.e. in } \Omega. \tag{38}$$

4a. Existence of a solution $\varrho = \mathcal{T}_1(\tilde{\theta})$ to Problem 1

Lemma 4.1. *Let* (H4)–(H6) *hold. Then Problem* 1 *admits one and only one solution ϱ such that*

$$\underline{\varrho} \leqslant \varrho \leqslant \overline{\varrho} \quad \text{a.e. in } Q_{\hat{T}}. \tag{39}$$

Furthermore, we have

$$\|\varrho_t\|_{L^{2p}(Q_{\hat{T}})}, \quad \|A\varrho\|_{L^{2p}(Q_{\hat{T}})} \leqslant c_0(\hat{T}^{1/2p} + R + 1), \tag{40}$$

where the constant c_0 is allowed to depend on $\underline{\varrho}$, $\overline{\varrho}$, $|\Omega|$ and on $\|\varrho_0\|_{W^{2-1/p,2p}(\Omega)}$.

It is established in [37] that there exists one and only one solution ϱ satisfying Problem 1 and such that (39)–(40) hold true. Now, we introduce the set

$$\Xi_p = \Xi_p(\hat{T}) = \{v \in W^{1,2p}(0,\hat{T}; L^{2p}(\Omega)) \cap L^{2p}(0,\hat{T}; W^{2,2p}(\Omega));\ \underline{\varrho} \leqslant v \leqslant \overline{\varrho} \text{ a.e.}\}, \tag{41}$$

which we endow with its natural norm. Moreover, paralleling the preceding step, we also introduce the convex and closed set $\Xi_p(\hat{T}, R)$ which consists of the functions $\varrho \in \Xi_p$ satisfying relation (40) with precisely this choice of c_0. Now, we introduce

Problem 2. Given $R, \hat{T} > 0$ and $\varrho \in \Xi_p(\hat{T}, R)$, find a function $\theta = \mathcal{T}_2(\varrho)\colon\ Q_{\hat{T}} \to \mathbb{R}$ such that

$$\theta \in H^1(0,\hat{T}; H) \cap L^\infty(0,\hat{T}; V), \quad u = \gamma(\theta) \in H^1(0,\hat{T}; V'), \tag{42}$$

$$u_t + J\theta = k + \theta\varrho\varrho_t + \varrho_t^2 \quad \text{in } V', \text{ a.e. in } (0,\hat{T}), \tag{43}$$

$$\theta(0) = \theta_0 \quad \text{a.e. in } \Omega. \tag{44}$$

4b. Existence of a solution $\theta = \mathcal{T}_2(\varrho)$ to Problem 2

Lemma 4.2. *Let* (C1)–(C2) *and* (H1)–(H3) *hold. Then Problem* 2 *admits one and only one solution θ such that the positivity condition* (34) *holds, for a, b depending only on the quantities specified in the statement of Theorem* 4.1. *Also, there exist m_0 and m_1 such that*

$$\|\theta\|_{H^1(0,\hat{T};H)} + \|\theta\|_{L^\infty(0,\hat{T};V)} \leqslant m_0(R)m_1(\hat{T}). \tag{45}$$

Proof. First, for the outcome of our results, we have to state the next basic lemma, which will be useful in the sequel.

Lemma 4.3. *Let η_ε, γ_ε, and α_ε be as above. Then there exists a constant $a \in (0,1)$ fulfilling*

$$a\varepsilon^{p-1} \leqslant \alpha'_\varepsilon(r) \leqslant \frac{1}{\varepsilon}, \tag{46}$$

$$\varepsilon \leqslant \gamma'_\varepsilon(r) \leqslant \frac{1}{a\varepsilon^{p-1}} \tag{47}$$

for any $\varepsilon \in (0,1)$ and $r \in \mathbb{R}$.

Proof. Taking $\varepsilon \in (0,1)$ and $r \in \mathbb{R}$, we have

$$\alpha'_\varepsilon(r) = \begin{cases} \dfrac{1}{\varepsilon + G(\gamma_\varepsilon^{-1}(r)) + \gamma_\varepsilon^{-1}(r)G'(\gamma_\varepsilon^{-1}(r))} & \text{if } 0 \leqslant \gamma_\varepsilon^{-1}(r) \leqslant \dfrac{1}{\varepsilon}, \\ \dfrac{1}{\varepsilon - 2\gamma_\varepsilon^{-1}(r)} & \text{if } -\dfrac{1}{\varepsilon} \leqslant \gamma_\varepsilon^{-1}(r) \leqslant 0, \\ \dfrac{1}{\varepsilon + G(1/\varepsilon)} & \text{if } \gamma_\varepsilon^{-1}(r) \geqslant \dfrac{1}{\varepsilon}, \\ \dfrac{1}{\varepsilon + 1/\varepsilon} & \text{if } \gamma_\varepsilon^{-1}(r) \leqslant -\dfrac{1}{\varepsilon}. \end{cases} \tag{48}$$

Put $X = \gamma_\varepsilon^{-1}(r)$ for $r \in \mathbb{R}$. We can easily deduce from (C1) that, taking $c_3 = c_1(1 + 1/(p-1))$ and $c_4 = c_2(1 + 1/(p-1))$, we have the relation

$$c_3 r^{p-1} \leqslant \gamma'(r) \leqslant c_4 r^{p-1} \quad \forall\, r > 0. \tag{49}$$

⋆ For $0 \leqslant X \leqslant 1/\varepsilon$, this yields that

$$\varepsilon \leqslant \varepsilon + \gamma'(X) \leqslant \frac{1}{\varepsilon} + \frac{c_4}{\varepsilon^{p-1}} \leqslant \frac{c_5}{\varepsilon^{p-1}}, \tag{50}$$

where $c_5 = 2\sup\{1, c_4\}$.

⋆ For $-1/\varepsilon \leqslant X \leqslant 0$, we get

$$\varepsilon \leqslant \varepsilon - 2X \leqslant \frac{3}{\varepsilon^{p-1}}. \tag{51}$$

⋆ Moreover,

$$\varepsilon \leqslant \varepsilon + G\Big(\frac{1}{\varepsilon}\Big) \leqslant \frac{1}{\varepsilon} + \frac{c_6}{\varepsilon^{p-1}} \leqslant \frac{c_7}{\varepsilon^{p-1}}, \tag{52}$$

where $c_7 = 2\sup\{1, c_6\}$. Finally,

$$\varepsilon \leqslant \varepsilon + \frac{1}{\varepsilon} \leqslant \frac{2}{\varepsilon^{p-1}}. \tag{53}$$

Setting $a = 1/\sup\{3, c_5, c_7\}$, it turns out from the above inequalities that for all $r \in \mathbb{R}$

$$a\varepsilon^{p-1} \leqslant \alpha'_\varepsilon(r) \leqslant \frac{1}{\varepsilon}. \tag{54}$$

We deduce immediately that for all $r \in \mathbb{R}$

$$\varepsilon \leqslant \gamma'_\varepsilon(r) \leqslant \frac{1}{a\varepsilon^{p-1}}. \tag{55}$$

Knowing that $\alpha_\varepsilon(0) = 0$ and integrating (54) with respect to r yields

$$\begin{cases} a\varepsilon^{p-1} r \leqslant \alpha_\varepsilon(r) \leqslant \dfrac{r}{\varepsilon} & \forall\, r \in \mathbb{R}^+, \\ \dfrac{r}{\varepsilon} \leqslant \alpha_\varepsilon(r) \leqslant a\varepsilon^{p-1} r & \forall\, r \in \mathbb{R}^-. \end{cases} \tag{56}$$

□

Let τ and μ be two positive constants. We define the set

$$\mathcal{H}_{p,\mu} = \mathcal{H}_{p,\mu}(\tau) = \{z \in L^{2p}(Q_\tau);\ \|z\|_{L^{2p}(Q_\tau)} \leqslant \mu\}, \tag{57}$$

which is convex and closed in $L^{2p}(Q_\tau)$. To prove the existence result for Problem 2, we introduce an approximation of this problem. Let us consider the following problem for $\varepsilon > 0$:

$$u_\varepsilon \in H^1(0,\tau;H) \cap L^\infty(0,\tau;V), \qquad \alpha_\varepsilon(u_\varepsilon) \in L^2(0,\tau;W), \tag{58}$$

$$u_{\varepsilon t} + J(\alpha_\varepsilon(u_\varepsilon)) = k + \varrho_t^2 + \alpha_\varepsilon(u_\varepsilon)\varrho\varrho_t \quad \text{in } V',\ \text{ a.e. in}(0,\tau), \tag{59}$$

$$u_\varepsilon(0) = \gamma_\varepsilon(\theta_0) \quad \text{a.e. in } \Omega. \tag{60}$$

The proof of existence of solutions to the approximating problem for (58)–(60) essentially consists of the following two lemmas. The first is a well-known result on Stefan problems (see [22, Theorem 3.3] and [23]).

Lemma 4.4. *Let* (C1)–(C2) *and* (H1)–(H3) *hold, let* $\mu, \tau > 0$, $\tilde{u} \in \mathcal{H}_{p,\mu}$. *Moreover, we assume that* $\varphi_1 \in L^p(Q_\tau)$, $\varphi_2 \in L^{2p}(Q_\tau)$, *and* $\varphi_1 \geqslant 0$ *almost everywhere. Then the problem*

$$u = u_\varepsilon \in H^1(0,\tau;H) \cap L^\infty(0,\tau;V), \quad \alpha_\varepsilon(u) \in L^2(0,\tau;W), \tag{61}$$

$$u_t + J(\alpha_\varepsilon(u)) = (k + \varphi_1) + \alpha_\varepsilon(\tilde{u})\varphi_2 \quad \textit{in } V',\ \text{a.e. } \textit{in } (0,\tau), \tag{62}$$

$$u(0) = \gamma_\varepsilon(\theta_0) \quad \text{a.e. } \textit{in } \Omega, \tag{63}$$

has one and only one solution.

The next step consists in showing that the operator $\tilde{u} \mapsto u$, where u is the solution furnished by the previous lemma, has a fixed point, at least for small times. We prove this by using again Schauder's theorem in the space $\mathcal{H}_{p,\mu}$. We note that we denote by τ this small final time to distinguish it from the final time $\hat{T}$ appearing in the statement of Theorem 4.1.

Lemma 4.5. *Let* (C1)–(C2) *and* (H1)–(H3) *hold and let* φ_1, φ_2 *be as above. Then, for every* $\varepsilon \in (0,1)$ *there exists* $\tau = \tau(p,\mu,\varepsilon) > 0$ *and at least one function* $u = u_\varepsilon \colon Q_\tau \to \mathbb{R}$ *fulfilling:*

$$u \in H^1(0,\tau;H) \cap L^\infty(0,\tau;V), \qquad \alpha_\varepsilon(u) \in L^2(0,\tau;W), \tag{64}$$

$$u_t + J(\alpha_\varepsilon(u)) = (k+\varphi_1) + \alpha_\varepsilon(u)\varphi_2 \quad \text{in } V', \ \text{ a.e. in } (0,\tau), \tag{65}$$

$$u(0) = \gamma_\varepsilon(\theta_0) \qquad \text{a.e. in } \Omega. \tag{66}$$

Proof. We fix an arbitrary $\mu > 0$ and denote by $\mathcal{S}$ the map $\tilde{u} \mapsto u$, where u is the solution of (61)–(63). We have to show the well-posedness, continuity, and compactness of the operator $\mathcal{S}$. Throughout the proof, all constants c (or $c(\varepsilon)$) will be allowed to depend on φ_1, φ_2, in addition to p and the parameters in (H1)–(H6). □

Lemma 4.6. *Under the hypotheses of Lemma* 4.5*, there exists* $\tau = \tau(p,\mu,\varepsilon) > 0$ *such that* $\mathcal{S}$ *maps* $\mathcal{H}_{p,\mu}$ *onto itself.*

Proof. Multiplying (62) by the time derivative of $\alpha_\varepsilon(u)$, then using Young's and Hölder's inequalities and recalling (46), yields

$$\frac{a\varepsilon^{p-1}}{2}|u_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\alpha_\varepsilon(u)\|_J^2 \leqslant c(\varepsilon)|\varphi_1 + \alpha_\varepsilon(\tilde{u})\varphi_2|_p^2 + \int_\Gamma g\frac{\partial\alpha_\varepsilon(u)}{\partial t}\,\mathrm{d}\sigma. \tag{67}$$

Now, we integrate the above relation between 0 and a generic $t \leqslant \tau$. Then, using the continuity of the trace operator $V \hookrightarrow L^2(\Gamma)$, integrating by parts in time and recalling (H1) and (56) leads to

$$\begin{aligned} &\|u_t\|^2_{L^2(Q_t)} + \|\alpha_\varepsilon(u(t))\|_J^2 \\ &\quad\leqslant c(\varepsilon)\{1 + t + \|\varphi_1\|^p_{L^p(Q_t)} + \|\alpha_\varepsilon(\tilde{u})\|^{2p}_{L^{2p}(Q_t)} \\ &\qquad + \|\varphi_2\|^{2p}_{L^{2p}(Q_t)} + \|g\|^2_{H^1(0,t;L^2(\Gamma))} + \|\alpha_\varepsilon(u)\|^2_{L^2(0,t;V)}\}. \end{aligned} \tag{68}$$

Applying Gronwall's lemma, it turns out from (56) and the continuous embedding $V \hookrightarrow L^{2p}(\Omega)$ that

$$\begin{aligned} \|u\|^2_{L^{2p}(Q_\tau)} &\leqslant \tau^{1/p}\|u\|^2_{L^\infty(0,\tau;L^{2p}(\Omega))} \leqslant c\tau^{1/p}\|u\|^2_{L^\infty(0,\tau;V)} \\ &\leqslant c(\varepsilon)\tau^{1/p}\|\alpha_\varepsilon(u)\|^2_{L^\infty(0,\tau;V)} \leqslant c_8(\varepsilon)\tau^{1/p}(1+\tau+\mu^{2p}). \end{aligned} \tag{69}$$

Thus, for any arbitrary μ, we can choose τ (depending on ε, p, and μ, of course) small enough so that

$$c_8(\varepsilon)\tau^{1/p}(1+\tau+\mu^{2p}) \leqslant \mu^2, \tag{70}$$

whence $\theta \in \mathcal{H}_{p,\mu}$. □

Lemma 4.7. *Let the hypotheses of Lemma* 4.5 *hold and let τ be as in* (70)*. Then the map $\mathcal{S}$ is continuous and compact (with respect to the natural topology induced in $\mathcal{H}_{p,\mu}$ by $L^{2p}(Q_\tau)$).*

Proof. We consider a sequence $(\tilde{u}_n)_n \subset \mathcal{H}_{p,\mu}$ and $\tilde{u} \in \mathcal{H}_{p,\mu}$ such that

$$\tilde{u}_n \underset{n\to\infty}{\longrightarrow} \tilde{u} \quad \text{strongly in } L^{2p}(Q_\tau), \tag{71}$$

and set $u_n = \mathcal{S}\tilde{u}_n$ and $u = \mathcal{S}\tilde{u}$. Then u and u_n fulfil (62) (in which $\tilde{u}$ will be substituted by $\tilde{u}_n$). Proceeding exactly as in the previous estimates (cf. (68)), we can find a positive constant not depending on n such that

$$\|u_n\|_{H^1(0,\tau;H)\cap L^\infty(0,\tau;V)} \leqslant c. \tag{72}$$

On the other hand, we can deduce from the generalized Aubin Lemma [40, Corollary 4] that

$$H^1(0,\tau;H)\cap L^\infty(0,\tau;V) \subset\subset L^{2p}(Q_\tau) \tag{73}$$

is a compact embedding. Thus, there exist a subsequence of n, denoted by n_k, $k\in\mathbb{N}$, and $u_1 \in H^1(0,\tau;H)\cap L^\infty(0,\tau;V)$ such that

$$u_{n_k} \longrightarrow u_1 \quad \text{strongly in } L^{2p}(Q_\tau). \tag{74}$$

The above convergences (71) and (74) allow us to pass to the limit in equation (62), as n goes to infinity. Moreover, thanks to the uniqueness result holding by Lemma 4.4, the whole sequence (u_n) converges to u_1 and we can identify $u_1 = u$. Finally, the proof of the compactness of $\mathcal{S}$ can be achieved similarly owing to (68) and (73). This yields the existence of a fixed-point for $\mathcal{S}$ which represents a solution to problem (64)–(66) and, hence, to the approximating problem (58)–(60). □

We now show the positivity of this approximating solution.

Lemma 4.8. *Let* (C1)–(C2) *and* (H1)–(H3) *hold and let* $\hat{T} > 0$, φ_1, φ_2 *be as in Lemma* 4.4, *and let* u *be any solution to* (64)–(66) *(with* $\tau = \hat{T}$*). Then* $u \geqslant 0$ *a.e. in* $Q_{\hat{T}}$.

P r o o f. We test (65) by $-u^-$ and obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^-|^2 &+ \int_\Omega {\alpha_\varepsilon}'(u)|\nabla u^-|^2\,\mathrm{d}x - n_0\int_\Gamma \alpha_\varepsilon(u)u^-\,\mathrm{d}\sigma \\ &+ \int_\Omega \varphi_1 u^-\,\mathrm{d}x + \int_\Gamma g u^-\,\mathrm{d}\sigma = -\int_\Omega \alpha_\varepsilon(u)\varphi_2 u^-\,\mathrm{d}x.\end{aligned} \tag{75}$$

First, it follows from (46) and (56) that

$$\int_\Omega \alpha_\varepsilon'(u)|\nabla u^-|^2\,\mathrm{d}x - n_0\int_\Gamma \alpha_\varepsilon(u)u^-\,\mathrm{d}x \geqslant a\varepsilon^{p-1}\|u^-\|_J^2. \tag{76}$$

Secondly, letting $q = 2p/(p-1)$ then

$$\begin{cases} q > 2 & \text{if } N \leqslant 2,\\ 2 < q < 6 & \text{if } N = 3,\end{cases} \tag{77}$$

and, hence, the embedding $V \hookrightarrow L^q(\Omega)$ is continuous. Finally, using the fact that g and φ_1 are nonnegative and combining (75) and (76) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^-|^2 + a\varepsilon^{p-1}\|u^-\|_J^2 &\leqslant a\varepsilon^{p-1}|\varphi_2|_{2p}|u^-|_q|u^-| \\ &\leqslant \frac{a\varepsilon^{p-1}}{2}\|u^-\|_J^2 + c|\varphi_2|_{2p}^2|u^-|^2.\end{aligned} \tag{78}$$

An application of Gronwall's Lemma implies that $u^- = 0$ a.e. in Q_τ, as desired, due to (H1). □

Now, we aim at establishing the lower bound for the approximating solutions holding by Lemma 4.5.

Lemma 4.9. *Let* (C1)–(C2) *and* (H1)–(H4) *hold and let* $\hat{T}, R > 0$. *Let also* $\varphi_1 = \varrho_t^2$ *and* $\varphi_2 = \varrho\varrho_t$, *where* $\varrho \in \Xi_p(\hat{T}, R)$. *Then, given any solution to* (64)–(66) *(with* $\tau = \hat{T}$*) and setting* $\theta = \alpha_\varepsilon(u)$, θ *satisfies* (34) *(where* a, b *depend on other quantities as specified in Theorem* 4.1*)*.

P r o o f. We start by noticing that the functions φ_1 and φ_2 satisfy the assumptions stated in Lemma 4.4. Then we rewrite relation (65) as

$$\frac{\partial\gamma_\varepsilon(\theta)}{\partial t} + J\theta = (k + \varphi_1) + \theta\varphi_2, \tag{79}$$

where we have set $\theta = \alpha_\varepsilon(u)$. Let $q > p$. We multiply (79) by $-\theta^{-q}$ and integrate over Q_t, $t \leqslant \hat{T}$. From (H3) we deduce the relation

$$
\begin{aligned}
(80) \qquad & -\int_0^t \int_\Omega \frac{\varepsilon + \eta_\varepsilon'(\theta)}{\theta^q}\, \theta_t \,\mathrm{d}x\,\mathrm{d}s \\
& + \frac{4q}{(q-1)^2} \int_0^t \int_\Omega \Big| \nabla \frac{1}{\theta^{(q-1)/2}} \Big|^2 \mathrm{d}x\,\mathrm{d}s + \underline{g} \int_0^t \int_\Gamma \frac{\mathrm{d}\sigma\,\mathrm{d}s}{\theta^q} + \int_0^t \int_\Omega \frac{\varphi_1}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s \\
& \qquad \leqslant n_0 \int_0^t \int_\Gamma \frac{\mathrm{d}\sigma\,\mathrm{d}s}{\theta^{q-1}} - \int_0^t \int_\Omega \frac{\varphi_2}{\theta^{q-1}}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

We treat the above inequality term by term, separately. First, we have

$$
\begin{aligned}
(81) \qquad & -\int_0^t \int_\Omega \frac{\varepsilon + \eta_\varepsilon'(\theta)}{\theta^q}\, \theta_t \,\mathrm{d}x\,\mathrm{d}s \\
& = \frac{\varepsilon}{q-1} \int_\Omega \frac{\mathrm{d}x}{\theta^{q-1}(t)} - \frac{\varepsilon}{q-1} \int_\Omega \frac{\mathrm{d}x}{\theta^{q-1}(0)} - \int_0^t \int_\Omega \frac{\eta_\varepsilon'(\theta)}{\theta^q}\, \theta_t \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

The last term in the above equation is written as

$$
(82) \qquad -\int_0^t \int_\Omega \frac{\eta_\varepsilon'(\theta)}{\theta^q}\, \theta_t \,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \hat{\xi}_{\varepsilon,q}(\theta(t))\,\mathrm{d}x - \int_\Omega \hat{\xi}_{\varepsilon,q}(\theta(0))\,\mathrm{d}x,
$$

where

$$
(83) \qquad \hat{\xi}_{\varepsilon,q}(r) = -\int_1^r \frac{\eta_\varepsilon'(s)}{s^q}\,\mathrm{d}s, \quad \forall\, r \in (0, +\infty).
$$

Let $w\colon\ \Omega \to [0, +\infty)$ be a positive measurable function. Then

$$
\begin{aligned}
(84) \int_\Omega \hat{\xi}_{\varepsilon,q}(w(x))\,\mathrm{d}x = & -\int_\Omega \int_1^{1/\varepsilon} \frac{\eta_\varepsilon'(s)}{s^q}\,\mathrm{d}s\,\mathrm{d}x - \int_\Omega \int_{1/\varepsilon}^{w(x)} \frac{\eta_\varepsilon'(s)}{s^q}\,\mathrm{d}s\,\mathrm{d}x \\
= & -|\Omega| \int_1^{1/\varepsilon} \frac{\gamma'(s)}{s^q}\,\mathrm{d}s + \int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}} \int_{w(x)}^{1/\varepsilon} \frac{\gamma'(s)}{s^q}\,\mathrm{d}s\,\mathrm{d}x \\
& - G\Big(\frac{1}{\varepsilon}\Big) \int_{\{x\in\Omega;\ w(x)\geqslant 1/\varepsilon\}} \int_{1/\varepsilon}^{w(x)} \frac{\mathrm{d}s\,\mathrm{d}x}{s^q}.
\end{aligned}
$$

On the one hand, from (84), (49), and (C1) we obtain

$$
\begin{aligned}
(85)\quad & \int_\Omega \hat{\xi}_{\varepsilon,q}(w(x))\,\mathrm{d}x \\
&\geqslant -c_4|\Omega|\int_1^{1/\varepsilon}\frac{\mathrm{d}s}{s^{q-p+1}} + c_3\int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}}\int_{w(x)}^{1/\varepsilon}\frac{\mathrm{d}s}{s^{q-p+1}}\,\mathrm{d}x \\
&\quad - G\Big(\frac{1}{\varepsilon}\Big)\int_\Omega\int_{1/\varepsilon}^{w(x)}\frac{\mathrm{d}s}{s^q}\,\mathrm{d}x \\
&\geqslant -\frac{c_4|\Omega|}{q-p}(1-\varepsilon^{q-p}) + \frac{c_3}{q-p}\int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}}\Big(\frac{1}{w^{q-p}(x)} - \varepsilon^{q-p}\Big)\,\mathrm{d}x \\
&\quad + \frac{c/\varepsilon^{p-1}}{q-1}\int_\Omega\Big(\frac{1}{w^{q-1}(x)} - \varepsilon^{q-1}\Big)\,\mathrm{d}x \\
&\geqslant -\frac{|\Omega|(c_3+c_4)}{q-p} - \frac{c\varepsilon^{q-p}|\Omega|}{q-1} \\
&\quad + \frac{c_3}{q-p}\int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}}\frac{\mathrm{d}x}{w^{q-p}(x)} + \frac{c}{q-1}\int_\Omega\frac{\mathrm{d}x}{w^{q-1}(x)} \\
&\geqslant -c + \frac{c_3}{q-p}\int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}}\frac{\mathrm{d}x}{w^{q-p}(x)} + \frac{c}{q-1}\int_\Omega\frac{\mathrm{d}x}{w^{q-1}(x)}.
\end{aligned}
$$

On the other hand, it follows from (84) and (49) that

$$
\begin{aligned}
(86)\qquad \int_\Omega \hat{\xi}_{\varepsilon,q}(w)\,\mathrm{d}x &\leqslant c_4\int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}}\int_{w(x)}^{1/\varepsilon}\frac{\mathrm{d}s}{s^{q-p+1}}\,\mathrm{d}x \\
&\leqslant \frac{c_4}{q-p}\int_{\{x\in\Omega;\ w(x)\leqslant 1/\varepsilon\}}\frac{1}{w^{q-p}(x)}\,\mathrm{d}x.
\end{aligned}
$$

At this stage, due to (80)–(86) and (H1), we have

$$
\begin{aligned}
(87)\qquad & \frac{\varepsilon}{q-1}\int_\Omega\frac{\mathrm{d}x}{\theta^{q-1}(t)} + \frac{4q}{(q-1)^2}\int_0^t\int_\Omega\Big|\nabla\frac{1}{\theta^{(q-1)/2}}\Big|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{c_3}{q-p}\int_{\{x\in\Omega;\ \theta(x,t)=\theta(t)\leqslant 1/\varepsilon\}}\frac{\mathrm{d}x}{\theta^{q-p}(t)} + \frac{c}{q-1}\int_\Omega\frac{\mathrm{d}x}{\theta^{q-1}(t)} \\
&\quad + \underline{g}\int_0^t\int_\Gamma\frac{\mathrm{d}\sigma\,\mathrm{d}s}{\theta^q} + \int_0^t\int_\Omega\frac{\varphi_1}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant n_0\int_0^t\int_\Gamma\frac{\mathrm{d}\sigma\,\mathrm{d}s}{\theta^{q-1}} - \int_0^t\int_\Omega\frac{\varphi_2}{\theta^{q-1}}\,\mathrm{d}x\,\mathrm{d}s + \frac{\varepsilon|\Omega|}{(q-1)\underline{\theta}^{q-1}} \\
&\quad + \frac{c_4|\Omega|}{(q-p)\underline{\theta}^{q-p}} + c.
\end{aligned}
$$

Secondly, the first term of the right-hand side of the above inequality is bounded as follows:

$$n_0 \int_0^t \int_\Gamma \frac{\mathrm{d}\sigma}{\theta^{q-1}}\,\mathrm{d}s \leqslant \frac{\underline{\underline{g}}}{2} \int_0^t \int_\Gamma \frac{\mathrm{d}\sigma\,\mathrm{d}s}{\theta^q} + \frac{c}{q}\Big(\frac{2n_0}{\underline{g}}\Big)^q t. \tag{88}$$

We obtain owing to Young's inequality and (41) that

$$\begin{aligned}
-\int_0^t \int_\Omega \frac{\varphi_2}{\theta^{q-1}}\,\mathrm{d}x\,\mathrm{d}s &= -\int_0^t \int_\Omega \frac{\varrho\varrho_t}{\theta^{(q-2)/2}\theta^{q/2}}\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \frac{1}{2}\int_0^t \int_\Omega \frac{\varrho_t^2}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_\Omega \frac{\varrho^2}{\theta^{q-2}}\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \frac{1}{2}\int_0^t \int_\Omega \frac{\varphi_1}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + c\int_0^t \int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-2}} \\
&\leqslant \frac{1}{2}\int_0^t \int_\Omega \frac{\varphi_1}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + \frac{c}{p-1}\int_0^t \int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-p}} + c\,\frac{p-2}{p-1}\int_0^t \int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-1}} \\
&\leqslant \frac{1}{2}\int_0^t \int_\Omega \frac{\varphi_1}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s + c\int_0^t \int_{\{x\in\Omega;\ \theta(x,s)\leqslant 1/\varepsilon\}} \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-p}(x,s)} \\
&\quad + c\int_0^t \int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-1}} + ct.
\end{aligned} \tag{89}$$

It follows from (87)–(89) that

$$\begin{aligned}
&\frac{\varepsilon}{q-1}\int_\Omega \frac{\mathrm{d}x}{\theta^{q-1}(t)} + \frac{4q}{(q-1)^2}\int_0^t \int_\Omega \Big|\nabla \frac{1}{\theta^{(q-1)/2}}\Big|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{c_3}{q-p}\int_{\{x\in\Omega;\ \theta(x,t)=\theta(t)\leqslant 1/\varepsilon\}} \frac{\mathrm{d}x}{\theta^{q-p}(t)} + \frac{c}{q-1}\int_\Omega \frac{\mathrm{d}x}{\theta^{q-1}(t)} \\
&\quad + \frac{\underline{\underline{g}}}{2}\int_0^t \int_\Gamma \frac{\mathrm{d}\sigma\,\mathrm{d}s}{\theta^q} + \frac{1}{2}\int_0^t \int_\Omega \frac{\varphi_1}{\theta^q}\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant c\int_0^t \int_{\{x\in\Omega;\ \theta(x,s)\leqslant 1/\varepsilon\}} \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-p}(x,s)} + c\int_0^t \int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-1}} + \frac{c}{q}\Big(\frac{2n_0}{\underline{g}}\Big)^q t + ct \\
&\quad + \frac{c}{(q-1)\underline{\theta}^{q-1}} + \frac{c}{(q-p)\underline{\theta}^{q-p}} + c
\end{aligned} \tag{90}$$

for all $\varepsilon \in (0,1)$. The positive constants c are independent of t, ε, and q. In order to reduce the notation, we set

$$\sigma_q(s) = \int_{\{x\in\Omega;\ \theta(x,s)\leqslant 1/\varepsilon\}} \frac{\mathrm{d}x}{\theta^{q-p}(x,s)}, \quad s \in (0,\hat{T}). \tag{91}$$

Thus, (90) entails

$$
\tag{92}
\begin{aligned}
&\frac{c_3}{q-p}\sigma_q(t)+\frac{c}{q-1}\int_\Omega \frac{\mathrm{d}x}{\theta^{q-1}(t)}\\
&\qquad\leqslant c\int_0^t \sigma_q(s)\,\mathrm{d}s + c\int_0^t\int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-1}}+\frac{c}{q}\Big(\frac{4n_0}{\underline{g}}\Big)^q t + ct\\
&\qquad\quad+\frac{c}{(q-1)\underline{\theta}^{q-1}}+\frac{c}{(q-p)\underline{\theta}^{q-p}}+c.
\end{aligned}
$$

We set

$$
\begin{aligned}
D(t)&=\frac{1}{q}\Big(\frac{4n_0}{\underline{g}}\Big)^q t + t + \frac{1}{(q-1)\underline{\theta}^{q-1}}+\frac{1}{(q-p)\underline{\theta}^{q-p}}+1,\\
y(t)&=\int_0^t \sigma_q(s)\,\mathrm{d}s,\\
z(t)&=\int_0^t\int_\Omega \frac{\mathrm{d}x\,\mathrm{d}s}{\theta^{q-1}}.
\end{aligned}
$$

We deduce from (92) an ordinary differential inequality of the form

$$
\tag{93}
\frac{1}{q-1}(y'(t)+z'(t))\leqslant c(y(t)+z(t))+cD(t).
$$

We multiply by $\mathrm{e}^{-c(q-1)t}$ and then integrate between 0 and $t\leqslant \hat{T}$ to obtain

$$
\tag{94}
y(t)+z(t)\leqslant c(\hat{T})\mathrm{e}^{c(q-p)t}.
$$

Consequently, (92) becomes, for any $q>p$ and any $t\in(0,\hat{T})$,

$$
\tag{95}
\begin{aligned}
&\sigma_q(t)+\int_\Omega \frac{\mathrm{d}x}{\theta^{q-1}(t)}\\
&\quad\leqslant c\Big\{\Big(\frac{2n_0}{\underline{g}}\Big)^q t + (q-1)t + \frac{1}{\underline{\theta}^{q-1}}+\frac{q-1}{q-p}\frac{1}{\underline{\theta}^{q-p}}+(q-1)\Big\}\mathrm{e}^{c(q-1)t}.
\end{aligned}
$$

Finally, it follows from the above estimate that

$$
\tag{96}
\begin{aligned}
\|\theta^{-1}(x,t)\|_{L^{q-p}(\Omega)}&\leqslant 2^{1/(q-p)-1}\sigma_q^{1/(q-p)}(t)+2^{1/(q-p)-1}|\Omega|^{1/(q-p)}\varepsilon\\
&\leqslant c2^{3/(q-p)}\Big\{\Big(\frac{2n_0}{\underline{g}}\Big)^{q/(q-p)}t^{1/(q-p)}+(q-1)^{1/(q-p)}t^{1/(q-p)}\\
&\quad+\Big(\frac{q-1}{q-p}\Big)^{1/(q-p)}\frac{1}{\underline{\theta}}+\frac{1}{\underline{\theta}^{(q-1)/(q-p)}}+(q-1)^{1/(q-p)}\Big\}\\
&\quad\times \mathrm{e}^{ct(q-1)/(q-p)}+2^{1/(q-p)-1}|\Omega|^{1/(q-p)}\varepsilon,
\end{aligned}
$$

where the positive constants c are still independent of t, q, and ε. Now, letting $q \to \infty$ yields

$$\|\theta^{-1}\|_{L^\infty(\Omega)} \leqslant c_9 \Big\{ \frac{1}{\underline{\theta}} + \frac{n_0}{\underline{g}} + 1 \Big\} \mathrm{e}^{c_{10}\hat{T}} + \frac{\varepsilon}{2} \quad \forall\, \varepsilon \in (0,1), \tag{97}$$

where c_9, c_{10} only depend on $\underline{\varrho}$, $\overline{\varrho}$, $|\Omega|$ and $|\Gamma|$. In particular, (97) holds true for $\varepsilon = 0$. Thus, denoting $\theta_* = a\mathrm{e}^{-b\hat{T}}$ with $1/a = c_9\{1/\underline{\theta} + n_0/\underline{g} + 1\}$ and $b = c_{10}$, we deduce that $\theta \geqslant \theta_*$ a.e. in $Q_{\hat{T}}$. $\square$

The next lemma concerns the global existence of the solution to the system (61)–(63).

Lemma 4.10. *Let* (C1)–(C2) *and* (H1)–(H3) *hold and let* $\hat{T} > 0$. *Let also* φ_1, φ_2 *be as in Lemma* 4.9. *Then there exists at least one solution* $u\colon Q_{\hat{T}} \to \mathbb{R}$ *satisfying*

$$u \in H^1(0,\hat{T};H) \cap L^\infty(0,\hat{T};V), \tag{98}$$

$$u_t + J(\alpha(u)) = (k + \varphi_1) + \alpha(u)\varphi_2 \quad \text{in } V', \ \text{a.e. in } (0,\hat{T}), \tag{99}$$

$$u(0) = \gamma(\theta_0) \quad \text{a.e. in } \Omega. \tag{100}$$

Moreover, $\theta = \alpha(u)$ *satisfies* (34).

P r o o f. We will now denote by u_ε the approximating solution given by Lemma 4.5. Our aim is to derive a priori bounds on u_ε which allow us to pass to the limit when ε goes to zero.

We first test (65) by $\alpha_\varepsilon(u_\varepsilon) = \theta_\varepsilon$ and then use the continuous embeddings $L^p(\Omega) \hookrightarrow V'$ and $V \hookrightarrow L^q(\Omega)$ for $q = 2p/(p-1)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \hat{\alpha}_\varepsilon(u_\varepsilon)\,\mathrm{d}x + \frac{1}{2}\|\alpha_\varepsilon(u_\varepsilon)\|_J^2 \leqslant c|\varphi_1|_p^2 + \|k\|_{V'}^2 + c|\varphi_2|_{2p}^2 |\alpha_\varepsilon(u_\varepsilon)|^2. \tag{101}$$

By (C2) we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|\hat{\alpha}_\varepsilon(u_\varepsilon)|_1 + \frac{1}{2}\|\alpha_\varepsilon(u_\varepsilon)\|_J^2 \\ &\qquad \leqslant c|\varphi_1|_p^2 + \|k\|_{V'}^2 + c|\varphi_2|_{2p}^2 |\hat{\alpha}_\varepsilon(u_\varepsilon)|_1 + c|\varphi_2|_{2p}^2. \end{aligned} \tag{102}$$

From Gronwall's Lemma we infer that

$$\begin{aligned} &|\hat{\alpha}_\varepsilon(u_\varepsilon)(t)|_1 + \|\alpha_\varepsilon(u_\varepsilon)\|_{L^2(0,t;V)}^2 \\ &\quad \leqslant c\Big(1 + \int_0^t |\varphi_1(s)|_p^2\,\mathrm{d}s + \|k\|_{L^2(0,t;V')}^2 + \int_0^t |\varphi_2(s)|_{2p}^2\,\mathrm{d}s\Big)\mathrm{e}^{c\int_0^{\hat{T}} |\varphi_2(s)|_{2p}^2\,\mathrm{d}s}. \end{aligned} \tag{103}$$

Moreover, applying Hölder's inequality with respect to time and using the fact that $\varrho \in \Xi_p(\hat{T}, R)$ yields that, for any $t \in (0, \hat{T})$,

$$\begin{aligned} (104) \qquad \int_0^t |\varphi_1(s)|_p^2 \, \mathrm{d}s &= \int_0^t |\varrho_t|_{2p}^4 \, \mathrm{d}s \leqslant \hat{T}^{(p-2)/p} \|\varrho_t\|_{L^{2p}(Q_t)}^4 \\ &\leqslant c\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1), \\ (105) \qquad \int_0^t |\varphi_2(s)|_{2p}^2 \, \mathrm{d}s &= \int_0^t |\varrho\varrho_t|_{2p}^2 \, \mathrm{d}s \leqslant c\hat{T}^{(p-1)/p} \|\varrho_t\|_{L^{2p}(Q_t)}^2 \\ &\leqslant c\hat{T}^{(p-1)/p}(\hat{T}^{1/p} + R^2 + 1). \end{aligned}$$

Finally, it turns out from (103)–(105) that

$$\begin{aligned} (106) \qquad & \|\hat{\alpha}_\varepsilon(u_\varepsilon)\|_{L^\infty(0,\hat{T};L^1(\Omega))}^2 + \|\alpha_\varepsilon(u_\varepsilon)\|_{L^2(0,\hat{T};V)}^2 \\ & \quad \leqslant c\big(1 + \hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) + \hat{T}^{(p-1)/p}(\hat{T}^{1/p} + R^2 + 1)\big) \\ & \quad \times \mathrm{e}^{c\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)}. \end{aligned}$$

Next, we consider the (equivalent) expression (79) and test it by $\theta_{\varepsilon t}$ to obtain

$$\begin{aligned} (107) \qquad & \varepsilon|\theta_{\varepsilon t}|^2 + \int_\Omega \theta_{\varepsilon t}^2 \eta_\varepsilon'(\theta_\varepsilon) \, \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_\varepsilon\|_J^2 \\ & \quad \leqslant \int_\Omega \varphi_1 \theta_{\varepsilon t} \, \mathrm{d}x + \int_\Gamma g\theta_{\varepsilon t} \, \mathrm{d}\sigma + \int_\Omega \varphi_2 \theta_\varepsilon \theta_{\varepsilon t} \, \mathrm{d}x. \end{aligned}$$

Now, we aim to estimate separately the three terms on the right-hand side of (107). Concerning the first, it follows from (C1), (49), Hölder's and Young's inequalities, and Lemma 4.9 that

$$\begin{aligned} (108) \quad \int_\Omega \varphi_1 \theta_{\varepsilon t} \, \mathrm{d}x &= \int_\Omega \frac{\varphi_1}{[\eta_\varepsilon'(\theta_\varepsilon)]^{1/2}} [\eta_\varepsilon'(\theta_\varepsilon)]^{1/2} \theta_{\varepsilon t} \, \mathrm{d}x \\ &\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2 \eta_\varepsilon'(\theta_\varepsilon) \, \mathrm{d}x + \int_\Omega \frac{\varphi_1^2}{\eta_\varepsilon'(\theta_\varepsilon)} \, \mathrm{d}x \\ &\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2 \eta_\varepsilon'(\theta_\varepsilon) \, \mathrm{d}x + \int_{\{\theta_\varepsilon \leqslant 1/\varepsilon\}} \frac{\varphi_1^2}{\gamma'(\theta_\varepsilon)} \, \mathrm{d}x + \int_{\{\theta_\varepsilon \geqslant 1/\varepsilon\}} \frac{\varphi_1^2}{G(1/\varepsilon)} \, \mathrm{d}x \\ &\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2 \eta_\varepsilon'(\theta_\varepsilon) \, \mathrm{d}x + c\int_{\{\theta_\varepsilon \leqslant 1/\varepsilon\}} \frac{\varphi_1^2}{\theta_\varepsilon^{p-1}} \, \mathrm{d}x + c\varepsilon^{p-1}\int_{\{\theta_\varepsilon \geqslant 1/\varepsilon\}} \varphi_1^2 \, \mathrm{d}x \\ &\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2 \eta_\varepsilon'(\theta_\varepsilon) \, \mathrm{d}x + c\mathrm{e}^{b(p-1)\hat{T}} |\varphi_1|_p^2, \end{aligned}$$

where the positive constant c is independent of ε. The second term can be controlled as the first:

$$
\begin{aligned}
(109)\qquad \int_\Omega \varphi_2\theta_\varepsilon\theta_{\varepsilon t}\,\mathrm{d}x &\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x + \int_\Omega \frac{(\varphi_2\theta_\varepsilon)^2}{\eta_\varepsilon'(\theta_\varepsilon)}\,\mathrm{d}x \\
&\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x + c\mathrm{e}^{b(p-1)\hat{T}}\int_\Omega (\varphi_2\theta_\varepsilon)^2\,\mathrm{d}x \\
&\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x + c\mathrm{e}^{b(p-1)\hat{T}}|\varphi_2|_{2p}^2|\theta_\varepsilon|_{2p/(p-1)}^2 \\
&\leqslant \frac{1}{4}\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x + c\mathrm{e}^{b(p-1)\hat{T}}|\varphi_2|_{2p}^2\|\theta_\varepsilon\|_J^2,
\end{aligned}
$$

where c is independent of ε. Combining (107)–(109) yields

$$
\begin{aligned}
(110)\qquad \varepsilon|\theta_{\varepsilon t}|^2 &+ \frac{1}{2}\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_\varepsilon\|_J^2 \\
&\leqslant \int_\Gamma g\theta_{\varepsilon t}\,\mathrm{d}\sigma + c\mathrm{e}^{b(p-1)\hat{T}}(|\varphi_2|_{2p}^2\|\theta_\varepsilon\|_J^2 + |\varphi_1|_p^2).
\end{aligned}
$$

By applying Gronwall's Lemma, we deduce from (104)–(105) that

$$
\begin{aligned}
(111)\quad &\int_0^t\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \|\theta_\varepsilon(t)\|_J^2 \\
&\qquad \leqslant c\Big(\int_0^t\int_\Gamma g\theta_{\varepsilon t}\,\mathrm{d}\sigma\,\mathrm{d}s + c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) + 1\Big) \\
&\qquad\quad \times \mathrm{e}^{c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)}.
\end{aligned}
$$

Now, we integrate by parts with respect to time. It turns out that

$$
\begin{aligned}
(112)\quad &\int_0^t\int_\Omega \theta_{\varepsilon t}^2\eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \|\theta_\varepsilon(t)\|_J^2 \\
&\leqslant c\Big(\int_\Gamma g(t)\theta_\varepsilon(t)\,\mathrm{d}\sigma - \int_0^t\int_\Gamma \theta_\varepsilon\frac{\partial g}{\partial t}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) + 1\Big)\mathrm{e}^{c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)} \\
&\leqslant \frac{1}{4}\|\theta_\varepsilon(t)\|^2 + c\big(\|g\|_{H^1(0,t;L^2(\Gamma))}^2 + \|\theta_\varepsilon\|_{L^2(0,t;V)}^2 \\
&\quad + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) + 1\big)\mathrm{e}^{c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)}.
\end{aligned}
$$

From (H2) and (106) we deduce that

$$(113)\quad \int_0^t \int_\Omega \theta_{\varepsilon t}^2 \eta_\varepsilon'(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \|\theta_\varepsilon\|^2_{L^\infty(0,\hat{T};V)} \\ \leqslant c\big(1 + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) \\ + \hat{T}^{(p-1)/p}(\hat{T}^{1/p} + R^2 + 1)\big)\mathrm{e}^{c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)}.$$

Moreover, due to (49) and (C1), the above inequality can be written as

$$(114)\quad c\int_0^t \int_{\{\theta_\varepsilon \leqslant 1/\varepsilon\}} \theta_\varepsilon^{p-1}\theta_{\varepsilon t}^2\,\mathrm{d}x\,\mathrm{d}s \\ + \frac{c}{\varepsilon^{p-1}}\int_0^t \int_{\{\theta_\varepsilon \geqslant 1/\varepsilon\}} \theta_{\varepsilon t}^2\,\mathrm{d}x\,\mathrm{d}s + \|\theta_\varepsilon\|^2_{L^\infty(0,\hat{T};V)} \\ \leqslant c\big(1 + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) + \hat{T}^{(p-1)/p}(\hat{T}^{1/p} + R^2 + 1)\big) \\ \times \mathrm{e}^{c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)},$$

where the positive constants c are independent of ε. Finally, we deduce from Lemma 4.9 that

$$(115)\quad \|\theta_\varepsilon\|^2_{H^1(0,\hat{T};H)} + \|\theta_\varepsilon\|^2_{L^\infty(0,\hat{T};V)} \\ \leqslant c\big(\mathrm{e}^{b(p-1)\hat{T}} + \mathrm{e}^{2b(p-1)\hat{T}}\hat{T}^{(p-2)/p}(\hat{T}^{2/p} + R^4 + 1) \\ + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p} + R^2 + 1)\big)\mathrm{e}^{c\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)}.$$

The right-hand side of (115) does not explode as R or $\hat{T}$ goes to 0, since the preceding c's do not. Thus, by the generalized Aubin Lemma (see, e.g., [40, Corollary 4]), we have (here and below, all the convergences are understood up to the extraction of subsequences, not relabelled)

$$(116)\qquad \theta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \theta \quad \text{strongly in } C^0([0,\hat{T}];H)$$

for some limit θ. Next, thanks to (115) and the continuous embedding $V \hookrightarrow L^6(\Omega)$, we have

$$(117)\qquad \theta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \theta \quad \text{weakly in } L^\infty(0,\hat{T};L^6(\Omega)).$$

Thus, it follows from (117) and (C1) that

$$(118)\qquad u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \quad \text{weakly in } L^\infty(0,\hat{T};L^{6/p}(\Omega))$$

for some limit u. This allows us to pass to the limit in (65) and obtain (99); indeed, using e.g. [7, Proposition 1, p. 42], we have $\theta = \alpha(u)$ a.e. □

Lemma 4.11. *Let* (C1)–(C2) *and* (H1)–(H3) *hold and let* $\hat{T} > 0$*. Let also* φ_1, φ_2 *be as in Lemma 4.9. Then system* (98)–(100) *has a unique solution defined in* $Q_{\hat{T}}$.

Proof. We consider a pair of solutions $u_1 = \gamma(\theta_1)$, $u_2 = \gamma(\theta_2)$ to the system (98)–(100).

Let $u = u_1 - u_2$, $\theta = \theta_1 - \theta_2$. Then we have the equality

$$u_t + J\theta = \varphi_2\theta. \tag{119}$$

We note that if $v \in V$ and $w \in H$ then

$$\langle Jv, J^{-1}w\rangle = ((Jv, w))_* = ((w, Jv))_* = (w, J^{-1}(Jv)) = (w, v).$$

Hence, testing (119) by $J^{-1}u$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V',J} + \int_\Omega \theta u \,\mathrm{d}x = \langle\varphi_2\theta, J^{-1}u\rangle. \tag{120}$$

On the one hand, it is easy to see that

$$\gamma(\theta_1) - \gamma(\theta_2) = \int_0^1 \gamma'(s\theta_1 + (1-s)\theta_2)\theta \,\mathrm{d}s. \tag{121}$$

Then, in view of (49) and (34), it turns out that

$$\int_\Omega \theta u \,\mathrm{d}x \geqslant c_{11}|\theta|^2, \tag{122}$$

where c_{11} depends on p and θ_*. On the other hand, combining (120) and (122) and then using the continuous embedding $L^{2p/(p+1)}(\Omega) \hookrightarrow V'$ yields

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V',J} + c_{11}|\theta|^2 &\leqslant \|u\|_{V',J}\|\theta\varphi_2\|_{V',J} \\ &\leqslant c\|u\|_{V',J}|\theta\varphi_2|_{2p/(p+1)} \\ &\leqslant c\|u\|_{V',J}|\varphi_2|_{2p}|\theta| \\ &\leqslant c(\hat{T})\|u\|^2_{V',J}|\varphi_2|^2_{2p} + \frac{c_{11}}{2}|\theta|^2. \end{aligned} \tag{123}$$

Thus, the thesis follows by integrating the above inequality in time and applying once more Gronwall's Lemma.

Moreover, (45) holds by properly choosing m_0 and m_1, since all the above constants and in particular, $c_1(R, \hat{T})$, i.e. the right-hand side of (115), do not explode as R or $\hat{T}$ or both become small. □

The proof of Lemma 4.2. is completed. $\square$

4c. Existence result in Theorem 4.1: Fixed-point argument applied to the operator $\mathcal{T}_2 \circ \mathcal{T}_1$

Having $\mathcal{T}_1$ and $\mathcal{T}_2$, we define the operator $\mathcal{T}$ as the composition $\mathcal{T}_2 \circ \mathcal{T}_1$. We have to show that, at least for small times, Schauder's theorem applies to the map $\mathcal{T}$ from $\Theta_p(\hat{T}, R)$ into itself. In other words, we will prove that there exists $\hat{T} = \hat{T}(R) > 0$ such that $\mathcal{T}$ possesses the properties stated in the following two lemmas.

Lemma 4.12. *There exist R, $\hat{T} = \hat{T}(R) > 0$ such that*

$$\mathcal{T}(\Theta_p(\hat{T}, R)) \subset \Theta_p(\hat{T}, R). \tag{124}$$

P r o o f. Our aim here is to find $\hat{T} > 0$ such that the operator $\mathcal{T}\colon \Theta_p(\hat{T}, R) \to \Theta_p(\hat{T}, R)$ is well-defined. Exploiting the preceding estimates (cf. (115)), by the Sobolev embedding $V \hookrightarrow L^{2p}(\Omega)$, we have

$$\begin{aligned} \|\theta\|_{L^{2p}(0,\hat{T};L^{2p}(\Omega))} &\leqslant c\|\theta\|_{L^{2p}(0,\hat{T};V)} \\ &\leqslant c\hat{T}^{1/2p}\|\theta\|_{L^{\infty}(0,\hat{T};V)} \\ &\leqslant c_{12}\hat{T}^{1/2p}\big(\mathrm{e}^{\frac{1}{2}b(p-1)\hat{T}} + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/2p}(\hat{T}^{1/p} + R^2 + 1) \\ &\quad + \mathrm{e}^{\frac{1}{2}b(p-1)\hat{T}}\hat{T}^{(p-1)/2p}(\hat{T}^{1/2p} + R + 1)\big) \\ &\qquad\qquad \times \mathrm{e}^{c_{13}\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)}. \end{aligned} \tag{125}$$

Hence, for any $R > 0$, we can choose $\hat{T}$ sufficiently small such that

$$\begin{aligned} &c_{12}\hat{T}^{1/2p}\big(\mathrm{e}^{\frac{1}{2}b(p-1)\hat{T}} + \mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-2)/2p}(\hat{T}^{1/p} + R^2 + 1) \\ &\quad + \mathrm{e}^{\frac{1}{2}b(p-1)\hat{T}}\hat{T}^{(p-1)/2p}(\hat{T}^{1/2p} + R + 1)\big) \\ &\qquad \times \mathrm{e}^{c_{13}\mathrm{e}^{b(p-1)\hat{T}}\hat{T}^{(p-1)/p}(\hat{T}^{1/p}+R^2+1)} \leqslant R, \end{aligned} \tag{126}$$

and ensure that θ belongs to $\Theta_p(\hat{T}, R)$. $\square$

Lemma 4.13. *Let $\hat{T} > 0$ be as in* (126). *Then $\mathcal{T}$ is continuous and compact with respect to the L^{2p}-norm.*

P r o o f. We start by showing that $\mathcal{T}$ is continuous with respect to the natural topology induced in $\Theta_p(\hat{T}, R)$ by $L^{2p}(Q_{\hat{T}})$. To this aim, we consider a sequence $(\tilde{\theta}_n)_n \in \Theta_p(\hat{T}, R)$ such that

$$\tilde{\theta}_n \underset{n\to\infty}{\longrightarrow} \tilde{\theta} \quad \text{strongly in } \Theta_p(\hat{T}, R) \tag{127}$$

and consider the sequence $(\varrho_n)_n$ of solutions to (36)–(38) with $\tilde{\theta}$ substituted by $\tilde{\theta}_n$, i.e.

$$\varrho_n = \mathcal{T}_1(\tilde{\theta}_n). \tag{128}$$

The standard energy estimates for the parabolic equations give a positive constant c not depending on n such that

$$\|\varrho_n\|_{H^1(0,\hat{T};H)\cap L^\infty(0,\hat{T};V)\cap L^2(0,\hat{T};W)} \leqslant c. \tag{129}$$

By the well-known weak and weak* compactness results, there exists a subsequence of n still denoted by n, for the sake of brevity, such that

$$\varrho_n \underset{n\to\infty}{\longrightarrow} \varrho \quad \text{weakly in } H^1(0,\hat{T};H)\cap L^2(0,\hat{T};W), \tag{130}$$

$$\varrho_n \underset{n\to\infty}{\longrightarrow} \varrho \quad \text{weakly}^* \text{ in } L^\infty(0,\hat{T};V). \tag{131}$$

Moreover, by the Aubin-Lions Theorem (see [30] and [31]), we also obtain the strong convergence

$$\varrho_n \underset{n\to\infty}{\longrightarrow} \varrho \quad \text{in } L^2(0,\hat{T};H). \tag{132}$$

The above convergences (127) and (132) allow us to pass to the limit in equation (37). Moreover, thanks to the uniqueness result holding by Lemma 4.1, we conclude that the whole sequence $(\varrho_n)_n$ converges to ϱ and we can identify $\varrho = \mathcal{T}_1(\tilde{\theta})$.

In the second step, we consider the sequence $(\theta_n)_n$ of solutions to (42)–(44) with ϱ substituted by ϱ_n, i.e., we consider, in particular,

$$\theta_n = \mathcal{T}_2(\varrho_n) = \mathcal{T}_2 \circ \mathcal{T}_1(\tilde{\theta}_n) = \mathcal{T}(\tilde{\theta}_n). \tag{133}$$

Proceeding as in the previous estimates (cf. (68) and (115)), we can find a positive constant c not depending on n such that

$$\|\theta_n\|_{H^1(0,\hat{T};H)\cap L^\infty(0,\hat{T};V)} \leqslant c. \tag{134}$$

Hence, there exists a subsequence of n still denoted by n such that

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta \quad \text{weakly}^* \text{ in } H^1(0,\hat{T};H)\cap L^\infty(0,\hat{T};V). \tag{135}$$

Furthermore, we deduce from the generalized Aubin Theorem (see [40, Corollary 4]) the strong convergence

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta \quad \text{in } C^0([0,\hat{T}];L^{2p}(\Omega)), \tag{136}$$

which implies that

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta \quad \text{in } L^{2p}(0,\hat{T};L^{2p}(\Omega)). \tag{137}$$

The above convergences (132) and (137) allow us to pass to the limit in relation (43). Again, thanks to the uniqueness result furnished by Lemma 4.2, the whole sequence $(\theta_n)_n$ converges to θ and we can identify

$$\theta = \mathcal{T}_2(\varrho) = \mathcal{T}_2 \circ \mathcal{T}_1(\tilde{\theta}) = \mathcal{T}(\tilde{\theta}). \tag{138}$$

Finally, by (137), we have

$$\mathcal{T}(\theta_n) \longrightarrow \mathcal{T}(\theta) \quad \text{strongly in } L^{2p}(0,\hat{T};L^{2p}(\Omega)). \tag{139}$$

This completes the proof of continuity of the operator $\mathcal{T}$.

It remains to show that the operator $\mathcal{T}$ is compact. Since $\mathcal{T}$ is continuous and $\Theta_p(\hat{T},R)$ is a closed set, it suffices to show that $\mathcal{T}(\Theta_p(\hat{T},R))$ is compact. To this aim, we consider a sequence $(\theta_n)_n \subset \mathcal{T}(\Theta_p(\hat{T},R))$. Then, proceeding exactly as for the previous estimates (cf. (68) and (115)), we deduce the existence of a positive constant depending neither on n nor on the choice of $\tilde{\theta}_n$ in $\Theta_p(\hat{T},R)$ such that

$$\|\theta_n\|_{H^1(0,\hat{T};H)\cap L^\infty(0,\hat{T};V)} \leqslant c. \tag{140}$$

We note that, owing the generalized Aubin Lemma (see [40, Corollary 4]),

$$H^1(0,\hat{T};H) \cap L^\infty(0,\hat{T};V) \subset\subset L^{2p}(Q_{\hat{T}}). \tag{141}$$

Consequently, (140) together with (141) imply that there exists a subsequence of n still denoted by n and $\theta \in H^1(0,\hat{T};H) \cap L^\infty(0,\hat{T};V)$ such that

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta \quad \text{in } L^{2p}(Q_{\hat{T}}), \tag{142}$$

which ensures that the operator $\mathcal{T}$ is compact. □

Thus, we have proved that $\mathcal{T}$ admits a fixed point in $\Theta_p(\hat{T},R)$, i.e. there exists at least a local in time solution to system (24)–(30) defined on the interval $]0,\hat{T}[$. The proof of Theorem 4.1. is complete. □

Now, we have to discuss the extension of this solution to the whole interval $]0,T[$ for an arbitrary final time $T > 0$. To this end, we derive some additional a priori estimates which yield suitable global bounds on the solution.

5. Positivity and boundedness

Lemma 5.1. *Let* (C1)–(C2) *and* (H1)–(H6) *hold, let* $T > 0$ *be any arbitrary final time and let* $(\theta, \varrho)\colon Q_T \to \mathbb{R}^2$ *be a solution to system* (24)–(30)*. Then* (31) *and* (32) *hold.*

P r o o f. We note that for $\theta < 0$ we have assumed that $\gamma(\theta) = -\theta^2$, hence we omit the details of the proof here, since it is similar to that of Lemma 4.1 in [37]. In the rest of the paper we denote by c a universal constant which depends on p, the data mentioned in (H1)–(H6) and on T. $\square$

6. Uniform a priori estimates

Lemma 6.1. *For every* $t \in [0, T]$, *we have*

$$\|\theta\|^p_{L^\infty(0,t;L^p(\Omega))} + \|\varrho\|^2_{L^\infty(0,t;V)} \leqslant c. \tag{143}$$

P r o o f. We can consider, due to (32), that $\gamma(\theta) = \Phi(\theta)$. Multiplying (25) by ϱ_t and (24) by 1, integrating over Q_t for $t \leqslant \nu$ and adding the resulting equalities yields

$$\begin{aligned}\frac{1}{2}\|\varrho(t)\|^2 &+ \int_\Omega F(\varrho(t))\,\mathrm{d}x + \int_\Omega \Phi(\theta(t))\,\mathrm{d}x \\ &= \frac{1}{2}\|\varrho_0\|^2 + \int_\Omega F(\varrho_0)\,\mathrm{d}x + \int_\Omega \Phi(\theta_0)\,\mathrm{d}x + \int_0^t\int_\Gamma g\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned} \tag{144}$$

Then, it turns out from (C1), (H2), (H6), and (31) that

$$\frac{1}{2}\|\varrho(t)\|^2 + c(p)|\theta(t)|^p_p \leqslant c, \tag{145}$$

from which the conclusion of this lemma follows. $\square$

Lemma 6.2. *We have*

$$\|\varrho\|_{H^1(0,t;H)} \leqslant c. \tag{146}$$

P r o o f. Since $p \geqslant 2$, from (143) we have

$$\|\theta\|_{L^2(0,t;H)} \leqslant |\Omega|^{1/2-1/p}T^{1/2}\|\theta\|_{L^\infty(0,t;L^p(\Omega))} \leqslant c. \tag{147}$$

We first multiply (25) by ϱ_t, then integrate with respect to x and t and finally use (31) to deduce that

$$\begin{aligned}(148)\quad \|\varrho_t\|_{L^2(0,t;H)} + \frac{1}{2}\|\varrho(t)\|^2 &\leqslant \int_0^t\int_\Omega |\theta\varrho\varrho_t|\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega |F(\varrho(t))|\,\mathrm{d}x \\ &\leqslant \max(-\underline{\varrho},\overline{\varrho})\|\theta\|_{L^2(0,t;H)}\|\varrho_t\|_{L^2(0,t;H)} + c(\underline{\varrho},\overline{\varrho},|\Omega|).\end{aligned}$$

Thus, (146) follows from (148) by using (147) and applying Young's inequality. □

Lemma 6.3. *We have*

$$\|\varrho\|_{L^2(0,t;W)} \leqslant c. \tag{149}$$

P r o o f. We test (25) by $A\varrho$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|^2 + \frac{1}{2}|A\varrho|^2 \leqslant c|\theta|^2 + c, \tag{150}$$

which results in (149) by integrating with respect to t and using (147). □

Now, we give a basic lemma which plays an important role in the sequel.

Lemma 6.4. *We have*

$$\|\varrho_t\|^2_{L^\infty(0,t;H)} + \|\varrho_t\|^2_{L^2(0,t;V)} \leqslant c + c\|\theta_t\|_{L^2(0,t;H)}. \tag{151}$$

P r o o f. We differentiate (25) with respect to time,

$$\frac{\partial^2\varrho}{\partial t^2} + A\varrho_t + F''(\varrho)\varrho_t = -\theta\varrho_t - \varrho\theta_t, \tag{152}$$

then we multiply (152) by ϱ_t and integrate over Q_t, $t \leqslant \nu$. Thus, it follows from (31) and (146) that

$$\begin{aligned}(153)\quad &\frac{1}{2}|\varrho_t(t)|^2 + \|\varrho_t\|^2_{L^2(0,t;V)} \\ &\leqslant -\int_0^t\int_\Omega F''(\varrho)(\varrho_t)^2\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \theta(\varrho_t)^2\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \varrho\varrho_t\theta_t\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant c\|\varrho_t\|^2_{L^2(0,t;H)} + \int_0^t \|\varrho\|_{L^\infty(\Omega)}|\varrho_t||\theta_t|\,\mathrm{d}s \\ &\leqslant c\|\varrho_t\|^2_{L^2(0,t;H)} + \|\varrho\|_{L^\infty(Q_t)}\|\varrho_t\|_{L^2(0,t;H)}\|\theta_t\|_{L^2(0,t;H)} \\ &\leqslant c + c\|\theta_t\|_{L^2(0,t;H)}.\end{aligned}$$

□

Lemma 6.5. *We have*

$$\|\theta\|_{H^1(0,t;H)} + \|\theta\|_{L^\infty(0,t;V)} \leqslant c. \tag{154}$$

P r o o f. We distinguish three cases according to the dimension N and the exponent p:

First case ($N = 2$, $2 \leqslant p < 5$) or ($N = 3$, $2 \leqslant p < 3$):

Let

$$\begin{cases} \psi_1 = g, \\ \psi_2 = \theta\varrho\varrho_t + (\varrho_t)^2, \\ \psi = \psi_1 + \psi_2. \end{cases} \tag{155}$$

To prove (154) in this case, we have to use the so-called renormalized solution (see [8], [9], and [10]). We recall the definition of the renormalized solutions.

Definition 6.1. Let $K > 0$ and $r \in \mathbb{R}$. The quantity

$$T_K(r) = \max\{-K, \min\{K, r\}\} \tag{156}$$

is called the *truncation function at height K.*

Definition 6.2. We assume that

(A1) B is the field of symmetric coercive matrices defined on Q_T with bounded coefficients; i.e.,
- $\star$ $(B)_{ij} = b_{ij} \in L^\infty(Q_T)$,
- $\star$ $b_{ij} = b_{ji}$ for $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$,
- $\star$ there exists $\lambda > 0$ such that $B(x,t)\xi \cdot \xi \geqslant \lambda\|\xi\|^2_{\mathbb{R}^N}$ for any $\xi \in \mathbb{R}^N$ and for almost every $(x,t) \in Q_T$;

(A2) $b\colon \mathbb{R} \to \mathbb{R}$, is a $\mathcal{C}^1$ increasing function such that
- $\star$ $b(0) = 0$,
- $\star$ there exist $\delta, s > 0$ such that $|b(r)| \geqslant \delta|r|^s$, $\forall\,|r| \geqslant 1$;

(A3) v_0 is a measurable function defined on Ω such that $b(v_0) \in L^1(\Omega)$;

(A4) $G \in L^1(Q_T)$.

Then a measurable function v defined on Q_T is a renormalized solution of the problem

$$\mathcal{P}(G, v_0) \begin{cases} \dfrac{\partial b(v)}{\partial t} - \operatorname{div}(BDv) = G & \text{in } Q_T, \\ b(v)|_{t=0} = b(v_0) & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma \times (0,T), \end{cases}$$

if v satisfies

$$b(v) \in L^\infty(0,T;L^1(\Omega)); \tag{157}$$

$$T_K(v) \in L^2(0,T;H^1_0(\Omega)) \quad \text{for any } K \geqslant 0 \tag{158}$$

for any function $S \in \mathcal{C}^\infty(\mathbb{R})$ such that S' has a compact support (i.e., $S' \in S \in \mathcal{C}^\infty_0(\mathbb{R})$),

$$\frac{\partial S(b(v))}{\partial t} - \operatorname{div}[S'(b(v))BDv] + S''(b(v))b'(v)BDv \cdot Dv = GS'(b(v)) \quad \text{in } D'(Q_T); \tag{159}$$

$$S(b(v))|_{t=0} = S(b(v_0)); \tag{160}$$

$$\lim_{n\to+\infty} \int_{\{(x,t)\,;\; n<|b(v)|<n+1\}} b'(v)|Dv|^2 \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{161}$$

R e m a r k 6.1. Under hypotheses (A1)–(A3) and for any $G \in L^1(Q_T)$, it is established in [10] that there exists at least one solution v satisfying Definition 6.2.

The next lemma can be found in [9]. It establishes the most important property of renormalized solutions of the nonlinear parabolic problem $\mathcal{P}(G, v_0)$.

Lemma 6.6. *Let v be a measurable function defined on Q_T and let $\sigma > 0$ be such that*

$$|v|^\sigma \in L^\infty(0,T;L^1(\Omega)), \tag{162}$$

$$\forall\, K > 0 \;\; T_K(v) \in L^2(0,T;H^1_0(\Omega)) \;\; \textit{with} \;\; \int_{Q_T} |DT_K(v)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant KM. \tag{163}$$

Then for all $1 \leqslant q < 1 + 2\sigma/N$ there exists a constant c depending only on T, Ω, q, and σ such that

$$\|v\|_{L^q(Q_T)} \leqslant c\||v|^\sigma|^{2/(2\sigma+N)}_{L^\infty(0,T;L^1(\Omega))} M^{N/(2\sigma+N)}. \tag{164}$$

P r o o f. For any positive real number r, we first write

$$\begin{aligned}\int_{Q_T} |v|^q \,\mathrm{d}x\,\mathrm{d}t &= \int_0^{+\infty} \operatorname{meas}\{(x,t);\ |v(x,t)|^q > s\}\,\mathrm{d}s \\ &= \int_0^r \operatorname{meas}\{(x,t);\ |v(x,t)|^q > s\}\,\mathrm{d}s \\ &\quad + \int_r^{+\infty} \operatorname{meas}\{(x,t);\ |v(x,t)|^q > s\}\,\mathrm{d}s,\end{aligned} \tag{165}$$

which leads to

$$\|v\|^q_{L^q(Q_T)} \leqslant rT|\Omega| + \int_r^{+\infty} \operatorname{meas}\{(x,t);\ |v(x,t)|^q > s\}\,\mathrm{d}s. \tag{166}$$

The last term in the above inequality is bounded as follows:

$$\operatorname{meas}\{(x,t);\ |v(x,t)|^q > s\} \leqslant \frac{c}{s^{2(N+\sigma)/Nq}} \int_{Q_T} |T_{s^{1/q}}(v)|^{2(N+\sigma)/N}\,\mathrm{d}x\,\mathrm{d}t. \tag{167}$$

In fact, we have

$$\begin{aligned} &\frac{c}{s^{2(N+\sigma)/Nq}} \int_{Q_T} |T_{s^{1/q}}(v)|^{2(N+\sigma)/N}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \geqslant \frac{c}{s^{2(N+\sigma)/Nq}} \int_{\{(x,t)\,;\ |v(x,t)|^q > s\}} |T_{s^{1/q}}(v)|^{2(N+\sigma)/N}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{168}$$

Rewriting T_K in the form

$$T_K(r) = \begin{cases} r & \text{if } |r| \leqslant K, \\ K\operatorname{sign}(r) & \text{if } |r| \geqslant K, \end{cases} \tag{169}$$

yields, for any $|v|^q > s$,

$$T_{s^{1/q}}(v) = s^{1/q}. \tag{170}$$

Thus (167) follows by combining (168) and (170).

On the one hand, it turns out from the classical interpolation inequality (for $N = 3$) that

$$\|T_{s^{1/q}}(v)\|_{L^{2(N+\sigma)/N}(Q_T)} \leqslant c\|T_{s^{1/q}}(v)\|^{1-\alpha}_{L^\infty(0,T;L^r(\Omega))}\|T_{s^{1/q}}(v)\|^{\alpha}_{L^2(0,T;L^6(\Omega))} \tag{171}$$

with

$$\begin{cases} \dfrac{N}{2(N+\sigma)} = \dfrac{\alpha}{2}, \\[2mm] \dfrac{N}{2(N+\sigma)} = \dfrac{\alpha}{6} + \dfrac{1-\alpha}{r}, \end{cases} \tag{172}$$

which leads to

$$\begin{cases} \alpha = \dfrac{N}{N+\sigma}, \\ r = \sigma. \end{cases} \tag{173}$$

Moreover, we know that, for any K and v, we have

$$T_K(v) \leqslant |v|. \tag{174}$$

Then, using the continuous embedding $V \hookrightarrow L^6(\Omega)$, we infer that

$$\begin{aligned} \|T_{s^{1/q}}(v)\|_{L^{2(N+\sigma)/N}(Q_T)} &\leqslant c\|T_{s^{1/q}}(v)\|^{\sigma/(N+\sigma)}_{L^\infty(0,T;L^\sigma(\Omega))}\|T_{s^{1/q}}(v)\|^{N/(N+\sigma)}_{L^2(0,T;V)} \\ &\leqslant c\||v|^\sigma\|^{1/(N+\sigma)}_{L^\infty(0,T;L^1(\Omega))}\|DT_{s^{1/q}}(v)\|^{N/(N+\sigma)}_{L^2(Q_T)}. \end{aligned} \tag{175}$$

On the other hand, Gagliardo-Nirenberg's ([15, p. 194]) and the classical interpolation inequalities (for $N = 2$) imply

$$\|T_{s^{1/q}}(v)\|_{L^{2(N+\sigma)/N}(Q_T)} \leqslant c\|T_{s^{1/q}}(v)\|^{1-\alpha}_{L^\infty(0,T;L^r(\Omega))}\|T_{s^{1/q}}(v)\|^{\alpha}_{L^2(0,T;V)} \tag{176}$$

with

$$\begin{cases} \alpha = 1 - \dfrac{rN}{2(N+\sigma)}, \\ 1-\alpha = 1 - \dfrac{2N}{2(N+\sigma)}, \end{cases} \tag{177}$$

which leads to

$$\begin{cases} \alpha = \dfrac{N}{N+\sigma}, \\ r = \sigma. \end{cases} \tag{178}$$

Thus, we get

$$\|T_{s^{1/q}}(v)\|_{L^{2(N+\sigma)/N}(Q_T)} \leqslant c\|T_{s^{1/q}}(v)\|^{\sigma/(N+\sigma)}_{L^\infty(0,T;L^\sigma(\Omega))}\|T_{s^{1/q}}(v)\|^{N/(N+\sigma)}_{L^2(0,T;V)}. \tag{179}$$

Finally, we conclude that, in the cases $N = 2$ or $N = 3$, we obtain the same form of estimate which reads

$$\int_{Q_T} |T_{s^{1/q}}(v)|^{2(N+\sigma)/N}\,\mathrm{d}x\,\mathrm{d}t \leqslant c\||v|^\sigma\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}\|DT_{s^{1/q}}(v)\|^2_{L^2(Q_T)}. \tag{180}$$

Consequently, returning to (167), it follows from (163) and (180) that

$$\operatorname{meas}\{(x,t);\ |v|^q > s\} \leqslant \frac{c}{s^{2(N+\sigma)/Nq}}\||v|^\sigma\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}Ms^{1/q}. \tag{181}$$

Therefore, (166) becomes

$$\|v\|^q_{L^q(Q_T)} \leqslant rT|\Omega| + c\||v|^\sigma\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}M\int_r^{+\infty} s^{-(N+2\sigma)/Nq}\,\mathrm{d}s. \tag{182}$$

The second term on the right-hand side of the above inequality represents a generalized integral which converges only if

$$-\frac{N+2\sigma}{Nq} < -1, \tag{183}$$

i.e.

$$q < 1 + \frac{2\sigma}{N}, \tag{184}$$

and, hence, we have

$$\int_r^{+\infty} s^{-(N+2\sigma)/Nq}\,\mathrm{d}s = -\frac{Nq}{N(1-q)-2\sigma} r^{(N(1-q)-2\sigma)/(Nq)}. \tag{185}$$

Thus, it follows that, for all $1 \leqslant q < 1 + 2\sigma/N$, we have

$$\|v\|^q_{L^q(Q_T)} \leqslant rT|\Omega| + c\||v|^\sigma\|^{2/N}_{L^\infty(0,T;L^1(\Omega))} M r^{(N(q-1)-2\sigma)/(Nq)}. \tag{186}$$

Now, let

$$g(r) = rT|\Omega| + c\||v|^\sigma\|^{2/N}_{L^\infty(0,T;L^1(\Omega))} M r^{(N(q-1)-2\sigma)/(Nq)}, \tag{187}$$

and let $r^* > 0$ be the point at which $g(r)$ achieves its minimum. Then we deduce from (186) that

$$\|v\|^q_{L^q(Q_T)} \leqslant g(r^*). \tag{188}$$

It is easy to see that $g'(r^*) = 0$ corresponds to

$$r^* = c\left(\frac{M(N(q-1)-2\sigma)\||v|^\sigma\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}}{qNT|\Omega|}\right)^{qN/(N+2\sigma)}, \tag{189}$$

and then (164) is established by combining (188) and (189). □

The previous lemma allows us to establish the following estimate to our solution θ. More precisely, we use a variant of this lemma, since we have different boundary conditions; however, the proof is the same.

Lemma 6.7. *For all $1 \leqslant q < 1 + 2p/N$ there exists a positive constant $c(q)$ independent of ν such that*

$$\|\theta\|_{L^q(Q_t)} \leqslant c(q). \tag{190}$$

P r o o f. This result is a direct consequence of the previous lemma. In fact, we have $g \in H^1(0,t;L^2(\Gamma))$ and $L^2(\Gamma) \hookrightarrow V'$ is a continuous embedding. Then we deduce that

$$\|\psi_1\|_{H^1(0,t;V')} \leqslant c. \tag{191}$$

On the other hand, (143) allows us to obtain that

$$\|\psi_2\|_{L^1(Q_t)} \leqslant c. \tag{192}$$

So, in order to make sure that θ is a renormalized solution of the nonlinear parabolic equation (24), whose right-hand side ψ is in $L^1(Q_t)$ and the initial data $\Phi(\theta_0) \in L^1(\Omega)$, we have to show that (163) holds. To this aim, we test (24) by $T_K(\theta) = \min(K,\theta)$ and obtain

$$\int_\Omega T_K(\theta)\frac{\partial \Phi(\theta)}{\partial t}\,\mathrm{d}x + c_{14}\|T_K(\theta)\|^2 \leqslant \langle \psi_1, T_K(\theta)\rangle + \int_\Omega \psi_2 T_K(\theta)\,\mathrm{d}x. \tag{193}$$

Note that we are concerned only with the case $T_K(\theta) = \theta$ (because it is very simple to verify that (163) holds for $T_K(\theta) = K$), hence the above inequality becomes

$$\begin{aligned} c(p)\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_{p+1}^{p+1} + c_{14}\|T_K(\theta)\|^2 &\leqslant \|\psi_1\|_{V'}\|T_K(\theta)\| + K|\psi_2|_1 \\ &\leqslant \frac{c_{14}}{2}\|T_K(\theta)\|^2 + c\|\psi_1\|_{V'}^2 + K|\psi_2|_1. \end{aligned} \tag{194}$$

Thus, integrating with respect to t and using (191) and (192) leads to

$$\|T_K(\theta)\|_{L^2(0,t;V)}^2 \leqslant c(1+K). \tag{195}$$

Moreover, it follows from (C1) and (143) that

$$\Phi(\theta) \in L^\infty(0,t;L^1(\Omega)). \tag{196}$$

Consequently, (190) follows from (195), (196), and a simple application of Lemma 6.6. $\square$

Lemma 6.8. *Let*

$$p < \begin{cases} 5 & \text{if } N = 2, \\ 3 & \text{if } N = 3, \end{cases} \qquad s = \begin{cases} \dfrac{6p}{5} & \text{if } N = 2, \\ p & \text{if } N = 3. \end{cases} \tag{197}$$

Then for all $s \leqslant q < 1 + 2p/N$ *there exists a positive constant* $c(q)$ *such that*

$$\|\psi_2\|_{L^{q/2}(Q_T)} \leqslant c(q). \tag{198}$$

P r o o f. We can easily verify that, when $N = 2$, we have $6p/5 < 1 + p$ and, when $N = 3$, we have $p < 1 + 2p/3$.

Now, let $h = F'(\varrho) + \theta\varrho$. Then (25) becomes

$$\varrho_t + A\varrho = -h. \tag{199}$$

Since $V \hookrightarrow L^q(\Omega)$ is a continuous embedding, we deduce from (143) that

$$\|\varrho\|_{L^q(Q_t)} \leqslant c. \tag{200}$$

Thus, (190) and (200) yield

$$\|h\|_{L^q(Q_t)} \leqslant c. \tag{201}$$

On the other hand, it follows from the Agmon-Douglis-Nirenberg estimates (see [28, Theorem 9.1, pp. 341–342]) that

$$\sum_{j=0}^{2} \sum_{\substack{2r+s=j \\ r,s\geqslant 0}} \|D_t^r D_x^s \varrho\|_{L^q(Q_{\overline{T}})} \leqslant c(\|h\|_{L^q(Q_{\overline{T}})} + \|\varrho_0\|_{W^{2-2/q,q}(\Omega)}), \tag{202}$$

where D_t^r and D_x^s denote respectively any derivative of ϱ with respect to t and x of order r and s. Moreover, we can easily verify that $q < 2p$, which implies the Sobolev embedding

$$W^{2-1/p,2p}(\Omega) \hookrightarrow W^{2-2/q,q}(\Omega). \tag{203}$$

Therefore, taking, in particular, $r = 0$ and $s = 2$ in (202), it follows from (H6), (202), and (203) that

$$\|A\varrho\|_{L^q(Q_t)} \leqslant c, \tag{204}$$

which leads, together with (199) and (201), to

$$\|\varrho_t\|_{L^q(Q_t)}. \tag{205}$$

Thus,

$$(\varrho_t)^2 \in L^{q/2}(\Omega). \tag{206}$$

Finally, (198) holds true owing to estimates (190), (200), and (206). $\square$

Lemma 6.9. *The following estimate holds:*

$$\|\psi_2\|_{L^2(Q_t)} \leqslant c. \tag{207}$$

Proof. We know that the two dimensional case corresponds to $p \geqslant 2$ and if we have also $4 \in (6p/5, p+1)$, then (207) will be obtained by a simple deduction from (198) and the continuous embedding $L^{q/2}(Q_t) \hookrightarrow L^2(Q_t)$. It suffices to take $4 \leqslant q \leqslant p+1$. In the three dimensional case, we are sure that $4 \notin (p, 1+2p/3)$, since assuming that $4 \leqslant 1+2p/3$ gives $p \geqslant 9/2$, which contradicts our assumptions. Hence, we have to show this lemma in the following two cases: $N=3$ or $N=2$ with $4 \notin (6p/5, p+1)$. Our idea is to combine Moser iterations estimates and (198) which holds by Lemma 6.8.

Let $q_0 \in (s, 1+2p/N)$. Then testing (24) by θ^{q_i-p} yields, for $i=0$,

$$\begin{aligned} \frac{p}{q_i}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^{q_i}\,\mathrm{d}x + \frac{4(q_i-p)}{q_i-p+1}\int_\Omega \bigl|\nabla\bigl(\theta^{(q_i-p+1)/2}\bigr)\bigr|^2\,\mathrm{d}x + n_0\int_\Gamma \theta^{q_i-p+1}\,\mathrm{d}\sigma \\ \leqslant \int_\Gamma g\theta^{q_i-p}\,\mathrm{d}\sigma + \int_\Omega \psi_2\theta^{q_i-p}\,\mathrm{d}x. \end{aligned} \tag{208}$$

We estimate separately the terms of (208). On the one hand, we note that

$$\frac{4(q_i-p)}{q_i-p+1}\int_\Omega \bigl|\nabla\bigl(\theta^{(q_i-p+1)/2}\bigr)\bigr|^2\,\mathrm{d}x + n_0\int_\Gamma \theta^{q_i-p+1}\,\mathrm{d}\sigma \geqslant c_{15}\bigl\|\theta^{(q_i-p+1)/2}\bigr\|^2, \tag{209}$$

where c_{15} depends of course on q_i and p. On the other hand, it follows from Young's inequality that

$$\int_\Omega \psi_2\theta^{q_i-p}\,\mathrm{d}x \leqslant \int_\Omega \theta^{q_i}\,\mathrm{d}x + \int_\Omega \psi_2^{q_i/p}\,\mathrm{d}x. \tag{210}$$

Finally, from (H2) we obtain

$$\int_\Gamma g\theta^{q_i-p}\,\mathrm{d}\sigma \leqslant \|g\|_{L^2(\Gamma)}\|\theta^{q_i-p}\|_{L^2(\Gamma)} \leqslant c\bigl\|\theta^{(q_i-p+1)/2}\bigr\|^{2(q_i-p)/(q_i-p+1)}_{L^{4(q_i-p)/(q_i-p+1)}(\Gamma)}. \tag{211}$$

We note that $(q_i - p)/(q_i - p + 1) < 1$. Then we deduce from the continuity of the embedding $V \hookrightarrow L^s(\Gamma)$, for any $s < 4$ (Γ is an $(N-1)$-dimensional set), that

$$\int_\Gamma g\theta^{q_i - p}\,\mathrm{d}\sigma \leqslant c + \frac{c_{15}}{2}\left\|\theta^{(q_i-p+1)/2}\right\|^2, \tag{212}$$

where c depends on q_i and p. Then, by first combining (208)–(210) and (212) and then applying Gronwall's Lemma, we infer that for $i = 0$

$$\|\theta\|_{L^\infty(0,t;L^{q_i}(\Omega))} + \left\|\theta^{(q_i-p+1)/2}\right\|_{L^2(0,t;V)} \leqslant c(q_i). \tag{213}$$

Thus, (213) becomes, by using the continuous embedding $V \hookrightarrow L^6(\Omega)$,

$$\|\theta\|_{L^\infty(0,t;L^{q_i}(\Omega))} + \|\theta\|_{L^{q_i-p+1}(0,t;L^{3(q_i-p+1)}(\Omega))} \leqslant c(q_i). \tag{214}$$

Now, we use the classical interpolation inequality written in the $L^q(0,t;L^r(\Omega))$-space with

$$\begin{cases} \dfrac{1}{q} = \dfrac{1-\delta}{q_i - p + 1}, \\[2mm] \dfrac{1}{r} = \dfrac{\delta}{q_i} + \dfrac{1-\delta}{3(q_i - p + 1)} \end{cases} \tag{215}$$

for $\delta \in (0,1)$. Letting $q = r$, we deduce that

$$\delta = \frac{2q_i}{5q_i - 3(p-1)}, \tag{216}$$

which corresponds to

$$q = \frac{5}{3}q_i - p + 1. \tag{217}$$

Consequently, we obtain the corresponding estimate

$$\|\theta\|_{L^q(Q_t)} \leqslant c(q_i), \quad \text{with} \quad q = \frac{5}{3}q_i - p + 1 \tag{218}$$

for $i = 0$. It becomes clear that the right-hand side of (24) is bounded, independently in time, in $L^q(Q_T)$ for q given by (217). Agmon-Douglis-Nirenberg estimates allow us to improve (198), but with this new q, which is bigger than that given by Lemma 6.8. In fact, we can easily verify that

$$\begin{cases} q > 1 + p & \text{if } N = 2, \\ q > 1 + \dfrac{2p}{3} & \text{if } N = 3. \end{cases} \tag{219}$$

Indeed, for $N = 2$ we have assumed that $\frac{6}{5}p \leqslant q_0 \leqslant 1 + p$, so that

$$1 + p \leqslant q \leqslant \frac{8}{3} + \frac{2}{3}p, \tag{220}$$

and for $N = 3$ we have assumed that $p \leqslant q_0 \leqslant 1 + \frac{2}{3}p$, which implies

$$1 + \frac{2}{3}p \leqslant q \leqslant \frac{8}{3} + \frac{1}{9}p. \tag{221}$$

Remark 6.2. This explains the reason why we have restricted q_0 to $[\frac{6}{5}p, 1 + p]$ when $N = 2$ and to $[p, 1 + \frac{2}{3}p]$ when $N = 3$. Indeed, Lemma 6.7 allows us to have q_0 between 1 and $1 + 2p/N$. Then we can restrict q_0 to $[s, 1 + 2p/N]$, $s \geqslant 1$. And, in order to have (219), we must choose s such that

$$\begin{cases} 1 + p \leqslant \frac{5}{3}s - p + 1 & \text{if } N = 2, \\ 1 + \frac{2}{3}p \leqslant \frac{5}{3}s - p + 1 & \text{if } N = 3, \end{cases} \tag{222}$$

whence our choices

$$\begin{cases} s = \frac{6}{5}p & \text{if } N = 2, \\ s = p & \text{if } N = 3. \end{cases} \tag{223}$$

Furthermore, the conditions

$$\begin{cases} p < 5 & \text{if } N = 2, \\ p < 3 & \text{if } N = 3, \end{cases} \tag{224}$$

can be explained by the fact that we must have

$$\begin{cases} \frac{6}{5}p < p + 1 & \text{if } N = 2, \\ p < 1 + \frac{2}{3}p & \text{if } N = 3. \end{cases} \tag{225}$$

Now, let $q_1 \in (q_0, q)$. Then applying the same procedure we deduce the existence of a third new $q = \frac{5}{3}q_1 - p + 1$, which is even bigger. We repeat these steps of Moser iterations on $i \in \mathbb{N}$ with $q_{i+1} \in (q_i, q)$. We note that

$$q = \frac{5}{3}q_i - p + 1 > 4 \tag{226}$$

if and only if we have

$$q_i > \frac{3(3+p)}{5}. \tag{227}$$

Consequently, we arrive at (213) with precisely $q_i = 3(3+p)/5+1$, which corresponds to $q = \frac{17}{3} > 4$ in (218) (indeed, higher exponents are not allowed, since k belongs to L^2, see (H2), and we cannot go over $q = 4$ in (198)). Finally, (207) holds true due to the continuity of the embedding $L^{q/2}(Q_T) \hookrightarrow L^2(Q_T)$. □

End of the proof of the first case. We infer from the above estimates that the right-hand side of (24) is bounded, uniformly in time, in $H^1(0,t;V') \cap L^2(0,t;H)$. Furthermore, it follows from (C1) that

$$\int_0^t \int_\Omega \Phi'(\theta)(\theta_t)^2 \,\mathrm{d}x \,\mathrm{d}s \geqslant c_3 \int_0^t \int_\Omega \theta^{p-1}(\theta_t)^2 \,\mathrm{d}x \,\mathrm{d}s \geqslant c_3 \theta_*^{p-1} \|\theta_t\|^2_{L^2(0,t;H)}. \tag{228}$$

Testing (24) by θ_t, then integrating and integrating by parts with respect to time yields

$$\begin{aligned} &\frac{c_3\theta_*^{p-1}}{2}\|\theta_t\|^2_{L^2(0,t;H)} + \frac{1}{4}\|\theta(t)\|^2 \\ &\qquad \leqslant c + c\|\theta\|^2_{L^2(0,t;V)} + c\|g\|^2_{H^1(0,t;L^2(\Gamma))} + c\|\psi_2\|^2_{L^2(0,t;H)}, \end{aligned} \tag{229}$$

which results in (154) owing to Gronwall's Lemma.

Second case ($N = 1$):

We first multiply (24) by θ_t, then integrate over Q_t, $t \leqslant \nu$ and use (33), (228), and the continuous embedding $V \hookrightarrow L^\infty(\Omega)$ to obtain

$$\begin{aligned} &c_3\theta_*^{p-1}\|\theta_t\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\theta(t)\|^2 \\ &\qquad \leqslant c + \int_0^t\int_\Gamma g\theta_t \,\mathrm{d}\sigma\,\mathrm{d}s + \int_0^t\int_\Omega (\varrho_t)^2\theta_t \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \theta\varrho\varrho_t\theta_t \,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \leqslant c + \frac{1}{4}\|\theta(t)\|^2 + c\|\theta\|^2_{L^2(0,t;V)} + \int_0^t \|\varrho_t\|_{L^\infty(\Omega)}|\varrho_t||\theta_t| \,\mathrm{d}s \\ &\qquad\quad + c\int_0^t |\theta|_p\|\varrho_t\|_{L^\infty(\Omega)}|\theta_t| \,\mathrm{d}s + c\|g\|^2_{H^1(0,t;L^2(\Gamma))} \\ &\qquad \leqslant c + \frac{1}{4}\|\theta(t)\|^2 + c\|\theta\|^2_{L^2(0,t;V)} + \|\varrho_t\|_{L^\infty(0,t;H)}\|\varrho_t\|_{L^2(0,t;V)}\|\theta_t\|_{L^2(0,t;H)} \\ &\qquad\quad + c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\varrho_t\|_{L^2(0,t;V)}\|\theta_t\|_{L^2(0,t;H)}. \end{aligned} \tag{230}$$

Inequality (143) and Young's inequality yield

$$\begin{aligned} &\frac{c_3}{2}\theta_*^{p-1}\|\theta_t\|^2_{L^2(0,t;H)} + \frac{1}{4}\|\theta(t)\|^2 \\ &\qquad \leqslant c + c\|\theta\|^2_{L^2(0,t;V)} + c\|\varrho_t\|^2_{L^\infty(0,t;H)}\|\varrho_t\|^2_{L^2(0,t;V)} + c\|\varrho_t\|^2_{L^2(0,t;V)}. \end{aligned} \tag{231}$$

Now, we have to show that θ and ϱ_t are bounded in $L^2(0,t;V)$. So, we establish the following lemma.

Lemma 6.10. *In one space dimension we have*

$$\|\theta\|^2_{L^2(0,t;V)} + \|\theta\|^{p+1}_{L^\infty(0,t;L^{p+1}(\Omega))} + \|\varrho\|^2_{L^\infty(0,t;W)} + \|\varrho_t\|^2_{L^2(0,t;V)} \leqslant c. \tag{232}$$

Proof. We test (24) by $A\varrho_t$, then integrate between 0 and $t \leqslant \nu$ and use Agmon's inequality

$$\|u\|_{L^\infty(\Omega)} \leqslant c|u|^{1/2}\|u\|^{1/2} \quad \forall\, u \in V, \tag{233}$$

(31), (143), and (149) to obtain

$$\begin{aligned}
&\|\varrho_t\|^2_{L^2(0,t;V)} + \frac{1}{2}|A\varrho(t)|^2 \\
&\quad\leqslant c + c\int_0^t \|F'(\varrho)\|\|\varrho_t\|\,\mathrm{d}s + c\int_0^t \|\theta\varrho\|\|\varrho_t\|\,\mathrm{d}s \\
&\quad\leqslant c + c\int_0^t \{|F''(\varrho)\nabla\varrho| + |F'(\varrho)|\}\|\varrho_t\|\,\mathrm{d}s \\
&\qquad + c\int_0^t \{|\varrho\nabla\theta + \theta\nabla\varrho| + |\theta\varrho|\}\|\varrho_t\|\,\mathrm{d}s \\
&\quad\leqslant c + c\int_0^t |\nabla\varrho|\|\varrho_t\|\,\mathrm{d}s + c\int_0^t \|\varrho_t\|\,\mathrm{d}s + c\int_0^t \|\varrho\|_{L^\infty(\Omega)}|\nabla\theta|\|\varrho_t\|\,\mathrm{d}s \\
&\qquad + c\int_0^t |\theta|\|\nabla\varrho\|_{L^\infty(\Omega)}\|\varrho_t\|\,\mathrm{d}s + c\int_0^t \|\varrho\|_{L^\infty(\Omega)}|\theta|\|\varrho_t\|\,\mathrm{d}s \\
&\quad\leqslant c + c\int_0^t \|\varrho\|\|\varrho_t\|\,\mathrm{d}s + c\int_0^t \|\varrho_t\|\,\mathrm{d}s \\
&\qquad + c\int_0^t \|\varrho\|_{L^\infty(\Omega)}(|\nabla\theta| + |\theta|)\|\varrho_t\|\,\mathrm{d}s + c\int_0^t |\theta|_p\|\nabla\varrho\|\|\varrho_t\|\,\mathrm{d}s \\
&\quad\leqslant c + cT^{1/2}(1 + \|\varrho\|_{L^\infty(0,t;V)})\|\varrho_t\|_{L^2(0,t;V)} \\
&\qquad + c\|\varrho\|_{L^\infty(0,t;V)}\|\theta\|_{L^2(0,t;V)}\|\varrho_t\|_{L^2(0,t;V)} \\
&\qquad + c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\varrho\|_{L^2(0,t;W)}\|\varrho_t\|_{L^2(0,t;V)} \\
&\quad\leqslant c + \frac{1}{2}\|\varrho_t\|^2_{L^2(0,t;V)} + c_{16}\|\theta\|^2_{L^2(0,t;V)}.
\end{aligned} \tag{234}$$

On the other hand, testing (24) by θ and noting that

$$\Big(\frac{\partial\Phi(\theta)}{\partial t}, \theta\Big) = \frac{\partial}{\partial t}\int_\Omega \psi(\theta)\,\mathrm{d}x,$$

where ψ is the antiderivative of the function defined on $\mathbb{R}^+$ by $r \mapsto r\Phi'(r)$ which vanishes at 0, we immediately infer that

$$\frac{c}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_{p+1}^{p+1} + \frac{1}{2}\|\theta\|_J^2 \leqslant c\|\theta\|_{L^\infty(\Omega)}|\theta|_p|\varrho_t| + c|\theta|_p|\varrho_t|_4^2 + c\|k\|_{V'}^2. \tag{235}$$

The Gagliardo-Nirenberg interpolation inequality (see [15, p. 194]) implies that

$$|\varrho_t|_4 \leqslant c|\varrho_t|^{3/4}\|\varrho_t\|^{1/4}. \tag{236}$$

Then by integrating with respect to time, we obtain

$$\begin{aligned}\frac{c}{p+1}|\theta(t)|_{p+1}^{p+1} &+ \frac{1}{2}\|\theta\|_{L^2(0,t;V)}^2 \\ &\leqslant c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\theta\|_{L^2(0,t;V)}\|\varrho_t\|_{L^2(0,t;H)} \\ &\quad + c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\varrho_t\|_{L^2(0,t;H)}^{3/2}\|\varrho_t\|_{L^2(0,t;V)}^{1/2} + c\|k\|_{L^2(0,t;V')}^2.\end{aligned} \tag{237}$$

Young's inequality and (143) yield

$$\begin{aligned}\frac{c}{p+1}|\theta(t)|_{p+1}^{p+1} &+ \frac{1}{4}\|\theta\|_{L^2(0,t;V)}^2 \\ &\leqslant \delta\|\varrho_t\|_{L^2(0,t;V)}^2 + c(\delta)\|\varrho_t\|_{L^2(0,t;H)}^2 + c\|k\|_{L^2(0,t;V')}^2\end{aligned} \tag{238}$$

for $\delta > 0$. Now, summing (234) and (238) multiplied by $8c_{16}$, we obtain the relation

$$\begin{aligned}\frac{1}{2}\|\varrho_t\|_{L^2(0,t;V)}^2 &+ \frac{1}{2}|A\varrho(t)|^2 + \frac{c}{p+1}|\theta(t)|_{p+1}^{p+1} + c_{16}\|\theta\|_{L^2(0,t;V)}^2 \\ &\leqslant 8c_{16}\delta\|\varrho_t\|_{L^2(0,t;V)}^2 + c(\delta)\|\varrho_t\|_{L^2(0,t;H)}^2 + c\|k\|_{L^2(0,t;V')}^2.\end{aligned} \tag{239}$$

□

Taking $\delta < 1/(16c_{16})$, (232) follows immediately from (146).

We return to (231) and deduce

$$\frac{c_3}{2}\theta_*^{p-1}\|\theta_t\|_{L^2(0,t;H)}^2 + \frac{1}{4}\|\theta(t)\|^2 \leqslant c + c\|\varrho_t\|_{L^\infty(0,t;H)}^2. \tag{240}$$

It turns out from Lemma 6.4 that

$$\frac{c_3}{2}\theta_*^{p-1}\|\theta_t\|_{L^2(0,t;H)}^2 + \frac{1}{4}\|\theta(t)\|^2 \leqslant c + c\|\theta_t\|_{L^2(0,t;H)}, \tag{241}$$

which results in (154).

Third case ($N = 2$, $p > 2$):

We multiply (24) by θ_t and then integrate over Q_t, $t \leqslant \nu$ to obtain

$$
\begin{aligned}
(242)\quad & c_3\theta_*^{p-1}\|\theta_t\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\theta(t)\|^2 \\
&\leqslant c + \int_0^t\int_\Gamma g\theta_t \,\mathrm{d}\sigma\,\mathrm{d}s + \int_0^t\int_\Omega (\varrho_t)^2\theta_t \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \theta\varrho\varrho_t\theta_t \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant c + \frac{1}{4}\|\theta(t)\|^2 + c\|\theta\|^2_{L^2(0,t;V)} + \int_0^t |\varrho_t|_4^2|\theta_t| \,\mathrm{d}s \\
&\quad + c\int_0^t |\theta|_p|\varrho_t|_{2p/(p-2)}|\theta_t| \,\mathrm{d}s + c\|g\|^2_{H^1(0,t;L^2(\Gamma))}.
\end{aligned}
$$

Owing to the Ladyzhenskaya interpolation inequality (see [28, Chapter II, (3.1)])

$$
(243)\qquad |\varrho_t|_4 \leqslant 2^{1/4}|\varrho_t|^{1/2}\|\varrho_t\|^{1/2},
$$

the continuous embedding $V \hookrightarrow L^{2p/(p-2)}(\Omega)$ and (242)–(243) we infer that

$$
\begin{aligned}
(244)\quad & c_3\theta_*^{p-1}\|\theta_t\|^2_{L^2(0,t;H)} + \frac{1}{4}\|\theta(t)\|^2 \\
&\leqslant c + c\|\theta\|^2_{L^2(0,t;V)} + \int_0^t \|\varrho_t\||\varrho_t||\varrho_t| \,\mathrm{d}s + c\int_0^t |\theta|_p\|\varrho_t\||\theta_t| \,\mathrm{d}s \\
&\leqslant c + c\|\theta\|^2_{L^2(0,t;V)} + \|\varrho_t\|_{L^\infty(0,t;H)}\|\varrho_t\|_{L^2(0,t;V)}\|\theta_t\|_{L^2(0,t;H)} \\
&\quad + c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\varrho_t\|_{L^2(0,t;V)}\|\theta_t\|_{L^2(0,t;H)}.
\end{aligned}
$$

Lemma 6.11. *In the two-space dimension case and when $p > 2$ we have*

$$
(245)\qquad \|\theta\|^2_{L^2(0,t;V)} + \|\theta\|^{p+1}_{L^\infty(0,t;L^{p+1}(\Omega))} + \|\varrho\|^2_{L^\infty(0,t;W)} + \|\varrho_t\|^2_{L^2(0,t;V)} \leqslant c.
$$

Proof. We test (25) by $A\varrho_t$, then integrate with respect to time and use relations (31), (143), and (149) to obtain

$$
\begin{aligned}
(246)\quad & \|\varrho_t\|^2_{L^2(0,t;V)} + \frac{1}{2}|A\varrho(t)|^2 \\
&\leqslant c + c\int_0^t \|F'(\varrho)\|\|\varrho_t\| \,\mathrm{d}s + c\int_0^t \|\theta\varrho\|\|\varrho_t\| \,\mathrm{d}s \\
&\leqslant c + c\int_0^t |\nabla\varrho|\|\varrho_t\| \,\mathrm{d}s + c\int_0^t \|\varrho_t\| \,\mathrm{d}s + c\int_0^t |\theta|_p|\nabla\varrho|_{2p/(p-2)}\|\varrho_t\| \,\mathrm{d}s \\
&\quad + c\int_0^t \|\varrho\|^2_{L^\infty(\Omega)}|\nabla\theta|\|\varrho_t\| \,\mathrm{d}s + c\int_0^t \|\varrho\|^2_{L^\infty(\Omega)}|\theta|\|\varrho_t\| \,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant c + c\int_0^t \|\varrho\|\|\varrho_t\| \,\mathrm{d}s + c\int_0^t \|\varrho_t\| \,\mathrm{d}s + c\int_0^t |\theta|_p \|\nabla\varrho\|\|\varrho_t\| \,\mathrm{d}s \\
&\quad + c\int_0^t \|\varrho\|^2_{L^\infty(\Omega)}(|\nabla\theta| + |\theta|)\|\varrho_t\| \,\mathrm{d}s \\
&\leqslant c + cT^{1/2}\|\varrho\|_{L^\infty(0,t;V)}\|\varrho_t\|_{L^2(0,t;V)} + cT^{1/2}\|\varrho_t\|_{L^2(0,t;V)} \\
&\quad + c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\varrho\|_{L^2(0,t;W)}\|\varrho_t\|_{L^2(0,t;V)} \\
&\quad + c\|\varrho\|^2_{L^\infty(Q_t)}\|\theta\|_{L^2(0,t;V)}\|\varrho_t\|_{L^2(0,t;V)} \\
&\leqslant c + \frac{1}{2}\|\varrho_t\|^2_{L^2(0,t;V)} + c_{17}\|\theta\|^2_{L^2(0,t;V)}.
\end{aligned}
$$

Now, we test (24) by θ and use the Ladyzhenskaya interpolation inequality to infer that

$$
\begin{aligned}
\frac{c}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|^{p+1}_{p+1} + \frac{1}{2}\|\theta\|^2_J &\leqslant c|\theta|_{2p/(p-2)}|\theta|_p|\varrho_t| + c|\theta|_p|\varrho_t|^2_4 + c\|k\|^2_{V'} \\
&\leqslant c\|\theta\||\theta|_p|\varrho_t| + c|\theta|_p|\varrho_t|\|\varrho_t\| + c\|k\|^2_{V'}.
\end{aligned} \tag{247}
$$

Then, integration of this inequality with respect to t gives

$$
\begin{aligned}
&\frac{c}{p+1}|\theta(t)|^{p+1}_{p+1} + \frac{1}{2}\|\theta\|^2_{L^2(0,t;V)} \\
&\qquad \leqslant c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\theta\|_{L^2(0,t;V)}\|\varrho_t\|_{L^2(0,t;H)} \\
&\qquad\quad + c\|\theta\|_{L^\infty(0,t;L^p(\Omega))}\|\varrho_t\|_{L^2(0,t;V)}\|\varrho_t\|_{L^2(0,t;H)} \\
&\qquad\quad + c\|k\|^2_{L^2(0,t;V')}.
\end{aligned} \tag{248}
$$

The remaining part of this proof is essentially the same as that of the previous lemma for (237) and, hence, we can omit the details. □

Finally, (154) follows from (244), Lemma 6.4, and Lemma 6.11.

Since all the above a priori estimates are independent of the time t, we deduce that our solution, furnished by Theorem 4.1, can be extended beyond $t = T$, which gives the global existence stated in Theorem 3.1.

7. Continuous dependence

This section is devoted to the proof of uniqueness in Theorem 3.1. More precisely, we prove the following continuous dependence result.

Lemma 7.1. *Let* (C1)–(C2) *hold and let* F *be as in* (H4)–(H5)*. Let us be given pairs of data* θ_{0i}*,* ϱ_{0i} *and* g_i*,* $i = 1, 2$*, satisfying* (H1)–(H3) *and* (H6)*. Denote by* (θ_i, ϱ_i)*,* $i = 1, 2$*, two corresponding solutions to system* (24)–(30)*. Then for each* $T > 0$ *there is a constant* $c(T)$ *which is allowed to depend on* T *and on the data in* (H1)–(H6)*, of course with* $i = 1, 2$*, such that*

$$
\begin{aligned}
(249) \quad \|\Phi(\theta_1) - \Phi(\theta_2)\|^2_{L^\infty(0,T;V')} + \|\varrho_{1t} - \varrho_{2t}\|^2_{L^2(0,T;H)} + \|\varrho_1 - \varrho_2\|^2_{L^\infty(0,T;V)} \\
+ \|\theta_1 - \theta_2\|^2_{L^2(0,T;H)} \\
\leqslant c(T)\{\|\Phi(\theta_{01}) - \Phi(\theta_{02})\|^2_{V'} + \|k_1 - k_2\|^2_{L^2(0,T;V')} + \|\varrho_{01} - \varrho_{02}\|^2\}.
\end{aligned}
$$

P r o o f. We set $\theta = \theta_1 - \theta_2$, $u = \Phi(\theta_1) - \Phi(\theta_2)$, $\varrho = \varrho_1 - \varrho_2$ and $k = k_1 - k_2$. We start by considering the difference between the corresponding equations (25) and we test the resulting formula by ϱ_t. Then, from (H4) and (31) we infer that

$$
(250) \qquad \frac{1}{4}|\varrho_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|^2 \leqslant c|\varrho|^2 + c_{17}|\theta|^2 + c\|\theta_2\|^2\|\varrho\|^2.
$$

Next, we consider the difference between the corresponding equations (24) and we test it by $J^{-1}u$. Then, Hölder's inequality, the continuous embedding $L^{6/5}(\Omega) \hookrightarrow V'$ and (122) imply

$$
\begin{aligned}
(251) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V',J} + c_{11}|\theta|^2 \\
\leqslant \|u\|_{V',J}\{|\varrho_t||\varrho_{1t} + \varrho_{2t}|_3 + c|\theta||\varrho_{1t}|_3 + |\theta_2|_6|\varrho|_6|\varrho_{1t}| \\
+ c|\theta_2|_6|\varrho_t| + \|k\|_{V',J}\}.
\end{aligned}
$$

Multiplying (250) by $\sigma > 0$, then adding together with (251) and using repeatedly Young's inequality yields

$$
\begin{aligned}
(252) \qquad \frac{\sigma}{8}|\varrho_t|^2 + \frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V',J} + \frac{c_{11}}{2}|\theta|^2 \\
\leqslant c\sigma|\varrho|^2 + c_{17}\sigma|\theta|^2 + c\sigma\|\theta_2\|^2\|\varrho\|^2 + \|k\|^2_{V',J} + \|\varrho\|^2|\varrho_{1t}|^2 \\
+ c(\sigma, \theta_*)\|u\|^2_{V',J}\{1 + |\varrho_{1t}|^2_3 + |\varrho_{2t}|^2_3 + \|\theta_2\|^2\}.
\end{aligned}
$$

Finally, using the regularity of $\partial_t\varrho_i$ given by (30), taking $\sigma < c_{11}/(2c_{17})$ and applying Gronwall's lemma, we have the assertion. □

Thus, the proof of Theorem 3.1 is complete. □

References

[1] *R. A. Adams*: Sobolev Spaces. Academic Press, New York, 1975.

[2] *R. A. Adams, J. Fournier*: Cone conditions and properties of Sobolev spaces. J. Math. Anal. Appl. *61* (1977), 713–734.

[3] *S. Agmon, A. Douglis, L. Nirenberg*: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II. Commun. Pure Appl. Math. *12* (1959), 623–727; *17* (1964), 35–92.

[4] *H. W. Alt, I. Pawlow*: A mathematical model of dynamics of non-isothermal phase separation. Physica D *59* (1992), 389–416.

[5] *H. Attouch*: Variational Convergence for Functions and Operators. Pitman, London, 1984.

[6] *C. Baiocchi*: Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert. Ann. Mat. Pura Appl. *76* (1967), 233–304. (In Italian.)

[7] *V. Barbu*: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff, Leyden, 1976.

[8] *D. Blanchard, G. A. Francfort*: A few results on a class of degenerate parabolic equations. Ann. Sc. Norm. Sup. Pisa *18* (1991), 213–249.

[9] *D. Blanchard, O. Guibé*: Existence of a solution for a nonlinear system in thermoviscoelasticity. Adv. Differ. Equ. *5* (2000), 1221–1252.

[10] *D. Blanchard, H. Redwane*: Renormalized solutions for a class of nonlinear evolution problems. J. Math. Pures Appl. *77* (1998), 117–151.

[11] *G. Bonfanti, M. Frémond, F. Luterotti*: Global solution to a nonlinear system for inversible phase changes. Adv. Math. Sci. Appl. *10* (2000), 1–24.

[12] *G. Bonfanti, M. Frémond, F. Luterotti*: Local solutions to the full model of phase transitions with dissipation. Adv. Math. Sci. Appl. *11* (2001), 791–810.

[13] *G. Bonfanti, M. Frémond, F. Luterotti*: Existence and uniqueness results to a phase transition model based on microscopic accelerations and movements. Nonlinear Anal., Real World Appl. *5* (2004), 123–140.

[14] *G. Bonfanti, F. Luterotti*: Global solution to a phase transition model with microscopic movements and accelerations in one space dimension. Commun. Pure Appl. Anal. *5* (2006), 763–777.

[15] *H. Brézis*: Analyse fonctionnelle. Théorie et applications. Masson, Paris, 1983. (In French.)

[16] *P. Colli*: On some doubly nonlinear evolution equations in Banach spaces. Japan J. Ind. Appl. Math. *9* (1992), 181–203.

[17] *P. Colli, M. Frémond, O. Klein*: Global existence of a solution to phase field model for supercooling. Nonlinear Anal., Real World Appl. *2* (2001), 523–539.

[18] *P. Colli, G. Gilardi, M. Grasselli*: Well-posedness of the weak formulation for the phase-field model with memory. Adv. Differ. Equ. *2* (1997), 487–508.

[19] *P. Colli, G. Gilardi, M. Grasselli, G. Schimperna*: Global existence for the conserved phase field model with memory and quadratic nonlinearity. Port. Math. (N.S.) *58* (2001), 159–170.

[20] *P. Colli, Ph. Laurençot*: Existence and stabilization of solutions to the phase-field model with memory. J. Integral Equations Appl. *10* (1998), 169–194.

[21] *P. Colli, F. Luterotti, G. Schimperna, U. Stefanelli*: Global existence for a class of generalized systems for irreversible phase changes. NoDEA, Nonlinear Differ. Equ. Appl. *9* (2002), 255–276.

[22] *A. Damlamian*: Some results on the multi-phase Stefan problem. Commun. Partial Differ. Equations *2* (1977), 1017–1044.

[23] *A. Damlamian, N. Kenmochi*: Evolution equations generated by subdifferentials in the dual space of $H^1(\Omega)$. Discrete Contin. Dyn. Syst. *5* (1999), 269–278.
[24] *R. J. DiPerna, J.-L. Lions*: On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Ann. Math. *130* (1989), 321–366.
[25] *R. J. DiPerna, J.-L. Lions*: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. *98* (1989), 511–547.
[26] *M. Frémond*: Non-Smooth Thermomechanics. Springer, Berlin, 2002.
[27] *M. Gurtin*: Generalized Ginzburg-Landau and Cahn-Hilliard equations based on microforce balance. Physica D *92* (1996), 178–192.
[28] *O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva*: Linear and Quasi-linear Equations of Parabolic Type. Translation of Mathematical Monographs, 23. AMS, Providence, 1968.
[29] *Ph. Laurençot, G. Schimperna, U. Stefanelli*: Global existence of a strong solution to the one-dimensional full model for irreversible phase transitions. J. Math. Anal. Appl. *271* (2002), 426–442.
[30] *J.-L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod/Gauthier-Villars, Paris, 1969. (In French.)
[31] *J.-L. Lions, E. Magenes*: Problèmes aux limites non homogènes et applications. Dunod, Paris, 1968. (In French.)
[32] *F. Luterotti, U. Stefanelli*: Existence result for the one-dimensional full model of phase transitions. Z. Anal. Anwend. *21* (2002), 335–350.
[33] *F. Luterotti, G. Schimperna, U. Stefanelli*: Existence result for a nonlinear model related to irreversible phase changes. Math. Models Methods Appl. Sci. *11* (2001), 808–825.
[34] *F. Luterotti, G. Schimperna, U. Stefanelli*: Global solution to a phase field model with irreversible and constrained phase evolution. Q. Appl. Math. *60* (2002), 301–316.
[35] *F. Luterotti, G. Schimperna, U. Stefanelli*: Local solution to Frémond's full model for irreversible phase transitions. In: Mathematical Models and Methods for Smart Materials. Proc. Conf., Cortona, Italy, June 25–29, 2001 (M. Fabrizio, B. Lazzari, A. Mauro, eds.). World Scientific, River Edge, 2002, pp. 323–328.
[36] *A. Miranville, G. Schimperna*: Nonisothermal phase separation based on a microforce balance. Discrete Contin. Dyn. Syst., Ser. B *5* (2005), 753–768.
[37] *A. Miranville, G. Schimperna*: Global solution to a phase transition model based on a microforce balance. J. Evol. Equ. *5* (2005), 253–276.
[38] *L. Nirenberg*: On elliptic partial differential equations. Ann. Sc. Norm. Sup. Pisa, III. Ser. *123* (1959), 115–162.
[39] *J. E. Rakotoson, J. M. Rakotoson*: Analyse fonctionnelle appliquée aux équations aux dérivées partielles. Presse Universitaires de France, 1999. (In French.)
[40] *J. Simon*: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl., IV. Ser. *146* (1978), 65–96.
[41] *R. Temam*: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York, 1988.

Author's address: *N. Fterich*, Laboratoire de Mathématiques et Applications, UMR CNRS 6086, Université de Poitiers-SP2MI, Boulevard Marie et Pierre Curie, F-86962 Chasseneuil Futuroscope Cedex, France, e-mail: `nesrine_ft@hotmail.fr`.