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NUMERICAL MODELLING OF SEMI-COERCIVE BEAM PROBLEM WITH UNILATERAL ELASTIC SUBSOIL OF WINKLER'S TYPE*

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Abstract. A non-linear semi-coercive beam problem is solved in this article. Suitable numerical methods are presented and their uniform convergence properties with respect to the finite element discretization parameter are proved here. The methods are based on the minimization of the total energy functional, where the descent directions of the functional are searched by solving the linear problems with a beam on bilateral elastic "springs". The influence of external loads on the convergence properties is also investigated. The effectiveness of the algorithms is illustrated on numerical examples.

Keywords: non-linear subsoil of Winkler's type, semi-coercive beam problem, approximation, iterative methods, convergence, projection, load stability

MSC 2010: 74B20, 74K10, 90C20, 90C31

1. Introduction

The semi-coercive problem of a beam on a unilateral elastic subsoil means to minimize a convex, differentiable and non-linear functional. The functional is coercive only if additional assumptions on external loads are formulated. The solvability and the finite element approximation of the problem have been investigated in [10]. There are some methods how to numerically solve the class of such problems. The methods based on linear complementarity are presented in [5]. The augmented Lagrangian method with different finite elements and meshes for the beam and the subsoil is investigated in [6], [7]. The methods for quadratic programming can also be used due to the dual formulations of the problems, see [9].

In this article, the total energy functional is minimized so that the descent directions of the functional are searched by solving the linear problems with a beam

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on bilateral elastic "springs". We obtain the so-called "descent direction method without projection" and prove its uniform convergence properties with respect to refinement of the partition. Since the problem is only semi-coercive, it is also useful to investigate the influence of the load on the convergence. Mainly for "unstable" cases of the load, the rate of convergence can be improved by adding the so-called "projection" step. We obtain the "descent direction method with projection", which has the same convergence properties as the former method.

In Section 2, the formulations of the problem, its approximation and the basic results of the article [10] are summarized. Moreover, two useful lemmas are added. In Section 3, auxiliary linear problems with bilateral elastic "springs" are defined and their uniform properties are derived. In Section 4, the descent direction methods with and without projection are introduced and their uniform convergence properties are proved. In Section 5, the approximated problem and algorithms are rewritten to their algebraical forms and the reason of the "projection" step is explained. And in Section 6, the effectiveness of the algorithms is illustrated by numerical examples.

2. Overview of the semi-coercive beam problem on unilateral elastic subsoil

2.1. Notation

We will use the Lebesgue spaces $L^p(\Omega)$, $p = 2, \infty$, Sobolev spaces $H^k(\Omega) \equiv W^{k,2}(\Omega)$, $k = 0, 1, 2, 3, 4$, and the spaces of continuously differentiable functions $C^k(\overline{\Omega})$, where Ω is an open, bounded and non-empty interval in $\mathbb{R}^1$. The spaces are described in the book [1]. Their standard norms are denoted as $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{k,2,\Omega}$ and $\|\cdot\|_{C^k(\overline{\Omega})}$, respectively. The ith seminorms, $i = 0, 1, \ldots, k$, of the spaces $H^k(\Omega)$ are denoted as $|\cdot|_{i,2,\Omega}$. The space of polynomials of the kth degree is denoted as P_k.

Since we will mainly use the interval $\Omega := (0, l)$ throughout the article, we will denote the norms and seminorms of the Sobolev spaces $H^k(\Omega)$, $k = 0, 1, 2, 3, 4$, without the symbol Ω for this particular choice of the interval.

With respect to the well-known imbedding theorem for the Sobolev space $H^2(\Omega)$, see [1], we will assume that the functions $v \in H^2(\Omega)$ also belong to $C^1(\overline{\Omega})$ to define the values $v(x)$, $v'(x)$, $x \in \overline{\Omega}$.

2.2. Setting of the problem

We consider a beam of the length l with free ends which is situated in the interval $\Omega = (0, l)$, and assume that the beam is supported by a unilateral elastic subsoil in the interval $\Omega_s := (x_l, x_r)$, $0 \leqslant x_l < x_r \leqslant l$. Such a subsoil is active only if the beam deflects against it. Let E, I and q denote functions that represent, respectively, Young's modulus of the beam material, the inertia moment of the cross-section of the beam, and the stiffness coefficient of the subsoil. The aim is to find the deflection w^* of the axes of the beam caused by the beam load. The situation is depicted in Fig. 1.

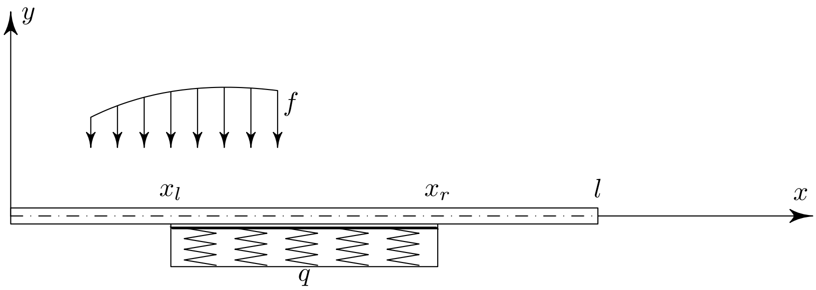

Figure 1. Scheme of the subsoiled beam with axes orientation.

We will assume that the functions E, I, q belong to the Lebesgue space $L^\infty(\Omega)$ and there exist positive constants E_0, I_0 and q_0 such that

$$E(x) \geqslant E_0, \;\; I(x) \geqslant I_0 \;\text{ a.e. in } \Omega, \quad \text{and} \quad q(x) \geqslant q_0 \;\text{ a.e. in } \Omega_s.$$

Then we can define forms

$$\begin{aligned} a(v_1, v_2) &:= \int_\Omega EIv_1''v_2''\,\mathrm{d}x, \quad v_1, v_2 \in H^2(\Omega), \\ b(v_1, v_2) &:= \int_{\Omega_s} qv_1v_2\,\mathrm{d}x, \quad v_1, v_2 \in H^1(\Omega), \end{aligned}$$

to represent the work of the inner forces and the subsoil, respectively. The forms a, b are bilinear and bounded on the space $H^2(\Omega)$.

The space of all continuous and linear functionals defined on $H^2(\Omega)$ will be denoted V^* and its corresponding norm by $\|\cdot\|_*$. The work of the beam load will be represented by a functional $L \in V^*$.

The total potential energy functional for the problem has the form

$$J(v) := \frac{1}{2}\big(a(v, v) + b(v^-, v^-)\big) - L(v), \quad v \in H^2(\Omega). \tag{2.1}$$

The functional J is Gâteaux differentiable and convex on the space $H^2(\Omega)$. Its Gâteaux derivative at any point $w \in H^2(\Omega)$ and any direction $v \in H^2(\Omega)$ has the form

$$J'(w; v) = a(w, v) + b(w^-, v) - L(v). \tag{2.2}$$

The variational formulation of the problem can be written as the minimization problem

$$\text{(P)} \qquad \text{find } w^* \in H^2(\Omega): \; J(w^*) \leqslant J(v) \;\; \forall\, v \in H^2(\Omega),$$

or equivalently, with respect to (2.2), as the non-linear variational equation

$$a(w^*, v) + b((w^*)^-, v) = L(v) \quad \forall\, v \in H^2(\Omega). \tag{2.3}$$

Notice that for sufficiently smooth data, the problem means to solve a non-linear differential equation of the fourth order with homogeneous Neumann boundary conditions.

2.3. Solvability and dependence on the load

Since the beam does not have fixed ends (it is only laid on the subsoil), the problem solvability depends on the beam load. The existence and uniqueness of the solution w^* of the problem (P) is ensured by the condition

$$L(p) < 0 \quad \forall\, p \in P_1, \;\; p > 0 \;\text{ in }\; \Omega_s, \tag{2.4}$$

where the polynomials of the first degree represent the rigid beam motions for which the subsoil is not active. Notice that the functional J is coercive on $H^2(\Omega)$ if this condition holds.

For further analysis, it will be useful to rewrite equivalently the condition (2.4) in the following way:

$$F < 0 \quad \text{and} \quad x_l < T < x_r, \tag{2.5}$$

where $F := L(1)$ is the load resultant and $T := L(x)/L(1)$ is the balance point of the load. The condition (2.5) means that the load resultant is situated in Ω_s and oriented against the subsoil, which causes that the beam deflection activates the subsoil on the set $M \subset \Omega_s$ with a positive one-dimensional Lebesgue measure, i.e. $w^* < 0$ in M. In addition, the balance point T lies in the convex closure of the set M.

To determine the dependence of the change of the solution of problem (P) on the change of the load, we will consider the class $\mathcal{S}_{\delta,\xi,\eta}$ of the loads $L \in V^*$ such that $T \in [x_l + \delta, x_r - \delta]$, $F \leqslant -\xi < 0$ and $\|L\|_* \leqslant \eta$, with respect to positive parameters δ, ξ, η. If we assume that $\mathcal{S}_{\delta,\xi,\eta}$ is non-empty then there exists a positive constant c which depends on the loads from $\mathcal{S}_{\delta,\xi,\eta}$ only through the parameters δ, ξ, η so that

$$\|w_1^* - w_2^*\|_{2,2} \leqslant c\|L_1 - L_2\|_* \quad \forall\, L_1, L_2 \in \mathcal{S}_{\delta,\xi,\eta}, \tag{2.6}$$

where $w_i^* = w_i^*(L_i)$ solves the problem (P) with respect to the load L_i, $i = 1, 2$.

The following lemma, which is also important for numerical modelling, describes the dependence of the constant c from the estimate (2.6) on the parameters δ, ξ, η for the limit cases $\delta \to 0$ and $\xi \to 0$.

Lemma 2.1. *Let $\eta > 0$ and $0 < \delta_{\max} < \frac{1}{2}(x_r - x_l)$. Then there exists a positive constant $\xi_{\max}$ depending on η such that for any sequences $\{\delta_k\}_k$, $0 < \delta_k \leqslant \delta_{\max}$, and $\{\xi_k\}_k$, $0 < \xi_k \leqslant \xi_{\max}$, $k \geqslant 0$, the following implication holds: if $\delta_k \to 0$ or $\xi_k \to 0$ then $c_k \to \infty$, where $c_k = c_k(\delta_k, \xi_k, \eta)$ is the smallest constant which satisfies* (2.6) *for the parameters δ_k, ξ_k, η.*

Proof. We will construct suitable sequences $\{L_{i,k}\}_k \subset V^*$, $i = 1, 2$, to prove the assertion. The corresponding load resultants, their balance points, and solutions of the problems (P) will be respectively denoted by $F_{i,k}$, $T_{i,k}$, and $w_{i,k}$, $i = 1, 2$. Subsequences of these sequences will be denoted in the same way. For the sake of brevity, some steps of the proof will be only sketched.

Case 1. Let $\eta > 0$ and $\delta_k \to 0$. Then there exists $\xi_{\max} > 0$ such that $\|L_{i,k}\|_* \leqslant \eta$, $i = 1, 2$, where

$$L_{1,k}(v) := \xi_k v((x_l + x_r)/2), \quad L_{2,k}(v) := \xi_k v(x_l + \delta_k), \quad \xi_k \leqslant \xi_{\max}, \quad k \geqslant 0.$$

We will assume that there exists $\xi_{\min} > 0$ such that $\xi_k \geqslant \xi_{\min}$, $k \geqslant 0$, in this first case. Then $F_{1,k} = F_{2,k} = \xi_k$, $T_{1,k} = \frac{1}{2}(x_l + x_r)$, and $T_{2,k} = x_l + \delta_k$. Therefore, $L_{i,k} \in \mathcal{S}_{\delta_k, \xi_{\min}, \eta}$, $i = 1, 2$. The sequence $\{w_{1,k}\}_k$ is bounded on $H^2(\Omega)$ by Theorem 3.2 in [10]. Suppose for a moment that some subsequence of $\{w_{2,k}\}_k$ is bounded on $H^2(\Omega)$. Then we can assume without loss of generality that there exists $w \in H^2(\Omega)$ such that $w_{2,k} \to w$ in $H^1(\Omega)$ by the Rellich theorem. The functions $w_{2,k}$ solve the equation

$$a(w_{2,k}, v) + b(w_{2,k}^-, v) = L_{2,k}(v) \quad \forall\, v \in H^2(\Omega). \tag{2.7}$$

The choice $v(x) = x - x_l \in P_1$ in (2.7) yields

$$b(w^-, v) = \lim_{k\to\infty} b(w_{2,k}^-, v) = \lim_{k\to\infty} L_{2,k}(v) = \lim_{k\to\infty} F_{2,k}(T_{2,k} - x_l) = 0.$$

Hence $w \geqslant 0$ in Ω_s. Then the choice $v(x) = 1 \in P_1$ in (2.7) yields a contradiction:

$$0 = \lim_{k\to\infty} b(w_{2,k}^-, 1) = \lim_{k\to\infty} L_{2,k}(1) = \lim_{k\to\infty} F_{2,k} \leqslant -\xi_{\min} < 0.$$

Therefore, $\|w_{2,k}\|_{2,2} \to \infty$ and by (2.6),

$$c_k \geqslant \frac{\|w_{1,k} - w_{2,k}\|_{2,2}}{\|L_{1,k} - L_{2,k}\|_*} \to \infty.$$

Case 2. Let $\eta > 0$, $0 < \delta_{\min} \leqslant \delta_k \leqslant \delta_{\max} < \frac{1}{2}(x_r - x_l)$ and $\xi_k \to 0$. Let us choose

$$\begin{aligned} L(v) &:= \eta_0 \Big[v(x_l) - 2v\Big(\frac{x_l + x_r}{2}\Big) + v(x_r) \Big], \\ L_{1,k}(v) &:= L(v) - \xi_k v\Big(\frac{x_l + x_r}{2}\Big), \\ L_{2,k}(v) &:= L_{1,k}(v) - \varepsilon_k v(x_l), \end{aligned}$$

where $\varepsilon_k = \xi_k\big(\frac{1}{2}(x_l + x_r) - (x_l + \delta_k)\big)/\delta_k > 0$ and $\eta_0 > 0$ is chosen such that $\|L_{i,k}\|_* \leqslant \eta$, $i = 1, 2$, for sufficiently large k. Then $L(1) = 0$, $L(x) = 0$, $F_{1,k} = -\xi_k \to 0$, $F_{2,k} = -\xi_k - \varepsilon_k$, $T_{1,k} = \frac{1}{2}(x_l + x_r)$, $T_{2,k} = x_l + \delta_k$, $L_{i,k} \in \mathcal{S}_{\delta_{\min},\xi_k,\eta}$, and $L_{i,k} \to L$ in V^*, $i = 1, 2$.

By Theorem 3.2 in [10], the sequences $\{w_{1,k}\}_k$, $\{w_{1,k}\}_k$ are bounded on $H^2(\Omega)$. Therefore, there exist subsequences $\{w_{i,k}\}_k$ and functions $w_i \in H^2(\Omega)$ such that $w_{i,k} \rightharpoonup w_i$ weakly in $H^2(\Omega)$ and $w_{i,k} \to w_i$ in $H^1(\Omega)$ (by the Rellich theorem), $i = 1, 2$. Since the functions $w_{i,k}$ solve the equations

$$a(w_{i,k}, v) + b(w_{i,k}^-, v) = L_{i,k}(v) \quad \forall\, v \in H^2(\Omega),\ \ i = 1, 2,\ \ k \geqslant 0,$$

the limit case $k \to \infty$ leads to

$$a(w_i, v) + b(w_i^-, v) = L(v) \quad \forall\, v \in H^2(\Omega),\ \ i = 1, 2.$$

The choice $v = 1$ yields $b(w_i^-, 1) = 0$. Thus $w_1, w_2 \geqslant 0$ in Ω_s and consequently, w_1, w_2 solve the Neumann problem

$$a(w_i, v) = L(v) \quad \forall\, v \in H^2(\Omega),\ \ i = 1, 2. \tag{2.8}$$

Hence, there exists a polynomial $p \in P_1$ such that $w_1 - w_2 = p$. Notice that if a function $v \in H^2(\Omega)$ is convex and $v \notin P_1$ in Ω_s then $L(v) > 0$. From this result and equation (2.8) it is possible to prove that $w_i'' > 0$ almost everywhere in Ω_s, $i = 1, 2$. It means that the functions w_1, w_2 are strictly convex in Ω_s and have just one minimum in $\overline{\Omega}_s$.

By Lemma 3.5 in [10], there exist sequences $\{x_{i,k}\}_k, \{y_{i,k}\}_k \subset \Omega_s$ and their limits x_i, y_i, $i = 1, 2$, such that

$$w_{i,k}(x_{i,k}) \leqslant 0, \quad w_{i,k}(y_{i,k}) \leqslant 0 \quad \text{and} \quad x_{i,k} \leqslant T_{i,k} \leqslant y_{i,k} \quad \forall\, k \geqslant 0,\ \ i = 1, 2.$$

Hence, $w_i(x_i) = w_i(y_i) = 0$, since w_i are non-negative in Ω_s, $i = 1, 2$. Consequently,

$$x_1 = y_1 = \frac{1}{2}(x_l + x_r), \quad x_2 = y_2 = \lim_{k \to \infty} T_{2,k} < \frac{1}{2}(x_l + x_r),$$

since w_i are strictly convex in Ω_s, $i = 1, 2$. Thus $w_1\bigl(\frac{1}{2}(x_l + x_r)\bigr) = 0$ and $w_2(x_l + \delta) = 0 < w_1(x_l + \delta)$. Therefore, $w_1 \neq w_2$ and consequently, by (2.6),

$$c_k \geqslant \frac{\|w_{1,k} - w_{2,k}\|_{2,2}}{\|L_{1,k} - L_{2,k}\|_*} \to \infty.$$

This result holds for any subsequences $\{w_{i,k}\}_k$ with weak limits $w_i \in H^2(\Omega)$, $i = 1, 2$, which means that the whole sequence $\{c_k\}_k$ converges to ∞.

Case 3. Let $\eta > 0$, $\delta_k \to 0$, $\xi_k \to 0$, and $0 < \delta_{\max} < \frac{1}{2}(x_r - x_l)$. Since $\mathcal{S}_{\delta_{\max},\xi_k,\eta} \subset \mathcal{S}_{\delta_k,\xi_k,\eta}$ for sufficiently large k, we have $c_k(\delta_{\max}, \xi_k, \eta) \leqslant c_k(\delta_k, \xi_k, \eta)$, which follows from the estimate (2.6). By Case 2, $c_k(\delta_{\max}, \xi_k, \eta) \to \infty$. Hence, $c_k(\delta_k, \xi_k, \eta) \to \infty$. $\square$

Notice that a small change of the load causes a relatively large "rigid" displacement of the beam in Case 2 of the proof.

With respect to Lemma 2.1, the loads for which the balance point T is close to the end points of the subsoil or the size of the load resultant is small in comparison to V^*-norm of the load, will be called *unstable*. Some unstable loads are illustrated in [11] on numerical examples.

2.4. Approximation of the problem

Let us define a partition τ_h,

$$0 = x_0 < x_1 < \ldots < x_N = l, \;\; h := \max_{j=1,\ldots,N}(x_j - x_{j-1}), \;\; h_{\min} := \min_{j=1,\ldots,N}(x_j - x_{j-1})$$

of the interval $\overline{\Omega} = [0, l]$, with nodal points x_j, $j = 0, 1, \ldots, N$, and parameters h, $h_{\min} > 0$. With respect to a positive parameter θ, we will consider the system $\mathcal{T}_\theta$ of such partitions τ_h for which the inequality $\theta h \leqslant h_{\min}$ holds.

For a partition $\tau_h \in \mathcal{T}_\theta$ with $N + 1$ nodal points, we define the function space

$$V_h \subset H^2(\Omega), \quad V_h := \{v_h \in C^1(\overline{\Omega})\colon\; v_h|_{(x_{j-1},x_j)} \in P_3, \; j = 1, 2, \ldots, N\},$$

i.e. the space of continuously differentiable and piecewise cubic functions.

For the sake of simplicity, we will assume that the function q, which represents the stiffness coefficient of the subsoil, is piecewise constant in the interval Ω_s and that the partitions $\tau_h \in \mathcal{T}_\theta$ take into account the points of discontinuity of q. Since the evaluation of the term $b(w_h^-, v_h)$, $w_h, v_h \in V_h$, cannot be computed exactly due to the non-linear term w_h^-, an approximation of the form b must be used. The form b will be approximated by a numerical quadrature on each subsoiled partition interval. Its approximation has the form

$$b_h(v_1, v_2) := \sum_{i=1}^{m(h)} r_i v_1(z_i) v_2(z_i), \quad v_1, v_2 \in H^2(\Omega), \tag{2.9}$$

where z_i, $z_1 < z_2 < \ldots < z_{m(h)}$, are the points of the numerical quadratures and the coefficients r_i are equal to the products of the stiffness coefficients and weights of the numerical quadrature. With respect to the assumption on $\tau_h \in \mathcal{T}_\theta$, there exist constants $c_1, c_2 > 0$ such that

$$c_1 q_0 \theta h \leqslant r_i \leqslant c_2 \|q\|_\infty h, \quad i = 1, 2, \ldots, m(h). \tag{2.10}$$

From a mechanical point of view, the subsoil is substituted by insulated "springs". We will suppose that the numerical quadrature is exact at least for polynomials of the first degree.

Putting

$$\mathcal{V}_M := \Big\{ v \in H^2(\Omega)\colon\ \exists p \leqslant M,\ \exists y_1, y_2, \ldots, y_{2p} \in \overline{\Omega}_s\colon \\ \{x \in \overline{\Omega}_s\colon\ v^-(x) = 0\} = \bigcup_{i=1}^{p} [y_{2i-1}, y_{2i}] \Big\},\ M > 0,$$

there exist positive constants c_1, c_2 and $c_3 = c_3(M)$, which are independent of the choice of τ_h, such that

$$|b_h(u,v)| \leqslant c_1 \|q\|_{\infty,\Omega_s} \|u\|_{1,2} \|v\|_{1,2} \quad \forall\, u, v \in H^1(\Omega), \tag{2.11}$$

$$|b(v^-, u) - b_h(v^-, u)| \leqslant c_2 h \|v\|_{1,2} \|u\|_{1,2} \quad \forall\, u, v \in H^1(\Omega), \tag{2.12}$$

$$|b(v^-, u) - b_h(v^-, u)| \leqslant c_3 h^2 \|v\|_{2,2} \|u\|_{2,2} \quad \forall\, u \in H^2(\Omega),\ \forall\, v \in \mathcal{V}_M. \tag{2.13}$$

Now, we set the approximated problem. For the sake of simplicity, we will not consider numerical quadrature of the forms a and L. The approximated problem corresponding to the partition $\tau_h \in \mathcal{T}_\theta$ has the form

$$(\mathrm{P}_h) \qquad \begin{cases} \text{find } w_h^* \in V_h\colon\ J_h(w_h^*) \leqslant J_h(v_h) \quad \forall\, v_h \in V_h, \\ J_h(v_h) := \dfrac{1}{2} a(v_h, v_h) + \dfrac{1}{2} b_h(v_h^-, v_h^-) - L(v_h). \end{cases}$$

Since the functional J_h is convex and has the Gâteaux derivative on the space V_h, the problem (P_h) can be rewritten equivalently to the nonlinear variational equation

$$a(w_h^*, v_h) + b_h((w_h^*)^-, v_h) = L(v_h) \quad \forall\, v_h \in V_h. \tag{2.14}$$

The existence of the solution of problem (P_h) is ensured by the condition

$$F < 0 \quad \text{and} \quad z_1 < T < z_{m(h)}. \tag{2.15}$$

This condition also ensures the uniqueness of the solution for sufficiently small h. Notice that if the condition (2.5) holds and the discretization parameter h is sufficiently small, then the condition (2.15) holds, too.

The set

$$A_h^* := \{i \in \{1, \ldots, m(h)\}\colon\ w_h^*(z_i) < 0\}, \tag{2.16}$$

which represents the active "springs", is non-empty. In addition, the balance point T belongs to the convex closure of the points $\{z_i;\ i \in A_h^*\}$.

For the approximated problems (P_h), we have the following estimates and convergence result:

$$\begin{aligned} \|w^* - w_h^*\|_{2,2} &\leqslant c_1(M)h^2\|w^*\|_{4,2}, \quad w^* \in H^4(\Omega) \cap \mathcal{V}_M,\ \forall\, \tau_h \in \mathcal{T}_\theta,\ h \leqslant h_0, \\ \|w^* - w_h^*\|_{2,2} &\leqslant c_2 h\|w^*\|_{3,2}, \quad w^* \in H^3(\Omega), \forall\, \tau_h \in \mathcal{T}_\theta,\ h \leqslant h_0, \\ \|w^* - w_h^*\|_{2,2} &\to 0, \quad w^* \in H^2(\Omega),\ h \to 0, \end{aligned} \tag{2.17}$$

where w^* and w_h^* are respectively the solutions of the problems (P) and (P_h), and h_0 is a sufficiently small parameter. The first of these estimates is numerically illustrated in [11] for some numerical quadratures.

At the end of this section we add a lemma which describes when the functionals J_h are uniformly coercive on $H^2(\Omega)$. The lemma will be also useful for the subsequent analysis.

Lemma 2.2. *Let $F < 0$, $x_l < T < x_r$, $0 < h_0 < \min\{T - x_l, x_r - T\}$, $c \in \mathbb{R}$, and $\theta > 0$. Then there exists a positive constant $\tilde{c}$ such that the following implication holds:*

$$J_h(u_h) \leqslant c \Longrightarrow \|u_h\|_{2,2} \leqslant \tilde{c} \quad \forall\, \tau_h \in \mathcal{T}_\theta,\ h \leqslant h_0,\ \forall\, u_h \in V_h.$$

Proof. Since the proof is similar to the first (existence) part of the proof of Theorem 3.1 in [10], some steps will be done more briefly.

Suppose that the lemma does not hold. Then, by the definition of J_h, there exist sequences $\{\tau_{h_k}\}_k$ and $\{u_k\}_k$, $u_k \in V_{h_k}$, $\|u_k\|_{2,2} \to \infty$ such that

$$0 \leqslant a(u_k, u_k) + b_{h_k}(u_k^-, u_k^-) \leqslant 2L(u_k) + 2c. \tag{2.18}$$

If we divide (2.18) by $\|u_k\|_{2,2}^2$, we obtain

$$a(v_k, v_k) + b_{h_k}(v_k^-, v_k^-) \to 0, \quad v_k := u_k/\|u_k\|_{2,2}.$$

Hence, by the Rellich theorem and (2.11), there exist a subsequence of $\{v_k\}_k$ (denoted in the same way) and a polynomial $p \in P_1$ such that $v_k \to p$ in $H^2(\Omega)$ and

$b_{h_k}(p^-, p^-) \to 0$. By the assumption on h_0, (2.10) or eventually (2.12) for $h_k \to 0$, we obtain $p \geqslant 0$ in the neighbourhood of the point T.

If we divide (2.18) by $\|u_k\|_{2,2}$, then $0 \leqslant L(p) = Fp(T)$. Therefore $p = 0$, since $F < 0$. However, this contradicts $\|v_k\|_{2,2} = 1$. $\square$

3. Linear problems with bilateral elastic springs

In this section we will define the family of linear problems with bilateral elastic "springs" and derive their uniform properties with respect to a refinement of the partition. Such problems will be solved in each iteration of the algorithms which will be presented below, in Section 4.

Let $\tau_h \in \mathcal{T}_\theta$ be a partition of $\overline{\Omega}$ and $A_h \subset \{1, \ldots, m(h)\}$ a non-empty set of indices. Let us define the bilinear form

$$b_h^{A_h}(v_1, v_2) := \sum_{i \in A_h} r_i v_1(z_i) v_2(z_i), \quad v_1, v_2 \in H^2(\Omega), \tag{3.1}$$

where the coefficients r_i and the spring points z_i have been described in the previous section. Let us define the functional

$$J_h^{A_h}(v_h) := \frac{1}{2} a(v_h, v_h) + \frac{1}{2} b_h^{A_h}(v_h, v_h) - L(v_h). \tag{3.2}$$

The corresponding linear problem $(\mathrm{P}_h^{A_h})$ with bilateral elastic springs has the form

$$\text{find } w_h = w_h(A_h) \in V_h : \; J_h^{A_h}(w_h) \leqslant J_h^{A_h}(v_h) \quad \forall\, v_h \in V_h, \tag{3.3}$$

or equivalently

$$\text{find } w_h = w_h(A_h) \in V_h : \; a(w_h, v_h) + b_h^{A_h}(w_h, v_h) = L(v_h) \quad \forall\, v_h \in V_h. \tag{3.4}$$

Lemma 3.1. *Let $\theta > 0$, $\tau_h \in \mathcal{T}_\theta$, and $\operatorname{card}(A_h) \geqslant 2$. Then the problem $(\mathrm{P}_h^{A_h})$ has a unique solution.*

If the condition (2.15) holds and $\operatorname{card}(A_h) = 1$ then $(\mathrm{P}_h^{A_h})$ has a solution if and only if $z_i = T$, where $i \in A_h$. In such a case, if $w_h(A_h)$ solves $(\mathrm{P}_h^{A_h})$ then $w_h(A_h) + p$, where $p \in P_1$, $p(T) = 0$, also solves $(\mathrm{P}_h^{A_h})$.

Proof. If $\tau_h \in \mathcal{T}_\theta$ and $\operatorname{card}(A_h) \geqslant 2$ then there exists $c > 0$ such that the inequality

$$c\|v\|_{2,2}^2 \leqslant a(v, v) + b_h^{A_h}(v, v) \quad \forall v \in H^2(\Omega) \tag{3.5}$$

holds. The proof of the inequality (3.5) is quite similar to the proof of the Poincaré inequality, see [4] and also the proof of Lemma 3.2. Notice that if $b_h^{A_h}(1,1) \to 0$ for $h \to 0$, then $c \to 0$.

The inequality (3.5) yields that the functional $J_h^{A_h}$ is coercive on V_h. Since J_h is also strictly convex and differentiable on V_h, the problem $(\mathrm{P}_h^{A_h})$ has a unique solution by the well-known theorems of the variational calculus, see for example [3].

Suppose that $A_h = \{i\}$, $i \in \{1, 2, \ldots, m(h)\}$. Then the choices $v_h = 1$ and $v_h = x$ in the equation (3.4) and the definitions of T, F yield that $z_i = T$ and $w_h(z_i) = F/r_i$, provided the problem $(\mathrm{P}_h^{A_h})$ has a solution w_h. Let us define an auxiliary Neumann problem

$$\text{find } \tilde{w}_h \in V_h: \ a(\tilde{w}_h, v_h) = L(v_h) - b_h^{A_h}(F/r_i, v_h) \quad \forall\, v_h \in V_h. \tag{3.6}$$

Such a problem has a solution, since

$$L(p) - b_h^{A_h}(F/r_i, p) = 0 \quad \forall\, p \in P_1.$$

If $\tilde{w}_h$ is a solution of the problem (3.6) then the other solutions have the form $\tilde{w}_h + p$, $p \in P_1$. Therefore, we can assume that there exists a solution w_h of (3.6) such that $w_h(z_i) = F/r_i$. Now, it is easy to show that the functions $w_h + p$, where $p \in P_1$, $p(T) = 0$, also solve $(\mathrm{P}_h^{A_h})$. □

Corollary 3.1. *Let the condition* (2.15) *hold. Then the solution* w_h^* *of the problem* (P_h) *also solves the problem* $(\mathrm{P}_h^{A_h^*})$, *where* A_h^* *is defined by* (2.16).

To show some uniform properties of the problems $(\mathrm{P}_h^{A_h})$ with respect to $\tau_h \in \mathcal{T}_\theta$ and A_h, we introduce the notation

$$\mathcal{A} := \bigcup_h \{A_h \subset \{1, \ldots, m(h)\}\colon \ \mathrm{card}(A_h) \geqslant 2\},$$
$$\mathcal{A}_\varrho := \bigcup_h \{A_h \subset \{1, \ldots, m(h)\}\colon \ \mathrm{card}(A_h) \geqslant \min\{m(h), \max\{2, \varrho/h\}\}\}, \ \varrho > 0.$$

Notice that the parameter ϱ means the "relative" number of the spring points, since

$$\exists\, c_1, c_2 > 0\colon \ c_1/h \leqslant m(h) \leqslant c_2/h \quad \forall\, \tau_h \in \mathcal{T}_\theta.$$

If $\{A_h\}_h \subset \mathcal{A}$ is such a sequence that $\mathrm{card}(A_h)h \to 0$, or equivalently $b_h^{A_h}(1,1) \to 0$ (see the estimate (2.10)), then $\{A_h\}_h \not\subset \mathcal{A}_\varrho$ for any $\varrho > 0$.

Lemma 3.2. *Let* $\theta, \varrho > 0$. *Then there exist positive constants* c_1, c_2 *depending on* $\theta, \varrho > 0$ *such that for any* $\tau_h \in \mathcal{T}_\theta$ *and any* $A_h \in \mathcal{A}_\varrho$ *the estimate*

$$c_1 \|v_h\|_{2,2}^2 \leqslant a(v_h, v_h) + b_h^{A_h}(v_h, v_h) \leqslant c_2 \|v_h\|_{2,2}^2 \quad \forall\, v_h \in V_h \tag{3.7}$$

holds.

P r o o f. The second inequality in (3.7) follows from (2.11), since $b_h^{A_h}(v_h, v_h) \leqslant b_h(v_h, v_h)$. Suppose that the first inequality in (3.7) does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $\{A_{h_k}\}_k$, and $\{v_{h_k}\}_k$ such that

$$a(u_k, u_k) + b_{h_k}^{A_{h_k}}(u_k, u_k) < \frac{1}{k}, \quad k \geqslant 1, \ u_k := \frac{v_{h_k}}{\|v_{h_k}\|_{2,2}}.$$

Hence, by the Rellich theorem and (2.11), we obtain

$$\exists \{u_{k'}\}_{k'} \subset \{u_k\}_k\colon \ u_{k'} \to p \in P_1 \text{ in } H^2(\Omega) \quad \text{and} \quad b_{h_{k'}}^{A_{h_{k'}}}(p, p) \to 0. \tag{3.8}$$

Since $\|u_{k'}\|_{2,2} = 1$, we find that $p \neq 0$, i.e. there exists at most one point $x \in \mathbb{R}$ such that $p(x) = 0$. Therefore, for a sufficiently small $\varepsilon > 0$ there exist $p_0 > 0$ and $0 < \tilde{\varrho} \leqslant \varrho$ such that

$$|p| \geqslant p_0 \text{ in } \tilde{\Omega}_s \quad \text{and} \quad \operatorname{card}(\tilde{A}_{h_{k'}}) \geqslant \tilde{\varrho}/h_{k'},$$

where $\tilde{\Omega}_s := \overline{\Omega}_s \setminus (x - \varepsilon, x + \varepsilon)$ and $\tilde{A}_{h_{k'}} := \{i \in A_{h_{k'}}\colon z_i^{k'} \in \tilde{\Omega}_s\}$, $z_i^{k'}$ being the spring points of the partition $\tau_{h_{k'}}$. Then, by the estimate (2.10), there exists a positive constant c such that

$$b_{h_{k'}}^{A_{h_{k'}}}(p, p) \geqslant c h_{k'} p_0^2 \sum_{i \in \tilde{A}_{h_{k'}}} 1 \geqslant c \tilde{\varrho} p_0^2 > 0.$$

However, this contradicts (3.8). Therefore, the estimate (3.7) holds. □

Corollary 3.2. *Let $\theta, \varrho > 0$. Then there exists a positive constant c depending on $\theta, \varrho > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$ and any $A_h \in \mathcal{A}_\varrho$*

$$\|w_h(A_h)\|_{2,2} \leqslant c\|L\|_*, \quad w_h(A_h) \textit{ solves } (\mathrm{P}_h^{A_h}). \tag{3.9}$$

P r o o f. The proof immediately follows from the equation (3.4) and the estimate (3.7). □

Let $\tau_h \in \mathcal{T}_\theta$ and $v \in H^2(\Omega)$. Then we can introduce the notation

$$A_h(v) := \{i \in \{1, \ldots, m(h)\}\colon \ v(z_i) < 0\}. \tag{3.10}$$

In particular, we will be interested in the relative cardinality of the set $A_h(w_h)$, where w_h solves the problem $(\mathrm{P}_h^{A_h})$ for some $A_h \in \mathcal{A}$.

Lemma 3.3. *Let $v \in H^2(\Omega)$ and $v < 0$ in a non-empty open interval $(y_1, y_2) \subset \Omega_s$. Then there exists a positive constant ϱ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant \frac{1}{2}(y_2 - y_1)$, we have $A_h(v) \in \mathcal{A}_\varrho$.*

P r o o f. The proof clearly follows from the definition of the partitions $\tau_h \in \mathcal{T}_\theta$. Notice that the size of the parameter ϱ depends on the length $y_2 - y_1$. $\square$

Lemma 3.4. *Let $F < 0$ and $\theta, \varrho > 0$. Then there exist positive constants $\tilde{\varrho}$ and h_0 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, and any $A_h \in \mathcal{A}_\varrho$,*

$$A_h \cap A_h(w_h) \in \mathcal{A}_{\tilde{\varrho}}, \tag{3.11}$$

where w_h solves the problem $(\mathrm{P}_h^{A_h})$.

P r o o f. Suppose that (3.11) does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$ and $\{A_k\}_k \subset \mathcal{A}_\varrho$, $A_k \equiv A_{h_k}$, such that

$$h_k \operatorname{card}(A_k \cap A_k(w_k)) \to 0, \quad A_k(w_k) \equiv A_{h_k}(w_{h_k}). \tag{3.12}$$

By Lemma 3.2, there exists $c_1 > 0$ such that $\|w_k\|_{2,2} \leqslant c_1$ for any $k \geqslant 0$. If we choose $v_h = 1$ in the equation (3.4) and denote the coefficients and spring points of the form b_{h_k} as r_i^k and z_i^k, then by the estimates (2.10) and (3.12) we obtain

$$\begin{aligned} F = b_{h_k}^{A_k}(w_k, 1) \geqslant & \sum_{i \in A_k \cap A_k(w_k)} r_i^k w_k(z_i^k) \\ \geqslant & - c_2 h_k \|w_k\|_{C(\overline{\Omega})} \operatorname{card}(A_k \cap A_k(w_k)) \to 0, \quad c_2 > 0. \end{aligned}$$

However, this contradicts $F < 0$. Therefore, (3.11) holds. $\square$

To show the other uniform properties of the problems $(\mathrm{P}_h^{A_h})$, we will define an auxiliary problem $(\mathrm{P}_{h,r}^{A_h})$ with the “rigid” beam:

$$\text{find } p_h \in P_1\colon\ J_h^{A_h}(p_h) \leqslant J_h^{A_h}(p) \quad \forall\, p \in P_1, \tag{3.13}$$

or equivalently

$$\text{find } p_h \in P_1\colon\ b_h^{A_h}(p_h, p) = L(p) \quad \forall\, p \in P_1. \tag{3.14}$$

Notice that the problem $(\mathrm{P}_{h,r}^{A_h})$ means to solve a linear system of two equations with two unknowns.

Lemma 3.5. *Let $\tau_h \in \mathcal{T}_\theta$ and $A_h \in \mathcal{A}$. Then $p_h(x) = t_1 x + t_2$, where*

$$t_1 = \frac{F}{\det} \sum_{i \in A_h} r_i (T - z_i) \quad \textit{and} \quad t_2 = \frac{-F}{\det} \sum_{i \in A_h} r_i z_i (T - z_i), \tag{3.15}$$

with

$$\det = \sum_{i,j \in A_h,\, i<j} r_i r_j (z_i - z_j)^2 > 0, \ \ F = L(1), \ \ T = L(x)/F. \tag{3.16}$$

P r o o f. The relations (3.15) can be easily derived if we choose $p = 1$ and $p = x$ in the equation (3.14). $\square$

Lemma 3.6. *Let $F < 0$ and $\theta > 0$. Let $\{\tau_{h_k}\}_k \subset \mathcal{T}_\theta$ and $\{A_k\}_k \subset \mathcal{A}$, $A_k \equiv A_{h_k}$ be such sequences that*

$$h_k \to 0 \quad \text{and} \quad h_k \operatorname{card}(A_k) \to 0. \tag{3.17}$$

Then there exists a positive constant c, which is independent of the choice of the above sequences with the property (3.17), *such that*

$$p_k(T) \to -\infty, \quad \|p_k\|_{2,2} \to \infty \quad \text{and} \quad \|p_k\|_{2,2} \leqslant c\frac{-p_k(T)}{h_k \operatorname{card}(A_k)}, \tag{3.18}$$

where $\{p_k\}_k$ is the corresponding sequence of the solutions of the problems $(\mathrm{P}^{A_k}_{h_k,r})$.

P r o o f. Since the polynomial space P_1 has a finite dimension and since

$$p(x) = p(T) + (x - T)p',$$

there exist $c_1, c_2 > 0$ such that

$$c_1\|p\|_{2,2} \leqslant \max\{|p(T)|, |p'|\} \leqslant c_2\|p\|_{2,2} \quad \forall\, p \in P_1. \tag{3.19}$$

Let us denote $n_k := \operatorname{card}(A_k) \geqslant 2$. The coefficients and spring points of the form $b^{A_k}_{h_k}$ will be denoted by r^k_i and z^k_i, $i = 1, \ldots, n_k$, $z^k_1 < \ldots < z^k_{n_k}$. The determinant (3.16) will be denoted by $\det_k$ for the problem $(\mathrm{P}^{A_k}_{h_k,r})$. Let

$$d^k_i := z^k_{i+1} - z^k_i, \;\; i = 1, \ldots, n_k - 1, \tag{3.20}$$

i.e.

$$z^k_i = z^k_1 + \sum_{j<i} d^k_j, \;\; i = 2, \ldots, n_k.$$

Since $\tau_{h_k} \in \mathcal{T}_\theta$, there exists $c_1 > 0$ such that

$$d^k_i \geqslant c_1 h_k, \quad \forall\, k \geqslant 0, \;\; i = 1, \ldots, n_k. \tag{3.21}$$

We will also use the notation

$$\sigma^k_0 := \sum_{i=1}^{n_k} r^k_i, \quad \sigma^k_1 := \sum_{i=1}^{n_k} r^k_i \sum_{j<i} d^k_j \quad \text{and} \quad \sigma^k_2 := \sum_{i=1}^{n_k} r^k_i \Big(\sum_{j<i} d^k_j\Big)^2, \tag{3.22}$$

where $d_0^k := 0$. Then

$$\begin{aligned}
(3.23)\qquad & \sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2 \\
&= \sigma_0^k (T - z_1^k)^2 - 2\sigma_1^k (T - z_1^k) + \sigma_2^k \\
&\geqslant \frac{1}{\sigma_0^k}(\sigma_0^k \sigma_2^k - (\sigma_1^k)^2) \\
&= \frac{1}{\sigma_0^k} \sum_{i_1, i_2 = 1}^{n_k} r_{i_1}^k r_{i_2}^k \Bigl(\sum_{j_1 < i_1} d_{j_1}^k\Bigr)\Bigl(\sum_{j_1 < i_1} d_{j_1}^k - \sum_{j_2 < i_2} d_{j_2}^k\Bigr) \\
&= \frac{1}{\sigma_0^k} \sum_{i_1, i_2\,;\; i_1 < i_2} r_{i_1}^k r_{i_2}^k \Bigl(\sum_{j_1 < i_1} d_{j_1}^k - \sum_{j_2 < i_2} d_{j_2}^k\Bigr)^2 \\
&= \frac{1}{\sigma_0^k} \sum_{i_1, i_2\,;\; i_1 < i_2} r_{i_1}^k r_{i_2}^k \Bigl(\sum_{i_1 \leqslant j < i_2} d_j^k\Bigr)^2 = \frac{1}{\sigma_0^k} \det{}_k\,.
\end{aligned}$$

Hence, by Lemma 3.5, the assumption (3.17) and the estimate (2.10), we obtain

$$p_k(T) = \frac{F}{\det_k} \sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2 \leqslant F/\sigma_0^k \leqslant cF/(h_k \operatorname{card}(A_k)) \to -\infty, \quad c > 0,$$

which implies $\|p_k\|_{2,2} \to \infty$. The estimates (3.23), (3.21), and (2.10) also yield

$$\begin{aligned}
\frac{\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2}{\sum_{i=1}^{n_k} r_i^k} &\geqslant c_2 \frac{h_k^2}{n_k^2} \sum_{i_1, i_2\,;\; i_1 < i_2} \Bigl(\sum_{i_1 \leqslant j < i_2} 1\Bigr)^2 \\
&= \frac{1}{12} c_2 h_k^2 (n_k^2 - 1), \quad c_2 > 0.
\end{aligned}$$

Hence, by the Cauchy-Schwarz inequality, Lemma 3.5, and the assumption (3.17), we obtain

$$\begin{aligned}
\frac{|p_k'|}{-p_k(T)} &= \frac{\Bigl|\sum_{i=1}^{n_k} r_i^k (T - z_i^k)\Bigr|}{\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2} \leqslant \left(\frac{\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2}{\sum_{i=1}^{n_k} r_i^k}\right)^{-1/2} \\
&\leqslant c_3 (h_k^2 (n_k^2 - 1))^{-1/2} \leqslant \frac{c_4}{h_k n_k}, \quad c_3 > 0,\ c_4 = \frac{2}{\sqrt{3}}\, c_3,
\end{aligned}$$

which implies (3.18) due to the estimate (3.19). □

Lemma 3.7. *Let $\theta > 0$. Then there exists a positive constant $c > 0$ such that the estimate*

$$c\|v_h\|_{2,2}^2 \leqslant a(v_h, v_h) + \left(\frac{b_h^{A_h}(v_h, 1)}{h^3}\right)^2 + \left(\frac{b_h^{A_h}(v_h, x)}{h^3}\right)^2 \quad \forall\, v_h \in V_h \tag{3.24}$$

holds for any $\tau_h \in \mathcal{T}_\theta$ and $A_h \in \mathcal{A}$.

The proof of Lemma 3.7 is based on the generalized Poincaré inequality, see [4]. The denominator h^3 in (3.24) ensures the validity of the estimate for $h \to 0$.

Corollary 3.3. *Let $\theta > 0$. Then there exists a positive constant $c > 0$ such that the estimates*

$$\|w_h - p_h\|_{2,2} \leqslant c\|L\|_* \quad \text{and} \quad a(w_h, w_h) \leqslant c\|L\|_*^2 \tag{3.25}$$

hold for any $\tau_h \in \mathcal{T}_\theta$ and $A_h \in \mathcal{A}$, where w_h, p_h solve respectively the problems $(\mathrm{P}_h^{A_h})$, $(\mathrm{P}_{h,r}^{A_h})$.

Proof. By Lemma 3.7 and the equations (3.4) and (3.14) we obtain

$$\begin{aligned} c\|w_h - p_h\|_{2,2}^2 &\leqslant a(w_h, w_h) \leqslant a(w_h, w_h) + b_h^{A_h}(w_h - p_h, w_h - p_h) \\ &= L(w_h - p_h) \leqslant \|L\|_* \|w_h - p_h\|_{2,2}, \end{aligned}$$

which yields the first estimate in (3.25) and consequently also the second. □

Corollary 3.4. *Let the assumptions of Lemma 3.6 be fulfilled. Then*

$$\frac{\|w_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1 \quad \text{and} \quad \frac{J_{h_k}^{A_k}(w_k)}{w_k(T)} \to -F/2, \quad k \to \infty, \tag{3.26}$$

where $\{w_k\}_k$, $\{p_k\}_k$ are respectively the corresponding sequences of the solutions of the problems $(\mathrm{P}_{h_k}^{A_k})$ and $(\mathrm{P}_{h_k,r}^{A_k})$.

Proof. By the estimate (3.25) and the limits (3.18) we obtain

$$\begin{aligned} \frac{\|w_k\|_{2,2}}{\|p_k\|_{2,2}} &\leqslant \frac{\|p_k\|_{2,2} + \|w_k - p_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1, \\ \frac{\|w_k\|_{2,2}}{\|p_k\|_{2,2}} &\geqslant \frac{\|p_k\|_{2,2} - \|w_k - p_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1, \end{aligned}$$

i.e. the first limit in (3.26) holds. Notice that due to (3.14),

$$\frac{J_{h_k}^{A_k}(p_k)}{p_k(T)} = \frac{-L(p_k)}{2p_k(T)} = -F/2,$$

which implies $J_{h_k}^{A_k}(p_k) \to -\infty$ by (3.18). In addition, due to (3.4) and (3.14),

$$\frac{J_{h_k}^{A_k}(w_k)}{J_{h_k}^{A_k}(p_k)} = \frac{J_{h_k}^{A_k}(p_k) - L(w_k - p_k)/2}{J_{h_k}^{A_k}(p_k)} \to 1$$

and by Lemma 3.6 and Corollary 3.3,

$$\lim_{k\to\infty} \frac{w_k(T)}{p_k(T)} = 1 + \lim_{k\to\infty} \frac{w_k(T) - p_k(T)}{p_k(T)} = 1. \tag{3.27}$$

Therefore,

$$\lim_{k\to\infty} \frac{J_{h_k}^{A_k}(w_k)}{w_k(T)} = \lim_{k\to\infty} \frac{J_{h_k}^{A_k}(p_k)}{p_k(T)} = -\frac{F}{2}.$$

□

Corollaries 3.3 and 3.4 show that the problems $(\mathrm{P}_{h_k}^{A_k})$ and $(\mathrm{P}_{h_k,r}^{A_k})$ have many common properties for the limit case $h_k \operatorname{card}(A_k) \to 0$. This fact will be used to prove the following theorems and lemmas.

Theorem 3.1. *Let $F < 0$, $x_l < T < x_r$, and $\theta > 0$. Then there exist positive constants ϱ and h_0 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, and any $A_h \in \mathcal{A}$ we have*

$$A_h(w_h) \in \mathcal{A}_\varrho,$$

where w_h solves the problem $(\mathrm{P}_h^{A_h})$.

Proof. Assume that Theorem 3.1 does not hold. Then there are sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, and $\{A_k\}_k \subset \mathcal{A}$, $A_k \equiv A_{h_k}$, such that

$$h_k \operatorname{card}(A_k(w_k)) \to 0, \quad A_k(w_k) \equiv A_{h_k}(w_{h_k}). \tag{3.28}$$

Let us denote $p_k := p_{h_k}$ as the solutions of the problems $(\mathrm{P}_{h_k,r}^{A_k})$, $k \geqslant 0$.

Suppose that there exist $\varrho_1 > 0$ and a subsequence of $\{A_k\}_k$ (denoted in the same way) such that

$$A_k \in \mathcal{A}_{\varrho_1} \quad \forall\, k \geqslant 0.$$

Then, by Lemma 3.4, there exists $\varrho_2 > 0$ such that $A_k(w_k) \in \mathcal{A}_{\varrho_2}$ for sufficiently large k, which contradicts (3.28).

Suppose that there exists a subsequence $\{A_k\}_k$ such that

$$h_k \operatorname{card}(A_k) \to 0.$$

By Lemma 3.6, $p_k(T) \to -\infty$. Therefore, $p_k \to -\infty$ in $[x_l, T]$ or in $[T, x_r]$, since $T \in \Omega_s = (x_l, x_r)$. Hence, by Corollary 3.3, there exists a sufficiently small $\varepsilon > 0$ such that $w_k < 0$ in $[x_l, T-\varepsilon]$ or in $[T+\varepsilon, x_r]$ for sufficiently large k, which contradicts (3.28) due to Lemma 3.3. □

Lemma 3.8. *Let $F < 0$ and $x_l < T < x_r$. Then there exist positive constants ϱ and h_0 such that $\{A_h(w_h^*)\}_{h \leqslant h_0} \subset \mathcal{A}_\varrho$, where w_h^* solves the problem* (P_h).

In addition, if $\tau_h \in \mathcal{T}_\theta$, $A_h \in \mathcal{A}$ and $A_h(w_h) = A_h$, where w_h solves the problem $(\mathrm{P}_h^{A_h})$, *then w_h also solves the problem* (P_h).

P r o o f. Let w_h^*, w^* solve respectively the problems (P_h) and (P). Since $w_h^* \to w^*$ in $H^2(\Omega)$ by (2.17) and since w^* is negative somewhere in Ω_s by Lemma 3.5 in [10], there exist $\varrho, h_0 > 0$ such that $A_h(w_h^*) \in \mathcal{A}_\varrho$ for $h \leqslant h_0$ by Lemma 3.3.

If $A_h(w_h) = A_h$ and w_h solves the problem $(\mathrm{P}_h^{A_h})$ then

$$L(v) = a(w_h, v) + b_h^{A_h}(w_h, v) = a(w_h, v) + b_h(w_h^-, v) \quad \forall\, v \in H^2(\Omega).$$

Thus the function w_h also solves the problem (P_h). □

By the next lemma, we estimate the difference between the solution w_h^* of the problem (P_h) and its approximations generated by the algorithms, which will be presented in Section 4, see the proof of Theorem 4.2.

Lemma 3.9. *Let $F < 0$, $x_l < T < x_r$, and $c, \theta > 0$. Then there exist positive constants $\tilde{c}$ and $h_0 > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, and any $u_h \in V_h$, $\|u_h\|_{2,2} \leqslant c$, we have*

$$\tilde{c}\|w_h^* - u_h\|_{2,2}^2 \leqslant a(w_h^* - u_h, w_h^* - u_h) + b_h\big((w_h^*)^- - u_h^-, w_h^* - u_h\big), \tag{3.29}$$

where w_h^* solves the problem (P_h).

P r o o f. Since the proof is similar to the proof of Theorem 4.5 in [10], some steps will be done more briefly. By Lemma 3.8 and Corollary 3.2, there exist $c_1, c_2 > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$ with sufficiently small h we have

$$\|w_h^*\|_{2,2} \leqslant c_1 \quad \text{and} \quad \|w_h^* - u_h\|_{2,2} \leqslant c_2. \tag{3.30}$$

Suppose that the lemma does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, $\{w_{h_k}^*\}_k$ and $\{u_{h_k}\}_k$ such that

$$a(w_k - u_k, w_k - u_k) + b_h(w_k^- - u_k^-, w_k - u_k) \to 0, \tag{3.31}$$

where

$$w_k := \frac{w_{h_k}^*}{\|w_{h_k}^* - u_{h_k}\|_{2,2}}, \quad u_k := \frac{u_{h_k}}{\|w_{h_k}^* - u_{h_k}\|_{2,2}}, \quad \|w_k - u_k\|_{2,2} = 1. \tag{3.32}$$

All subsequences of these sequences will be denoted in the same way. By the Rellich theorem, (3.31), and (3.32), there exist subsequences $\{w_k\}_k$ and $\{u_k\}_k$, and a polynomial $p \in P_1$, $p \neq 0$, such that $w_k - u_k \to p$ in $H^2(\Omega)$. By Lemma 3.8,

$$\exists\, \varrho_1 > 0\colon\ A_{h_k}(w^*_{h_k}) \in \mathcal{A}_{\varrho_1}. \tag{3.33}$$

Suppose that $\|w^*_{h_k} - u_{h_k}\|_{2,2} \to 0$. Then

$$\exists\, \varrho_2 > 0\colon\ A_{h_k}(w^*_{h_k}) \cap A_{h_k}(u_{h_k}) \in \mathcal{A}_{\varrho_2} \tag{3.34}$$

for sufficiently large k by (3.33). Since

$$b_{h_k}(w_k^- - u_k^-, w_k - u_k) \geqslant b_{h_k}^{A_{h_k}(w^*_{h_k}) \cap A_{h_k}(u_{h_k})}(w_k - u_k, w_k - u_k),$$

(3.31), (3.34), (2.11), and (2.10) yield that $p = 0$, which contradicts $p \neq 0$.

Therefore, we can assume that the sequences $\{w_k\}_k$ and $\{u_k\}_k$ are bounded due to (3.30). It means that there exist their subsequences which converge to functions w and $u = w - p$ in $H^1(\Omega)$ by the Rellich theorem. Then, by (3.31) and (2.12),

$$w^- - (w - p)^- = 0 \quad \text{in } \Omega_s. \tag{3.35}$$

Since $w^*_{h_k} \to w^*$ in $H^2(\Omega)$, w^* solves the problem (P), by (2.17), and since $w^* < 0$ somewhere in Ω_s, also $w < 0$ somewhere in Ω_s. Therefore, (3.35) yields that $p = 0$ which contradicts $p \neq 0$. $\square$

4. Descent direction methods with and without projection

In this section, two methods are presented as a numerical realization of the problem (P_h). The methods are based on the minimization of the total energy functional J_h, where the descent directions of the functional are searched by solving linear problems of type ($\mathrm{P}_h^{A_h}$) presented in the previous section. The difference between the methods is in the "projection step". The step is useful mainly for unstable loads as we will see in Section 5.

Since the uniform convergence properties of the methods with respect to refinement of the partition are derived, the corresponding algorithms are first described in the functional form. Their algebraical form will be presented later, in Section 5. We will assume that the solvability conditions (2.5) hold.

4.1. Descent direction method without projection

Let $\tau_h \in \mathcal{T}_\theta$ be a partition and z_i, $i \in \{1, 2, \dots, m(h)\}$, the corresponding set of springs.

Algorithm 1

Initialization

$$w_{h,0} = 0,$$
$$A_{h,0} = \{1, 2, \ldots, m(h)\}.$$

Iteration $k = 0, 1, \ldots$

$$s_{h,k} \in V_h,\ \ w_{h,k} + s_{h,k} \text{ solves } (\mathrm{P}_h^{A_{h,k}}),$$
$$\alpha_{h,k} = \arg\min_{0 \leqslant \alpha \leqslant 1} J_h(w_{h,k} + \alpha s_{h,k}),$$
$$w_{h,k+1} = w_{h,k} + \alpha_{h,k} s_{h,k},$$
$$A_{h,k+1} = A_h(w_{h,k+1}).$$

In the remaining part of this subsection, we show that Algorithm 1 is well-defined, i.e., the problems $(\mathrm{P}_h^{A_{h,k}})$ are uniquely solvable and $w_{h,k} \to w^*$ in $H^2(\Omega)$ uniformly with respect to sufficiently small h.

Let $u_h \in V_h$, $A_h(u_h) \in \mathcal{A}$, $w_h \in V_h$ solve the problem $(\mathrm{P}_h^{A_h(u_h)})$ and let $s_h := w_h - u_h$. It will be useful to introduce the notation $A_h^\alpha := A_h(u_h + \alpha s_h)$. Then $A_h^0 = A_h(u_h)$ and $A_h^1 = A_h(w_h)$. Notice that the equality

$$(u_h + \alpha s_h)(z_i) = \alpha w_h(z_i) + (1 - \alpha) u_h(z_i)$$

yields the inclusion

$$A_h^0 \cap A_h^1 \subset A_h^0 \cap A_h^\alpha \quad \forall\, \alpha \in [0, 1] \tag{4.1}$$

and the implication

$$A_h^1 \subset A_h^0 \Longrightarrow A_h^\alpha \subset A_h^0 \quad \forall\, \alpha \in [0, 1]. \tag{4.2}$$

Lemma 4.1. *Let $u_h \in V_h$, $A_h^0 \equiv A_h(u_h) \in \mathcal{A}$, $w_h \in V_h$ solve the problem $(\mathrm{P}_h^{A_h^0})$ and let $s_h := w_h - u_h$. Let*

$$\alpha_h := \arg\min_{0 \leqslant \alpha \leqslant 1} J_h(u_h + \alpha s_h).$$

Then

$$J_h'(u_h; s_h) = 2J_h^{A_h^0}(w_h) - 2J_h^{A_h^0}(u_h) \tag{4.3}$$
$$= -\,a(s_h, s_h) - b_h^{A_h^0}(s_h, s_h) \leqslant 0, \tag{4.4}$$

where $J_h'(u_h; s_h) = 0$ if and only if u_h solves the problem (P_h), and

$$\alpha_h \geqslant \frac{a(s_h, s_h) + b_h^{A_h^0}(s_h, s_h)}{a(s_h, s_h) + b_h^{A_h^0 \cup A_h^{\alpha_h}}(s_h, s_h)} > 0, \quad s_h \neq 0. \tag{4.5}$$

P r o o f. By Lemma 3.1, the problem $(\mathrm{P}_h^{A_h^0})$ has a unique solution w_h. Then the choice $v_h = s_h$ in the variational equation (3.4) yields

$$\begin{aligned} J_h'(u_h; s_h) &= a(u_h, s_h) + b_h(u_h^-, s_h) - L(s_h) \\ &= a(u_h, s_h) + b_h^{A_h^0}(u_h, s_h) - L(s_h) \\ &= -a(s_h, s_h) - b_h^{A_h^0}(s_h, s_h) \leqslant 0. \end{aligned}$$

The choices $v_h = u_h$ and $v_h = w_h$ in the variational equation (3.4) yield the equality (4.3). By the inequality (3.5), $J_h'(u_h; s_h) = 0$ if and only if $s_h = 0$, i.e. if $u_h = w_h$. By Lemma 3.8, it means that in such a case, u_h solves the problem (P_h).

Let us denote $\varphi(\alpha) := J_h(u_h + \alpha s_h)$ and let $s_h \neq 0$. Since J_h is a convex and differentiable functional on V_h, there exists α_h which minimizes φ in $[0, 1]$. The inequality (4.4) yields $\alpha_h > 0$ and $\varphi'(\alpha_h) \leqslant 0$. If $\alpha_h = 1$, then the inequality (4.5) holds. Otherwise,

$$\begin{aligned} 0 = \varphi'(\alpha_h) &= a(u_h + \alpha_h s_h, s_h) + b_h((u_h + \alpha_h s_h)^-, s_h) - L(s_h) \\ &= J_h'(u_h; s_h) + \alpha_h \Big[a(s_h, s_h) + b_h\Big(\frac{(u_h + \alpha_h s_h)^- - u_h^-}{\alpha_h}, s_h\Big)\Big]. \end{aligned} \tag{4.6}$$

Notice that

$$\begin{aligned} & b_h\Big(\frac{(u_h + \alpha_h s_h)^- - u_h^-}{\alpha_h}, s_h\Big) \\ &\quad = b_h^{A_h^0 \cap A_h^{\alpha_h}}(s_h, s_h) - b_h^{A_h^0 \setminus A_h^{\alpha_h}}(u_h, s_h)/\alpha_h + b_h^{A_h^{\alpha_h} \setminus A_h^0}(u_h + \alpha_h s_h, s_h)/\alpha_h \\ &\quad = b_h^{A_h^0 \cup A_h^{\alpha_h}}(s_h, s_h) + b_h^{A_h^{\alpha_h} \setminus A_h^0}(u_h, s_h)/\alpha_h - b_h^{A_h^0 \setminus A_h^{\alpha_h}}(u_h + \alpha_h s_h, s_h)/\alpha_h. \end{aligned}$$

If $i \in A_h^{\alpha_h} \setminus A_h^0$ then $u_h(z_i) \geqslant 0$ and $s_h(z_i) < 0$. If $i \in A_h^0 \setminus A_h^{\alpha_h}$ then $(u_h + \alpha_h s_h)(z_i) \geqslant 0$ and $s_h(z_i) > 0$. Therefore,

$$b_h^{A_h^{\alpha_h} \setminus A_h^0}(u_h, s_h) \leqslant 0 \quad \text{and} \quad b_h^{A_h^0 \setminus A_h^{\alpha_h}}(u_h + \alpha_h s_h, s_h) \geqslant 0.$$

Hence,

$$b_h\Big(\frac{(u_h + \alpha_h s_h)^- - u_h^-}{\alpha_h}, s_h\Big) \leqslant b_h^{A_h^0 \cup A_h^{\alpha_h}}(s_h, s_h)$$

and (4.6) yields the estimate (4.5). $\square$

Notice that if $A_h^1 \subset A_h^0$, then the implication (4.2) and the estimate (4.5) yield $\alpha_h = 1$.

By the next lemma we can estimate the relative cardinality of the sets $A_{h,k}$ which are generated by Algorithm 1, see the proof of Theorem 4.1.

Lemma 4.2. *Let c, θ be positive constants and let the solvability condition* (2.5) *hold. Then there exist positive constants h_0, ϱ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, and any $u_h \in V_h$, $\|u_h\|_{2,2} \leqslant c$, $A_h^0 \equiv A_h(u_h) \in \mathcal{A}_\varrho$ we have*

$$A_h^{\alpha_h} \equiv A_h(u_h + \alpha_h s_h) \in \mathcal{A}_\varrho, \tag{4.7}$$

where $\alpha_h = \arg\min_{0 \leqslant \alpha \leqslant 1} J_h(u_h + \alpha s_h)$, $s_h = w_h - u_h$, and $w_h \in V_h$ solves the problem $(\mathrm{P}_h^{A_h^0})$.

P r o o f. Assume that the lemma does not hold. Then there are sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, $\{\varrho_k\}_k$, $\varrho_k \to 0$, $\{u_k\}_k$, $u_k \in V_{h_k}$, $\|u_k\|_{2,2} \leqslant c$, $A_k^0 \equiv A_{h_k}(u_k) \in \mathcal{A}_{\varrho_k}$ such that

$$A_k^{\alpha_k} \equiv A_{h_k}(u_k + \alpha_k s_k) \notin \mathcal{A}_{\varrho_k} \quad \forall\, k \geqslant 0, \tag{4.8}$$

where $\{\alpha_k\}_k$, $\{s_k\}_k$, and $\{w_k\}_k$ are the corresponding sequences for the sequences $\{\tau_{h_k}\}_k$ and $\{u_k\}_k$. For the sake of simplicity, all subsequences of these sequences will be denoted in the same way. Relation (4.8) implies that

$$\operatorname{card}(A_k^{\alpha_k}) < \operatorname{card}(A_k^0) \quad \forall\, k \geqslant 0. \tag{4.9}$$

Suppose that there exist $\varrho_1 > 0$ and a subsequence $\{A_k^0\}_k$ such that $A_k^0 \in \mathcal{A}_{\varrho_1}$. Then, by Lemma 3.4, there exists $\varrho_2 > 0$ such that $A_k^0 \cap A_k^1 \in \mathcal{A}_{\varrho_2}$ for sufficiently large k. Hence, by (4.1) we obtain $A_k^{\alpha_k} \in \mathcal{A}_{\varrho_2}$, which contradicts (4.8). Therefore, we can assume that

$$h_k \operatorname{card}(A_k^0) \to 0, \quad k \to \infty. \tag{4.10}$$

Corollary 3.4, (4.10) and the boundedness of u_k yield

$$\|w_k\|_{2,2} \to \infty, \quad \|s_k\|_{2,2} \to \infty, \quad \text{and} \quad \frac{\|s_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1, \tag{4.11}$$

where $p_k \in P_1$ solves the problem $(\mathrm{P}_{h_k,r}^{A_k^0})$ defined in Section 3. Consequently, by Corollary 3.3 we obtain

$$a(s_k, s_k)/\|s_k\|_{2,2}^2 \to 0. \tag{4.12}$$

Since $\|u_k\|_{2,2} \leqslant c$, there exists $c_0 > 0$ such that $J_{h_k}(u_k) \leqslant c_0$ for any $k \geqslant 0$ and since $J_{h_k}(u_k) \geqslant J_{h_k}(u_k + \alpha_k s_k)$, we have

$$\exists\, c_1 > 0\colon\ \|u_k + \alpha_k s_k\|_{2,2} \leqslant c_1 \quad \forall\, k \geqslant 0 \tag{4.13}$$

by Lemma 2.2. The boundedness of $\{u_k\}_k$, (4.13), and (4.11) yield

$$\exists\, c_2 > 0\colon\ \|\alpha_k s_k\|_{2,2} \leqslant c_2 \quad \forall\, k \geqslant 0 \quad \text{and} \quad \alpha_k \to 0. \tag{4.14}$$

Suppose that

$$\exists\, c_3 > 0\colon\ \|\alpha_k s_k\|_{2,2} \geqslant c_3 \quad \forall\, k \geqslant 0. \tag{4.15}$$

Then by the Rellich theorem, (4.12), (4.14), and (4.15) there exist a subsequence $\{\alpha_k s_k\}_k$ and $p \in P_1$, $p \neq 0$, such that $\alpha_k s_k \to p$ and consequently $\alpha_k p_k \to p$ in $H^2(\Omega)$. Since the sequences $\{u_k\}_k$ and $\{u_k + \alpha_k s_k\}_k$ are bounded, there exist their subsequences with weak limits u and $u + p$ in $H^2(\Omega)$. We can also assume that $u_k \to u$ and $u_k + \alpha_k s_k \to u + p$ in $H^1(\Omega)$ by the Rellich theorem. The functions u and $u + p$ are non-negative in Ω_s by virtue of the assumptions (4.8), (4.10), and Lemma 3.3.

Due to the assumption $F < 0$ we have $A_k^0 \cap A_k^1 \neq \emptyset$, see the proof of Lemma 3.4. Hence and by (4.1) we obtain $A_k^0 \cap A_k^{\alpha_k} \neq \emptyset$, i.e., there exists a sequence $\{i_k\}_k$ such that $i_k \in A_k^0 \cap A_k^{\alpha_k}$. Therefore, there exist a subsequence $\{z_{i_k}^k\}_k$ and $z \in \overline{\Omega}_s$ such that $z_{i_k}^k \to z$. Non-negativity of u and $u + p$ yields

$$u(z) = 0 \quad \text{and} \quad p(z) = 0 \tag{4.16}$$

and consequently,

$$u'(z) \begin{cases} = 0, & z \neq x_l, x_r, \\ \geqslant 0, & z = x_l, \\ \leqslant 0, & z = x_r, \end{cases} \qquad \text{and} \quad u'(z) + p'(z) \begin{cases} = 0, & z \neq x_l, x_r, \\ \geqslant 0, & z = x_l, \\ \leqslant 0, & z = x_r. \end{cases} \tag{4.17}$$

Since $p \neq 0$, there exists just one such point z, by virtue of (4.16). Moreover, by (4.17), $z = x_l$ or $z = x_r$. In both cases, $p < 0$ in Ω_s, since $p_k(T) \to -\infty$ by Lemma 3.6.

Let $\varphi_k(\alpha) := J_{h_k}(u_k + \alpha s_k)$. Since $\alpha_k \to 0$, the definition of α_k yields

$$0 = \varphi_k'(\alpha_k) = a(u_k + \alpha_k s_k, s_k) + b_{h_k}\bigl((u_k + \alpha_k s_k)^-, s_k\bigr) - L(s_k)$$

for sufficiently large k. If we multiply this equality by α_k then for $k \to \infty$ we obtain a contradiction $0 = -L(p) = -Fp(T) < 0$ by (2.11) and the non-negativity of $u + p$.

Suppose that

$$\|\alpha_k s_k\|_{2,2} \to 0 \quad \text{for } k \to \infty. \tag{4.18}$$

Then by the estimates (4.5) and (4.3) we obtain

$$
\begin{aligned}
0 &\leqslant (1-\alpha_k) J'_{h_k}(u_k; s_k) + \alpha_k b_{h_k}^{A_k^{\alpha_k} \setminus A_k^0}(s_k, s_k) \\
&= 2(1-\alpha_k)\bigl(J_{h_k}^{A_k^0}(w_k) - J_{h_k}^{A_k^0}(u_k)\bigr) + \alpha_k b_{h_k}^{A_k^{\alpha_k} \setminus A_k^0}(s_k, s_k).
\end{aligned}
$$

If we divide this inequality by $-w_k(T)$, we obtain by Lemma 3.6, Corollary 3.4, (2.10), (3.27), (4.9), (4.10), (4.11), and (4.18)

$$
\begin{aligned}
0 &\leqslant F + \lim_{k\to\infty} \Bigl\{ \|\alpha_k s_k\|_{2,2} \frac{\|p_k\|_{2,2}}{-p_k(T)} \frac{p_k(T)}{w_k(T)} \frac{\|s_k\|_{2,2}}{\|p_k\|_{2,2}} b_{h_k}^{A_k^{\alpha_k} \setminus A_k^0} \Bigl(\frac{s_k}{\|s_k\|_{2,2}}, \frac{s_k}{\|s_k\|_{2,2}} \Bigr) \Bigr\} \\
&\leqslant F + c_4 \lim_{k\to\infty} \|\alpha_k s_k\|_{2,2} \frac{1}{h_k \operatorname{card}(A_k^0)} \sum_{A_k^{\alpha_k} \setminus A_k^0} r_i^k \\
&\leqslant F + c_5 \lim_{k\to\infty} \|\alpha_k s_k\|_{2,2} = F < 0,
\end{aligned}
$$

which is a contradiction. Therefore, (4.7) holds. $\square$

Theorem 4.1. *Let the condition* (2.5) *hold and let* $\theta > 0$. *Then there exist positive constants* ϱ, c, *and* h_1 *such that for any* $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_1$,

$$
A_{h,k} \in \mathcal{A}_\varrho \quad \text{and} \quad \|w_{h,k}\|_{2,2} \leqslant c \quad \forall\, k \geqslant 0, \tag{4.19}
$$

where the sets $A_{h,k}$ *and the functions* $w_{h,k}$ *are generated by Algorithm* 1.

Proof. The theorem will be proved by mathematical induction. By Lemma 2.2, there exist $c > 0$ and $h_0 > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, the implication

$$
J_h(u_h) \leqslant 0 \Longrightarrow \|u_h\|_{2,2} \leqslant c \quad \forall\, u_h \in V_h \tag{4.20}
$$

holds. Since $\|w_{h,0}\|_{2,2} = 0 \leqslant c$ and $A_{h,0} = \{1, \ldots, m(h)\}$, there exist $\varrho > 0$ and $0 < h_1 \leqslant h_0$ (which depend only on θ and c) such that $A_{h,1} \in \mathcal{A}_\varrho$ for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_1$, by Lemma 4.2. Suppose that

$$
A_{h,i} \in \mathcal{A}_\varrho \quad \forall\, \tau_h \in \mathcal{T}_\theta, \;\; h \leqslant h_1, \;\; i = 0, 1, \ldots, k.
$$

Since

$$
J_h(w_{h,k}) \leqslant \ldots \leqslant J_h(w_{h,1}) \leqslant J_h(w_{h,0}) \leqslant 0, \quad h \leqslant h_1,
$$

also $\|w_{h,k}\|_{2,2} \leqslant c$ by the implication (4.20), which by Lemma 4.2 yields $A_{h,k+1} \in \mathcal{A}_\varrho$ for any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_1$. $\square$

Lemma 4.3. *Let the condition* (2.5) *hold and let* $\theta > 0$. *Then there exist positive constants* c *and* h_0 *such that*

$$\alpha_{h,k} \geqslant c \quad \forall\, \tau_h \in \mathcal{T}_\theta,\ \ h \leqslant h_0,\ \ \forall\, k \geqslant 0,\ \ s_{h,k} \neq 0, \tag{4.21}$$

where the numbers $\alpha_{h,k}$ *and the functions* $s_{h,k}$ *are generated by Algorithm* 1.

P r o o f. Let $s_{h,k}$, $w_{h,k}$, $\alpha_{h,k}$, $A_{h,k}$, $k \geqslant 0$, be generated by Algorithm 1. By Theorem 4.1, there exist $\varrho, h_0 > 0$ such that $A_{h,k} \in \mathcal{A}_\varrho$, $h \leqslant h_0$, for any $k \geqslant 0$. Hence, by Lemma 3.2, there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} a(v,v) + b_h^{A_{h,k}}(v,v) &\geqslant c_1 \|v\|_{2,2}^2, \\ a(v,v) + b_h^{A_{h,k} \cup A_{h,k+1}}(v,v) &\leqslant c_2 \|v\|_{2,2}^2 \end{aligned}$$

for all $v \in H^2(\Omega)$ and for all $k \geqslant 0$. Then the estimate (4.5) in Lemma 4.1 yields

$$\alpha_{h,k} \geqslant \frac{a(s_{h,k}, s_{h,k}) + b_h^{A_{h,k}}(s_{h,k}, s_{h,k})}{a(s_{h,k}, s_{h,k}) + b_h^{A_{h,k} \cup A_{h,k+1}}(s_{h,k}, s_{h,k})} \geqslant \frac{c_1}{c_2} > 0 \quad \forall\, k \geqslant 0,\ \ s_{h,k} \neq 0.$$

□

Lemma 4.4. *Let the condition* (2.5) *hold and let* $\theta > 0$. *Then there exist positive constants* c *and* h_0 *such that*

$$J_h(w_{h,k+1}) \leqslant J_h(w_{h,k}) - c\|s_{h,k}\|_{2,2}^2 \quad \forall\, \tau_h \in \mathcal{T}_\theta,\ \ h \leqslant h_0,\ \ \forall\, k \geqslant 0, \tag{4.22}$$

where the functions $s_{h,k}$, $w_{h,k}$ *are generated by Algorithm* 1.

P r o o f. Let $s_k \equiv s_{h,k}$, $w_k \equiv w_{h,k}$, $\alpha_k \equiv \alpha_{h,k}$, $A_k \equiv A_{h,k}$, $k \geqslant 0$, be generated by Algorithm 1. Let $\varphi_k(\alpha) := J_h(w_k + \alpha s_k)$. By the definition of α_k,

$$0 \geqslant \varphi_k'(\alpha_k) = a(w_{k+1}, s_k) + b_h(w_{k+1}^-, s_k) - L(s_k).$$

Hence, by the definition of A_k, A_{k+1}, and w_{k+1},

$$\begin{aligned} J_h(w_{k+1}) = J_h(w_k) &+ \alpha_k \varphi_k'(\alpha_k) - \frac{1}{2}\alpha_k^2 a(s_k, s_k) \\ &+ \frac{1}{2} b_h(w_{k+1}^-, w_{k+1}) - \frac{1}{2} b_h(w_k^-, w_k) - \alpha_k b_h(w_{k+1}^-, s_k). \end{aligned}$$

Notice that

$$
\begin{aligned}
\frac{1}{2}b_h(w_{k+1}^-, w_{k+1}) - \frac{1}{2}b_h(w_k^-, w_k) - \alpha_k b_h(w_{k+1}^-, s_k)&\\
&= \frac{1}{2}b_h^{A_{k+1}}(w_k + \alpha_k s_k, w_k + \alpha_k s_k) - \frac{1}{2}b_h^{A_k}(w_k, w_k)\\
&\quad - \alpha_k b_h^{A_{k+1}}(w_k + \alpha_k s_k, s_k)\\
&= -\frac{1}{2}\alpha_k^2 b_h^{A_{k+1}}(s_k, s_k) + \frac{1}{2}b_h^{A_{k+1}}(w_k, w_k) - \frac{1}{2}b_h^{A_k}(w_k, w_k)\\
&= -\frac{1}{2}\alpha_k^2 b_h^{A_{k+1}\cap A_k}(s_k, s_k) - \frac{1}{2}\alpha_k^2 b_h^{A_{k+1}\setminus A_k}(s_k, s_k)\\
&\quad + \frac{1}{2}b_h^{A_{k+1}\setminus A_k}(w_k, w_k) - \frac{1}{2}b_h^{A_k\setminus A_{k+1}}(w_k, w_k)\\
&\leqslant -\frac{1}{2}\alpha_k^2 b_h^{A_{k+1}\cap A_k}(s_k, s_k),
\end{aligned}
$$

since $-\alpha_k s_k(z_i) > w_k(z_i)$ and consequently, $\alpha_k^2 s_k^2(z_i) > w_k^2(z_i)$ if $i \in A_{k+1} \setminus A_k$. Therefore,

$$
J_h(w_{k+1}) \leqslant J_h(w_k) - \frac{1}{2}\alpha_k^2\big(a(s_k, s_k) + b_h^{A_k\cap A_{k+1}}(s_k, s_k)\big). \tag{4.23}
$$

By Theorem 4.1 there exist $\varrho_1 > 0$ and $h_1 > 0$ such that $A_k \in \mathcal{A}_{\varrho_1}$ for any $k \geqslant 0$ and any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_1$. Therefore, by Lemma 3.4 there exist $0 < \varrho \leqslant \varrho_1$ and $0 < h_0 \leqslant h_1$ such that $A_k \cap A_k(w_k + s_k) \in \mathcal{A}_\varrho$ and consequently (see (4.1)), $A_k \cap A_{k+1} \in \mathcal{A}_\varrho$ for any $k \geqslant 0$ and any $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$. Then, by Lemma 3.2, there exists $c > 0$ such that

$$
c\|s_k\|_{2,2}^2 \leqslant a(s_k, s_k) + b_h^{A_k\cap A_{k+1}}(s_k, s_k) \quad \forall\, \tau_h \in \mathcal{T}_\theta,\ \ h \leqslant h_0,\ \ \forall\, k \geqslant 0.
$$

Hence, by (4.23) and Lemma 4.3, we obtain (4.22). □

Theorem 4.2. *Let the condition* (2.5) *hold and let* $\theta > 0$. *Then there exists* $h_0 > 0$ *such that the sequence* $\{w_{h,k}\}_k$ *generated by Algorithm* 1 *converges uniformly (with respect to* h*) to the function* w_h^* *solving the problem* (P_h) *in* $H^2(\Omega)$ *for any* $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$.

In addition, for any fixed $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, *there exists an iteration* $k_0 = k_0(h) \geqslant 0$ *such that* $w_{h,k_0} + s_{h,k_0} = w_h^*$.

P r o o f. Let $s_k \equiv s_{h,k}$, $w_k \equiv w_{h,k}$, $\alpha_k \equiv \alpha_{h,k}$, $A_k \equiv A_{h,k}$, $k \geqslant 0$, be generated by Algorithm 1. By Lemma 4.4, there exist $c_1 > 0$ and $h_0 > 0$ such that

$$
J_h(w_h^*) \leqslant J_h(w_k) \leqslant -c_1 \sum_{i=0}^{k-1} \|s_i\|_{2,2}^2 \quad \forall\, \tau_h \in \mathcal{T}_\theta,\ \ h \leqslant h_0,\ \ \forall\, k \geqslant 0. \tag{4.24}
$$

By (2.17),

$$J_h(w_h^*) = -L(w_h^*)/2 \to -L(w^*)/2 = J(w^*), \quad h \to 0,$$

where w^* solves the problem (P). Hence, by (4.24) there exists $c_2 > 0$ such that

$$\sum_{i=0}^{\infty} \|s_i\|_{2,2}^2 \leqslant c_2 \quad \forall\, \tau_h \in \mathcal{T}_\theta, \ \ h \leqslant h_0, \tag{4.25}$$

and consequently $\|s_k\|_{2,2} \to 0$ uniformly with respect to h for $k \to \infty$. Since $w_k + s_k$ solves the problem $(\mathrm{P}_h^{A_k})$, the variational equations (2.14) and (3.4) yield

$$a(w_h^* - w_k, w_h^* - w_k) + b_h\big((w_h^*)^- - w_k^-, w_h^* - w_k\big) = a(s_k, w_h^* - w_k) + b_h^{A_k}(s_k, w_h^* - w_k).$$

Hence, by Theorem 4.1, Lemma 3.9, and (2.11), there exists $c_3 > 0$ such that

$$\|w_h^* - w_k\|_{2,2} \leqslant c_3 \|s_k\|_{2,2} \to 0 \quad \forall\, \tau_h \in \mathcal{T}_\theta, \ \ h \leqslant h_0, \ \ \forall\, k \geqslant 0,$$

which implies the uniform convergence of the sequence $\{w_{h,k}\}_k$ to the function w_h^* solving the problem (P_h).

Since $w_k \to w_h^*$, also $A_k \to A_h^*$ and consequently, $A_k(w_k + s_k) \to A_h^*$. Since $\mathrm{card}(A_k) \leqslant m(h) < \infty$ for any fixed $h \leqslant h_0$, there exists $k_0 \geqslant 0$ such that $A_{k_0} = A_{k_0}(w_{k_0} + s_{k_0})$. Then, by Lemma 3.8, $w_{k_0} + s_{k_0} = w_h^*$. □

Remark 4.1. The convergence result of Algorithm 1 holds for parameters $h \leqslant h_0$, for some h_0. Taking into consideration the analysis in [10], we can assume that the size of h_0 depends on the stability of the load, i.e., how much the balance point T is close to the end points x_l, x_r of the subsoil and how much the size of the load resultant F is relatively close to zero.

Remark 4.2. Numerical examples show that Algorithm 1 converges for almost all initial choices of $A_{h,0}$. However, the initial choice $A_{h,0} = \{1, \ldots, m(h)\}$ ensures in the tested examples that $\alpha_{h,k} = 1$ for any $k \geqslant 0$ due to inclusions $A_{h,k+1} \subset A_{h,k}$. These inclusions are shown in [8] for a particular choice of the load.

Remark 4.3. We can also substitute $\alpha_{h,k}$ by

$$\tilde{\alpha}_{h,k} := \min_{\alpha \geqslant 0} J_h(w_{h,k} + \alpha s_{h,k}).$$

The corresponding algorithm will be denoted Algorithm 2 and it is shown on numerical examples that we can expect the same convergence properties as for Algorithm 1. However, it is necessary to generalize Lemma 4.2 to use Algorithm 2 correctly. The comparison of the algorithm will be illustrated by numerical examples in Section 6.

There are many numerical methods how to find the values $\alpha_{h,k}$ or $\tilde{\alpha}_{h,k}$ which do not depend on the parameter h. Here, the regula falsi method has been used.

Algorithms 1, 2 can also be used for coercive beam problems with the same convergence result which can be proved without Lemma 4.2 and without the restricted assumption on the parameter h.

R e m a r k 4.4. The descent direction method without projection can also be characterized as a semismooth Newton method with damping. The semismooth Newton method was introduced in [2].

4.2. Descent direction method with projection

First of all, we will define the class of auxiliary problems which are specified by a partition $\tau_h \in \mathcal{T}_\theta$ and by a function $v_h \in V_h$:

$$(\mathrm{P}_h^{v_h}) \qquad \text{find } p_h = p_h(v_h) \in P_1\colon\ J_h(v_h + p_h) \leqslant J_h(v_h + p) \quad \forall\, p \in P_1,$$

or equivalently

$$\text{find } p_h = p_h(v_h) \in P_1\colon\ b_h\big((v_h + p_h)^-, p\big) = L(p) \quad \forall\, p \in P_1. \tag{4.26}$$

The problem $(\mathrm{P}_h^{v_h})$ means to solve a system of two non-linear equations with two unknowns. Similarly to the problem (P_h), it is possible to prove that the condition (2.15) ensures the existence of a solution and the uniqueness of the solution holds for sufficiently small parameters h. Notice that if w_h^* solves the problem (P_h) then the problem $(\mathrm{P}_h^{w_h^*})$ solves the zero polynomial.

Lemma 4.5. *Let the solvability condition* (2.5) *hold and let* $c, \theta > 0$. *Then there exist positive constants* $\varrho > 0$ *and* h_0 *such that for any* $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, *and any* $v_h \in V_h$, $|v_h|_{2,2} \leqslant c$,

$$A_h(v_h + p_h) \in \mathcal{A}_\varrho, \tag{4.27}$$

where p_h *solves* $(\mathrm{P}_h^{v_h})$.

P r o o f. We start with the well-known inequality

$$\exists\, c_1 > 0\colon\ |v|_{2,2}^2 \geqslant c_1 \inf_{p \in P_1} \|v + p\|_{2,2}^2 \quad \forall\, v \in H^2(\Omega), \tag{4.28}$$

which can be proved by the Poincaré inequality. Notice that

$$v_h + p + p_h(v_h + p) = v_h + p_h(v_h) \quad \forall\, p \in P_1,$$

where $p_h(v_h+p)$ solves $(\mathrm{P}_h^{v_h+p})$. Thus $A_h(v_h+p_h(v_h)) = A_h(v_h+p+p_h(v_h+p))$. Therefore, by virtue of the assumption $|v_h|_{2,2} \leqslant c$ and the inequality (4.28) we can assume that $\|v_h\|_{2,2} \leqslant \tilde{c}$, $\tilde{c} > 0$, for any $v_h \in V_h$.

Suppose that Lemma 4.5 does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, and $\{v_k\}_k$, $v_k \equiv v_{h_k}$, $\|v_k\|_{2,2} \leqslant \tilde{c}$, such that

$$h_k \operatorname{card}(A_k) \to 0, \tag{4.29}$$

where $A_k \equiv A_{h_k}(v_k+p_k)$ and p_k solves $(\mathrm{P}_{h_k}^{v_k})$. The choice $p=1$ in the equation (4.26) and the estimate (2.10) yields

$$F = \sum_{i\in A_k} r_i^k(v_k+p_k)(z_i^k) \geqslant c_2 \min_{i\in A_k}(v_k+p_k)(z_i^k) h_k \operatorname{card}(A_k), \quad c_2 > 0.$$

Hence, by (4.29) and the boundedness of $\{v_k\}$, we obtain that there exists a point $z \in [x_l, x_r]$ such that $p_k(z) \to -\infty$. If $z \in \Omega_s$, then the assumption (4.29) cannot hold by virtue of Lemma 3.3. Therefore, $z = x_l$ or $z = x_r$.

Let us consider the former case. For the latter, we obtain a similar contradiction. Then $p_k(x_l) \to -\infty$ and $p_k(z) \not\to -\infty$ for $z > x_l$. Hence, $p_k(z) \to \infty$ for $z > x_l$. It means that $z_i^k \to x_l$ for all $i \in A_k$, since the functions v_k are uniformly bounded. Therefore, $z_i^k < T$ for all $i \in A_k$, where k is sufficiently large. If we choose $p = x$ in the equation (4.26), we obtain

$$T = \frac{\sum\limits_{i\in A_k} r_i^k(v_k+p_k)(z_i^k) z_i^k}{\sum\limits_{i\in A_k} r_i^k(v_k+p_k)(z_i^k)} \leqslant \max_{i\in A_k} z_i^k < T,$$

which is a contradiction. □

The descent direction method with projection is obtained from the previous method by adding the "projection" step, where the problem of type $(\mathrm{P}_h^{v_h})$ is solved in:

Algorithm 3

Initialization

$w_{h,0} = p_h(0)$, $p_h(0)$ solves (P_h^0),
$A_{h,0} = A_h(w_{h,0})$.

Iteration $k = 0, 1, \ldots$

$s_{h,k} \in V_h,\ \ w_{h,k} + s_{h,k}$ solves $(\mathrm{P}_h^{A_{h,k}})$,
$\alpha_{h,k} = \arg\min\limits_{0\leqslant\alpha\leqslant 1} J_h(w_{h,k} + \alpha s_{h,k})$,
$\tilde{w}_{h,k} = w_{h,k} + \alpha_{h,k} s_{h,k}$,

$$p_{h,k} = p_h(\tilde{w}_{h,k}),\ p_h(\tilde{w}_{h,k}) \text{ solves } (\mathrm{P}_h^{\tilde{w}_{h,k}}),$$
$$w_{h,k+1} = \tilde{w}_{h,k} + p_{h,k},$$
$$A_{h,k+1} = A_h(w_{h,k+1}).$$

Lemma 4.6. *Let the condition* (2.5) *hold and let* $\theta > 0$. *Then there exist positive constants* ϱ, c_1, c_2, *and* h_0 *such that for any* $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, *and any* $k \geqslant 0$, *we have*

$$A_{h,k} \in \mathcal{A}_\varrho, \tag{4.30}$$

$$\alpha_{h,k} \geqslant c_1, \tag{4.31}$$

$$J_h(w_{h,k+1}) \leqslant J_h(w_{h,k}) - c_2 \|s_{h,k}\|_{2,2}^2, \tag{4.32}$$

where $A_{h,k}$, $\alpha_{h,k}$, $s_{h,k}$, *and* $w_{h,k}$ *are generated by Algorithm* 3.

The proofs of (4.30)–(4.32) are quite similar to those of (4.19), (4.21), and (4.22) for Algorithm 1. Only instead of Lemma 4.2, we use Lemma 4.5 and the inequality

$$J_h(w_{h,k+1}) \leqslant J_h(\tilde{w}_{h,k}),$$

which follows from the definition of the problem $(\mathrm{P}_h^{\tilde{w}_{h,k}})$.

In the same way as for Algorithm 1, we obtain the following convergence result for Algorithm 3.

Theorem 4.3. *Let the condition* (2.5) *hold and let* $\theta > 0$. *Then there exists* $h_0 > 0$ *such that the sequence* $\{w_{h,k}\}_k$ *generated by Algorithm* 3 *converges uniformly (with respect to* h*) to the function* w_h^* *solving the problem* (P_h) *for any* $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$.

In addition, for any fixed $\tau_h \in \mathcal{T}_\theta$, $h \leqslant h_0$, *there exists an iteration* $k_0 = k_0(h) \geqslant 0$ *such that* $w_{h,k_0} + s_{h,k_0} = w_h^*$.

For an implementation of the "projection" step in Algorithm 3, i.e., for an implementation of the problem $(\mathrm{P}_h^{v_h})$, we can use a minor modification of Algorithm 1 with the same convergence results:

Initialization

$$p_{h,0} \in P_1,\ b_h(v_h + p_{h,0}, p) = L(p) \quad \forall\, p \in P_1,$$
$$A_{h,0} = A_h(v_h + p_{h,0}).$$

Iteration $k = 0, 1, \ldots$

$$\tilde{p}_{h,k} \in P_1,\ b_h^{A_{h,k}}(v_h + p_{h,k} + \tilde{p}_{h,k}, p) = L(p) \quad \forall\, p \in P_1,$$
$$\alpha_{h,k} = \arg\min_{0 \leqslant \alpha \leqslant 1} J_h(v_h + p_{h,k} + \alpha \tilde{p}_{h,k}),$$
$$p_{h,k+1} = p_{h,k} + \alpha_{h,k} \tilde{p}_{h,k},$$
$$A_{h,k+1} = A_h(v_h + p_{h,k+1}).$$

R e m a r k 4.5. Due to the projection step, the functions $w_{h,k}$ generated by Algorithm 3 have some common properties with the unknown function w_h^* as we see at the end of the next section.

Again, it is possible to substitute $\alpha_{h,k}$ by

$$\tilde{\alpha}_{h,k} = \arg\min_{\alpha \geqslant 0} J_h(w_{h,k} + \alpha s_{h,k})$$

in Algorithm 3.

The projection step cannot be applied for coercive problems, since the polynomials of the first degree do not belong to the tested functions for such problems.

5. Algebraic formulation of the problem

5.1. Rewriting the approximated problem

Let $\tau_h \in \mathcal{T}_\theta$ be a partition with nodal points

$$0 = x_0 < x_1 < \ldots < x_l = x_{j_l - 1} < \ldots < x_r = x_{j_r} < \ldots < x_N = l$$

and let $z_1 < z_2 < \ldots < z_m$ be the corresponding points which are obtained from the chosen numerical quadrature.

The functions $v_h \in V_h$ will be standardly represented by the vector $v \in \mathbb{R}^n$, $n = 2N + 2$. The form a and the functional L will be represented by the stiffness matrix $K \in \mathbb{R}^{n \times n}$ and by the load vector $f \in \mathbb{R}^n$. Notice that the matrix K is symmetric and positive semi-definite.

Let the polynomials $p = 1$ and $p = x$ be represented by the vectors $p_1, p_x \in \mathbb{R}^n$. Then the matrix $R := (p_1, p_x) \in \mathbb{R}^{n \times 2}$ represents all polynomials from P_1 and forms the kernel of K, i.e. $KR = 0$.

The matrix which transforms the function values and the values of the first derivatives at the nodal points x_j, $j = 0, 1, \ldots, N$, onto the points z_i, $i = 1, \ldots, m$, will be denoted by $B \in \mathbb{R}^{m \times n}$. Let $D \in \mathbb{R}^{m \times m}$ be a diagonal matrix containing the coefficients r_i, i.e. the products of the weights of the numerical quadrature and the stiffness coefficients of the subsoil.

The Euclidean scalar product and norm in $\mathbb{R}^k$, $k \geqslant 1$, will be denoted by $(\cdot, \cdot)_k$ and $\|\cdot\|_k$.

For the sake of simplicity, the corresponding functional and the unknown vector in the algebraic formulation will be denoted in the same way as in the continuous problem (P). Then the algebraic formulation of the problem (P_h) has the form

$$(\mathbb{P}) \qquad \begin{cases} \text{find } w^* \in \mathbb{R}^n\colon\ J(w^*) \leqslant J(w) \quad \forall\, w \in \mathbb{R}^n, \\ J(w) := \dfrac{1}{2}(Kw, w)_n + \dfrac{1}{2}\big(D(Bw)^-, (Bw)^-\big)_m - (f, w)_n, \end{cases}$$

where $u^- \in \mathbb{R}^m$ is the negative part of u, i.e.

$$(u^-)_i := \min\{0, u_i\}, \quad i = 1, 2, \ldots, m.$$

The problem $(\mathbb{P})$ can be rewritten equivalently as the non-linear system of equations:

$$\text{find } w^* \in \mathbb{R}^n\colon\ Kw^* + B^T D(Bw^*)^- = f. \tag{5.1}$$

Let a set $A_h \subset \{1, 2, \ldots, m\}$ of indices be represented by the diagonal matrix $A \in \mathbb{R}^{m\times m}$ such that $A_{ii} = 1$ if $i \in A_h$, otherwise $A_{ii} = 0$. The algebraic representation of a set $A_h(v_h)$ will be denoted by $A(v)$.

We also introduce the notation

$$G := BR = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_m \end{pmatrix}^T, \quad e := R^T f = F\begin{pmatrix} 1 \\ T \end{pmatrix}. \tag{5.2}$$

Then the auxiliary problems $(\mathrm{P}_h^{A_h})$ and $(\mathrm{P}h^{v_h})$ have the following algebraical forms:

$$(\mathbb{P}^A) \qquad \text{find } w = w(A) \in \mathbb{R}^n\colon\ (K + B^T DAB)w = f; \tag{5.3}$$

$$(\mathbb{P}^v) \qquad \text{find } c = c(v) \in \mathbb{R}^2\colon\ G^T D(Bv + Gc)^- = e. \tag{5.4}$$

The corresponding algebraical formulations of Algorithms 1, 3 are the following:

Algorithm 1

Initialization

$w^{(0)} = 0,$

$A_{(0)}$, $(A_{(0)})_{ii} = 1$, $i = \{1, \ldots, m\}$.

Iteration $k = 0, 1, \ldots$

$s^{(k)}$, $w^{(k)} + s^{(k)}$ solves $(\mathbb{P}^{A_{(k)}})$,

$\alpha_{(k)} = \arg\min\limits_{0\leqslant\alpha\leqslant1} J(w^{(k)} + \alpha s^{(k)})$,

$w^{(k+1)} = w^{(k)} + \alpha_{(k)} s^{(k)}$,

$A_{(k+1)} = A(w^{(k+1)})$.

Algorithm 3

Initialization

$w^{(0)} = Rc^{(0)}$, $c^{(0)}$ solves $(\mathbb{P}^0)$,

$A_{(0)} = A(w^{(0)})$,

Iteration $k = 0, 1, \ldots$

$s^{(k)}$, $w^{(k)} + s^{(k)}$ solves $(\mathbb{P}^{A_{(k)}})$,

$\alpha_{(k)} = \arg\min\limits_{0\leqslant\alpha\leqslant1} J(w^{(k)} + \alpha s^{(k)})$,

$\tilde{w}^{(k)} = w^{(k)} + \alpha_{(k)} s^{(k)}$,

$c^{(k)}$, $c^{(k)}$ solves $(\mathbb{P}^{\tilde{w}^{(k)}})$,

$w^{(k+1)} = \tilde{w}^{(k)} + Rc^{(k)}$,

$A_{(k+1)} = A(w^{(k+1)})$.

5.2. Analysis of the projection step

To explain the reason of the "projection step", we will consider the set

$$\Lambda := \{\lambda \in \mathbb{R}^m\colon\ \lambda \leqslant 0,\ G^T D\lambda = e\}. \tag{5.5}$$

First of all, we derive some basic properties of the set Λ. Clearly, the set Λ is closed and convex in $\mathbb{R}^m$.

Lemma 5.1. *Let $F < 0$ and $z_1 < T < z_m$. Then the set Λ is non-empty and bounded in $\mathbb{R}^m$.*

Proof. The assumptions of the lemma ensure that there exists a solution w^* of the problem $(\mathbb{P})$. If we multiply the equation (5.1) by the vectors in the form $(Ra)^T$, $a \in \mathbb{R}^2$, we obtain that $(Bw^*)^- \in \Lambda$ by (5.2).

The boundedness follows from the definition of the set Λ and the estimate (2.10):

$$-F = -e_1 = -(G^T D\lambda)_1 = \sum_{i=1}^{m} r_i|\lambda_i| \geqslant c\|\lambda\|_m, \quad c > 0.$$

$\square$

Lemma 5.2. *Let $F < 0$, $z_1 < T < z_m$, and $\lambda \in \Lambda$. Let*

$$A(\lambda) := \{i \in \{1, 2, \ldots, m\}\colon\ \lambda_i < 0\}.$$

Then

$$\min_{i\in A(\lambda)} z_i \leqslant T \leqslant \max_{i\in A(\lambda)} z_i. \tag{5.6}$$

Proof. The equation $G^T D\lambda = e$ yields that

$$T = \sum_{i\in A(\lambda)} r_i\lambda_i z_i \Big/ \sum_{i\in A(\lambda)} r_i\lambda_i.$$

Hence, we obtain (5.6). $\square$

The following lemma says that the diameter of the set Λ is small for unstable loads.

Lemma 5.3. *Let $\{F_k\}_k$, $\{T_k\}_k$ be the sequences of the load resultants and their balance points such that $F_k < 0$, $z_1 < T_k < z_m$ for any $k \geqslant 0$. Let $\{\Lambda_k\}_k$ be the sequence of the corresponding sets defined by (5.5). If $T_k \to z_1$ or $T_k \to z_m$ or $F_k \to 0$, then* $\operatorname{diam}(\Lambda_k) \to 0$.

Proof. Let $T_k \to z_1$. Then by the definition of the set Λ_k we obtain

$$0 = \sum_{i=1}^{m} r_i\lambda_i^k(z_i - T_k) = r_1\lambda_1^k(z_1 - T_k) + \sum_{i=2}^{m} r_i\lambda_i^k(z_i - T_k) \quad \forall\, \lambda^k \in \Lambda_k,\ \forall\, k \geqslant 1.$$

The first term on the right-hand side is non-negative and tends to zero for $k \to \infty$. The second term is non-positive for sufficiently large k and therefore, $\lambda_i^k \to 0$ for $i = 2, \ldots, m$. Since it also holds

$$F_k = \sum_{i=1}^{m} r_i \lambda_i^k \quad \forall\, \lambda^k \in \Lambda_k,\ \forall\, k \geqslant 1, \tag{5.7}$$

we obtain

$$\lambda_1^k - \tilde{\lambda}_1^k = -\frac{1}{r_1} \sum_{i=2}^{m} r_i (\lambda_i^k - \tilde{\lambda}_i^k) \to 0 \quad \forall\, \lambda^k,\ \ \tilde{\lambda}^k \in \Lambda_k,$$

which means that $\operatorname{diam}(\Lambda_k) \to 0$.

Similarly, we can prove the assertion for the case $T_k \to z_m$. For the case $F_k \to 0$ the assertion also holds, since the equation (5.7) yields $\lambda^k \to 0$ for any $\lambda^k \in \Lambda_k$. $\square$

Since Λ is closed, convex, and non-empty set, we can define uniquely the projection P of the space $\mathbb{R}^m$ onto the set Λ with respect to the scalar product $(D\cdot, \cdot)_m$ in $\mathbb{R}^m$:

$$\big(D(\eta - P(\eta)), \lambda - P(\eta)\big)_m \leqslant 0 \quad \forall\, \lambda \in \Lambda. \tag{5.8}$$

Let $v \in \mathbb{R}^n$ and let $c = c(v) \in \mathbb{R}^2$ solve the problem $(\mathbb{P}^v)$. Then the vector $(Bv+Gc)^-$ belongs to Λ and

$$\begin{aligned}
&\big(D(Bv - (Bv + Gc)^-), \lambda - (Bv + Gc)^-\big)_m = \\
&\qquad = \big(D((Bv + Gc)^+ - Gc), \lambda - (Bv + Gc)^-\big)_m \\
&\qquad = \big(D(Bv + Gc)^+, \lambda\big)_m + \big(c, G^T D((Bv + Gc)^- - \lambda)\big)_2 \\
&\qquad = \big(D(Bv + Gc)^+, \lambda\big)_m \leqslant 0 \quad \forall\, \lambda \in \Lambda.
\end{aligned}$$

Therefore, by Definition 5.8 of the projection P,

$$P(Bv) = (Bv + Gc)^-.$$

It means that for the vectors $w^{(k)}$, $k \geqslant 0$, generated by Algorithm 3, and for the solution w^*, we obtain $(Bw^{(k)})^-, (Bw^*)^- \in \Lambda$. Thus, these vectors have the common properties specified by the above lemmas. Mainly, for unstable loads, the vectors $(Bw^{(k)})^-$ are close to the vector $(Bw^*)^-$, which means that the vectors $Bw^{(k)}$ have a set of the active "springs" similar to that of the vector Bw^*. Therefore, we can expect better convergence properties for Algorithm 3 than for Algorithm 1 for such loads. This will be also demonstrated by numerical examples in the next section.

The set Λ is also important for the dual formulation of the problem, see [9], since the vectors $-\lambda$, where $\lambda \in \Lambda$, can represent admissible Lagrange multipliers.

6. Numerical examples

In this section, convergence results of Algorithms 1–3 will be demonstrated by numerical examples.

We will consider the beam of the length 1 m with the parameter $EI = 5*10^5\,\mathrm{Nm}^2$. The subsoil is situated in the interval (x_l, x_r), where $x_l = 0.1\,\mathrm{m}$ and $x_r = 0.9\,\mathrm{m}$, and its stiffness coefficient is $q = 5*10^8\,\mathrm{Nm}^{-2}$. At the end points 0, l of the beam, we will consider point loads F_0 and F_l, which will be specified for the particular examples. The interval $(0, l)$ will be divided into $10 * 2^j$, $j = 2, 3, \ldots, 8$, equidistant parts. The situation is depicted in Fig. 2.

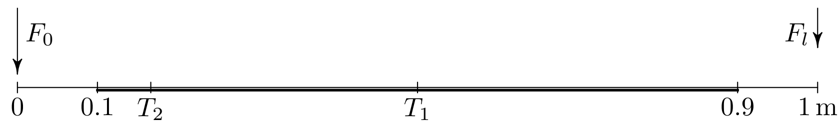

Figure 2. Scheme of the tested problem.

We use the following stopping criterion:

$$\frac{\|r^{(k)}\|_n}{\|f\|_n} \leqslant \varepsilon, \quad r^{(k)} := f - Kw^{(k)} - B^T D(Bw^{(k)})^-,$$

where $\varepsilon = 10^{-6}$ and $r^{(k)}$ is the kth residuum of the algorithms. For an approximation of the bilinear form b, the reference numerical quadrature

$$\int_{-1}^{1} \varphi(\xi)\,\mathrm{d}\xi \approx \varphi(-\sqrt{3}/3) + \varphi(\sqrt{3}/3)$$

is used. The linear problems with bilateral elastic springs are solved by the Cholesky factorization.

Example 1. Let $F_0 = -5000\,\mathrm{N}$ and $F_l = -5000\,\mathrm{N}$. Such a load fulfils the solvability condition (2.5) and is stable, since the balance point $T_1 = 0.5\,\mathrm{m}$ is situated in the centre of the subsoil interval. The dependence of the number of outer iterations on the refinement parameter j of the partition is shown in Tab. 1.

Notice that the number of outer iterations does not depend on j and is practically the same for all the algorithms. The number of iterations for the "projected" step in Algorithm 3 are about four. The approximated solution for $j = 8$, i.e. for 2560 elements, is depicted in Fig. 3.

E x a m p l e 2. Let $F_0 = -5000\,\mathrm{N}$ and $F_l = -1000\,\mathrm{N}$. Such a load fulfils the solvability condition (2.5) and is not too stable, since the balance point $T_2 = 0.1667\,\mathrm{m}$ is close to the end point x_l of the subsoil. The dependence of the number of outer iterations on the refinement parameter j of the partition is shown in Tab. 1.

Ex. 1	2	3	4	5	6	7	8
ALG1	4	3	4	4	4	4	4
ALG2	3	3	3	3	3	4	4
ALG3	3	3	3	3	3	3	3

Ex. 2	2	3	4	5	6	7	8
ALG1	6	6	7	8	7	8	8
ALG2	5	5	6	6	6	6	6
ALG3	2	2	2	2	2	2	2

Table 1. Numbers of outer iterations for Examples 1 and 2.

Notice that the number of outer iterations does not depend on j. The number of outer iterations for Algorithm 3 is smaller than for Algorithms 1, 2, which is the expected result.

The approximated solution for $j = 8$ is depicted in Fig. 3.

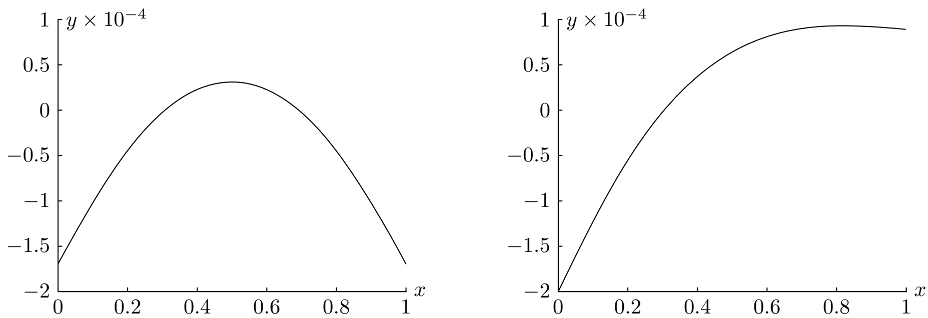

Figure 3. Approximated beam deflections w for Examples 1 and 2.

7. Conclusion

The descent direction methods with and without projection have been introduced and analysed. The methods can be generalized to the problems with more parts of the subsoil and also for two-dimensional models of thin elastic plates.

The methods have been illustrated by numerical examples. Other numerical examples, which confirm some theoretical results, can be found in [11].

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