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ON THE SEPARATION OF PARAMETRIC CONVEX POLYHEDRAL SETS WITH APPLICATION IN MOLP

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Abstract. We investigate diverse separation properties of two convex polyhedral sets for the case when there are parameters in one row of the constraint matrix. In particular, we deal with the existence, description and stability properties of the separating hyperplanes of such convex polyhedral sets. We present several examples carried out on PC. We are also interested in supporting separation (separating hyperplanes support both the convex polyhedral sets at given faces) and permanent separation (a hyperplane separates the convex polyhedral sets for all feasible parameters). Finally, we show how the developed theory is applicable in multiobjective linear programming.

Keywords: convex polyhedra, parameters, separating hyperplane, supporting hyperplane, solution set, stability set

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1. Introduction

Separation of convex sets is an important mathematical tool used both in theory and in practice. We find applications in economics (e.g. the second welfare theorem [12]), computer science (e.g. support vector machines [1]) and especially in optimization (see, e.g., Nožička et al. [16], Rockafellar [18]).

In practice, input data are often known only approximately due to measurement errors. To the best of our knowledge, combination of these two principles—separation and uncertainty—has never been studied systematically. We broke this in Hladík [10] and [11], where we derived the basic separation properties of the convex sets with parameters on the right-hand side or in one column of the constraint matrix, respectively. In the present paper we are concerned with the third case, when parameters are situated in one row of the constraint matrix. Nevertheless, the proposed theorems and their proofs are not always analogous to the previous ones and other approaches had to be used.

We aim at developing the theoretical background useful for diverse applications. One possible application—multiobjective linear programming—is mentioned in Section 8.

We now introduce some notation. Given a matrix $\boldsymbol{M}$, we use $\boldsymbol{M}_{i,\cdot}$ and $\boldsymbol{M}_{\cdot,j}$ to denote the ith row and the jth column of $\boldsymbol{M}$, respectively. For given vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$, the expression $\boldsymbol{a} < \boldsymbol{b}$ means $a_i < b_i$ for all $i = 1, \ldots, k$. For any set $\mathcal{X}$ we denote by $\operatorname{cl}\mathcal{X}$, $\operatorname{int}\mathcal{X}$, $\dim\mathcal{X}$, $\operatorname{conv}\mathcal{X}$, and $\mathcal{X}^\perp$ the closure, the interior, the dimension, the convex hull, and the orthogonal complement of $\mathcal{X}$, respectively. For the kth unit vector we use the symbol $\boldsymbol{e}_k \equiv (0, \ldots, 0, 1, 0, \ldots, 0)^T$.

2. Basic definitions and theorems

First, we recall some basic definitions and theorems concerning separation. There are several kinds of separability of convex sets (cf. Klee [14]). For the purpose of this paper it is convenient to use the following one requiring full dimension of the sets.

Definition 1. Convex sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$ are called *separable* if $\dim\mathcal{X} = \dim\mathcal{Y} = n$ and there exists a hyperplane $\mathcal{R} = \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{r}^T\boldsymbol{x} = s\}$ such that $\mathcal{X} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{r}^T\boldsymbol{x} \leqslant s\}$ and $\mathcal{Y} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{r}^T\boldsymbol{x} \geqslant s\}$; $\mathcal{R}$ is called a *separating hyperplane* of the sets $\mathcal{X}$, $\mathcal{Y}$.

We also adopt the following well-known separation theorem (see e.g. Grünbaum [4], Kemp and Kimura [13], Nožička et al. [16], Rockafellar [18]):

Theorem 1. *Convex sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$ are separable if and only if $\dim\mathcal{X} = \dim\mathcal{Y} = n$ and $\operatorname{int}\mathcal{X} \cap \operatorname{int}\mathcal{Y} = \emptyset$.*

Consider a family of convex polyhedral sets

$$\mathcal{M}_1(\boldsymbol{\lambda}, \mu) \equiv \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b},\ \boldsymbol{\lambda}^T\boldsymbol{x} \leqslant \nu\} \tag{2.1}$$

and a convex polyhedral set

$$\mathcal{M}_2 \equiv \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{C}\boldsymbol{x} \leqslant \boldsymbol{d}\}, \tag{2.2}$$

where $\boldsymbol{A} \in \mathbb{R}^{(m-1)\times n}$, $\boldsymbol{C} \in \mathbb{R}^{l\times n}$, $\boldsymbol{b} \in \mathbb{R}^{m-1}$, $\boldsymbol{d} \in \mathbb{R}^l$, $m > 1$, $l > 0$ are fixed and $\boldsymbol{\lambda} \in \mathbb{R}^n$, $\nu \in \mathbb{R}$ are parameters. Denote also

$$\mathcal{M}_1 \equiv \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\}. \tag{2.3}$$

Parametric convex polyhedral sets were also studied e.g. in Nožička et. al. [15], and the particular case of row parameters in Grygarová [9] (for systems of linear equations with non-negative variables).

In the following sections we derive diverse separation properties of $\mathcal{M}_1(\boldsymbol{\lambda},\mu)$ and $\mathcal{M}_2$, and particularly the set of parameters $\boldsymbol{\lambda}$, μ under which the corresponding properties are preserved. In Section 3 we derive the description of the solution set (a set of parameters for which given convex polyhedral sets are separable) and in Section 4 we describe all the separating hyperplanes. In Section 5 we define stability sets, derive their description by means of linear inequalities and present examples carried out on PC. In Section 6 we characterize the set of all parameters for which there exists a separating hyperplane supporting given convex polyhedral sets at fixed faces. In Section 7 we show a way how to check the existence of a permanent separating hyperplane (a hyperplane that separates convex polyhedral sets for all feasible parameters). Finally, in Section 8 we give an application in multiobjective linear programming.

3. Solution set

The solution set is the most important characteristic of separability of $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$ and $\mathcal{M}_2$, and we use it throughout the paper.

Definition 2. The *solution set* is the set of all $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$ for which the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$ and $\mathcal{M}_2$ are separable.

From now on we suppose that $\dim\mathcal{M}_1=\dim\mathcal{M}_2=n$. Otherwise, the solution set is empty.

We achieve a description of the solution set in several steps. In Theorem 2 we characterize the set of parameters $\boldsymbol{\lambda}$, ν for which $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$ is non-empty (we will use it in Section 6), and in Theorem 3 we do the same for the set of parameters for which $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$ has full dimension. The latter result together with Proposition 1 are needed to describe the solution set, which is done in Theorem 4.

Theorem 2. *Denote by $\boldsymbol{g}_k$, $k\in L$, any basis of the lineality space $\mathcal{L}\equiv\{\boldsymbol{x}\in\mathbb{R}^n\colon\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}$, denote by $\boldsymbol{x}_i$, $i\in V$, all vertices and by $\boldsymbol{h}_j$, $j\in H$, all extremal directions (vectors in directions of unbounded edges) of the convex polyhedral set $\mathcal{M}_1\cap\mathcal{L}^\perp$. Then the set of all $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$ satisfying $\mathcal{M}_1(\boldsymbol{\lambda},\nu)\neq\emptyset$ has the description*

$$\mathbb{R}^{n+1}\setminus\mathcal{P},$$

where

$$\mathcal{P}=\{(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}\colon\boldsymbol{x}_i^T\boldsymbol{\lambda}>\nu\ \forall\, i\in V,\ \boldsymbol{h}_j^T\boldsymbol{\lambda}\geqslant 0\ \forall\, j\in H,\ \boldsymbol{g}_k^T\boldsymbol{\lambda}=0\ \forall\, k\in L\}. \tag{3.1}$$

P r o o f. Let $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$ be given arbitrarily. Each point $\boldsymbol{x}\in\mathcal{M}_1$ can be (see Padberg [17, Assertions 7.2 c, 7.3 d]) expressed as

$$\boldsymbol{x}=\sum_{i\in V}\alpha_i\boldsymbol{x}_i+\sum_{j\in H}\beta_j\boldsymbol{h}_j+\sum_{k\in L}\gamma_k\boldsymbol{g}_k,$$

where $\sum\limits_{i\in V}\alpha_i=1$, $\alpha_i\geqslant 0$, $i\in V$, $\beta_j\geqslant 0$, $j\in H$, $\gamma_k\in\mathbb{R}$, $k\in L$, and $V\neq\emptyset$ (from the assumption that $\dim\mathcal{M}_1=n$). If $(\boldsymbol{\lambda},\nu)\in\mathcal{P}$, then

$$\boldsymbol{x}^T\boldsymbol{\lambda}=\sum_{i\in V}\alpha_i\boldsymbol{x}_i^T\boldsymbol{\lambda}+\sum_{j\in H}\beta_j\boldsymbol{h}_j^T\boldsymbol{\lambda}+\sum_{k\in L}\gamma_k\boldsymbol{g}_k^T\boldsymbol{\lambda}>\sum_{i\in V}\alpha_i\nu=\nu$$

and $\mathcal{M}_1(\boldsymbol{\lambda},\nu)=\emptyset$. Conversely, if $(\boldsymbol{\lambda},\nu)\notin\mathcal{P}$, then one of the following three situations occurs. Either $\boldsymbol{x}_i^T\boldsymbol{\lambda}\leqslant\nu$ holds for a certain $i\in V$, and therefore $\boldsymbol{x}_i\in\mathcal{M}_1(\boldsymbol{\lambda},\nu)\neq\emptyset$. Or $\boldsymbol{h}_j^T\boldsymbol{\lambda}<0$ holds for a certain $j\in H$. Consider a point $\boldsymbol{x}_c\equiv\boldsymbol{x}_1+c\boldsymbol{h}_j$, $c\geqslant 0$, where $\boldsymbol{x}_1$ is any vertex of $\mathcal{M}_1\cap\mathcal{L}^\perp$. Then for an arbitrary $c\geqslant\max\{(\nu-\boldsymbol{x}_1^T\boldsymbol{\lambda})/\boldsymbol{h}_j^T\boldsymbol{\lambda},0\}$ we have $\boldsymbol{x}_c^T\boldsymbol{\lambda}=\boldsymbol{x}_1^T\boldsymbol{\lambda}+c\boldsymbol{h}_j^T\boldsymbol{\lambda}\leqslant\nu$, whence $\boldsymbol{x}_c\in\mathcal{M}_1(\boldsymbol{\lambda},\nu)\neq\emptyset$. The third possibility is that $\boldsymbol{g}_k^T\boldsymbol{\lambda}\neq 0$ holds for a certain $k\in L$. We can assume without loss of generality that $\boldsymbol{g}_k^T\boldsymbol{\lambda}<0$. Denote by $\boldsymbol{x}_1$ any vertex of $\mathcal{M}_1\cap\mathcal{L}^\perp$. Then the point $\boldsymbol{x}_c\equiv\boldsymbol{x}_1+c\boldsymbol{g}_k$ belongs to $\mathcal{M}_1$ for all $c\in\mathbb{R}$, and for any $c\geqslant(\nu-\boldsymbol{x}_1^T\boldsymbol{\lambda})/\boldsymbol{g}_k^T\boldsymbol{\lambda}$ we have $\boldsymbol{x}_c\in\mathcal{M}_1(\boldsymbol{\lambda},\nu)\neq\emptyset$. □

Theorem 3. *Let* $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$ *be given arbitrarily. The set of all values of parameters* $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$ *satisfying* $\dim\mathcal{M}_1(\boldsymbol{\lambda},\nu)=n$ *is equal to*

$$\mathbb{R}^{n+1}\setminus\mathcal{P}',$$

where $\mathcal{P}'$ *is the closure of the convex cone from* (3.1) *without the origin described by*

$$\mathcal{P}'\equiv\{(\boldsymbol{\lambda},\nu)\neq(\mathbf{0},0)\colon\ \boldsymbol{x}_i^T\boldsymbol{\lambda}\geqslant\nu\ \forall\, i\in V,\ \boldsymbol{h}_j^T\boldsymbol{\lambda}\geqslant 0\ \forall\, j\in H,\ \boldsymbol{g}_k^T\boldsymbol{\lambda}=0\ \forall\, k\in L\}. \tag{3.2}$$

P r o o f. Let $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$ be given arbitrarily. Each point $\boldsymbol{x}\in\mathcal{M}_1$ can be expressed as

$$\boldsymbol{x}=\sum_{i\in V}\alpha_i\boldsymbol{x}_i+\sum_{j\in H}\beta_j\boldsymbol{h}_j+\sum_{k\in L}\gamma_k\boldsymbol{g}_k,$$

where $\sum\limits_{i\in V}\alpha_i=1$, $\alpha_i\geqslant 0$, $i\in V$, $\beta_j\geqslant 0$, $j\in H$, $\gamma_k\in\mathbb{R}\ \forall\, k\in L$, and $V\neq\emptyset$ (from the assumption $\dim\mathcal{M}_1=n$). If $(\boldsymbol{\lambda},\nu)\in\mathcal{P}'$, then

$$\boldsymbol{x}^T\boldsymbol{\lambda}=\sum_{i\in V}\alpha_i\boldsymbol{x}_i^T\boldsymbol{\lambda}+\sum_{j\in H}\beta_j\boldsymbol{h}_j^T\boldsymbol{\lambda}+\sum_{k\in L}\gamma_k\boldsymbol{g}_k^T\boldsymbol{\lambda}\geqslant\sum_{i\in V}\alpha_i\nu=\nu$$

and according to the description (2.1) of $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$ we have $\boldsymbol{x}^T\boldsymbol{\lambda}=\nu$. Therefore $\dim\mathcal{M}_1(\boldsymbol{\lambda},\nu)<n$. Conversely, if $(\boldsymbol{\lambda},\nu)\notin\mathcal{P}'$, then one of the following three situations occurs. Either $\boldsymbol{x}_i^T\boldsymbol{\lambda}<\nu$ holds for a certain $i\in V$, and therefore $\dim\mathcal{M}_1(\boldsymbol{\lambda},\nu)=n$. Or $\boldsymbol{h}_j^T\boldsymbol{\lambda}<0$ holds for a certain $j\in H$. Consider a point $\boldsymbol{x}_c\equiv\boldsymbol{x}_1+c\boldsymbol{h}_j$, $c>0$, where $\boldsymbol{x}_1$ is any vertex of $\mathcal{M}_1\cap\mathcal{L}^\perp$. Then for an arbitrary $c>\max\{(\nu-\boldsymbol{x}_1^T\boldsymbol{\lambda})/\boldsymbol{h}_j^T\boldsymbol{\lambda},0\}$ we have $\boldsymbol{x}_c^T\boldsymbol{\lambda}=\boldsymbol{x}_1^T\boldsymbol{\lambda}+c\boldsymbol{h}_j^T\boldsymbol{\lambda}<\nu$, and arbitrarily close to this point there is an interior point of $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$. Hence $\dim\mathcal{M}_1(\boldsymbol{\lambda},\nu)=n$. The third possibility is that $\boldsymbol{g}_k^T\boldsymbol{\lambda}\neq 0$ holds for a certain $k\in L$. We can assume without loss of generality that $\boldsymbol{g}_k^T\boldsymbol{\lambda}<0$. If $\boldsymbol{x}_1$ is any vertex of $\mathcal{M}_1\cap\mathcal{L}^\perp$, then the vector $\boldsymbol{x}_c\equiv\boldsymbol{x}_1+c\boldsymbol{g}_k$ belongs to $\mathcal{M}_1$ for all $c\in\mathbb{R}$. For any $c>(\nu-\boldsymbol{x}_1^T\boldsymbol{\lambda})/\boldsymbol{g}_k^T\boldsymbol{\lambda}$ we have $\boldsymbol{x}_c\in\mathcal{M}_1(\boldsymbol{\lambda},\nu)$, and arbitrarily close to this point there is an interior point of $\mathcal{M}_1(\boldsymbol{\lambda},\nu)$. □

In order to derive a characterization of the solution set, we have to introduce the set

$$\mathcal{U}\equiv\{(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}\colon\ \operatorname{int}\mathcal{M}_1(\boldsymbol{\lambda},\nu)\cap\operatorname{int}\mathcal{M}_2=\emptyset\}. \tag{3.3}$$

Then according to Theorems 1 and 3 we obtain the following description of the solution set.

Theorem 4 (Description of the solution set). *The solution set is equal to* $\mathcal{U}\setminus\mathcal{P}'$.

It remains to characterize the set $\mathcal{U}$. It is done in the next proposition.

Proposition 1. *Denote by* $\boldsymbol{g}_k$, $k\in L'$, *any basis of the lineality space* $\mathcal{L}_{12}\equiv\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{A}\boldsymbol{x}=\boldsymbol{0},\ \boldsymbol{C}\boldsymbol{x}=\boldsymbol{0}\}$. *Denote by* $\boldsymbol{x}_i$, $i\in V'$, *all vertices and by* $\boldsymbol{h}_j$, $j\in H'$, *all extremal directions (vectors in directions of the unbounded edges) of the convex polyhedral set* $\mathcal{M}_1\cap\mathcal{M}_2\cap\mathcal{L}_{12}^\perp$. *If* $\operatorname{int}\mathcal{M}_1\cap\operatorname{int}\mathcal{M}_2=\emptyset$, *then* $\mathcal{U}=\mathbb{R}^{n+1}$, *otherwise*

$$\mathcal{U}=\{(\boldsymbol{\lambda},\nu)\neq(\boldsymbol{0},0)\colon\ \boldsymbol{x}_i^T\boldsymbol{\lambda}\geqslant\nu\ \forall\, i\in V',\ \boldsymbol{h}_j^T\boldsymbol{\lambda}\geqslant 0\ \forall\, j\in H',\ \boldsymbol{g}_k^T\boldsymbol{\lambda}=0\ \forall\, k\in L'\}. \tag{3.4}$$

P r o o f. If $\operatorname{int}\mathcal{M}_1\cap\operatorname{int}\mathcal{M}_2=\emptyset$, then we have $\operatorname{int}\mathcal{M}_1(\boldsymbol{\lambda},\nu)\cap\operatorname{int}\mathcal{M}_2=\emptyset$ for all $(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{n+1}$, and hence $\mathcal{U}=\mathbb{R}^{n+1}$.

Let $\operatorname{int}\mathcal{M}_1\cap\operatorname{int}\mathcal{M}_2\neq\emptyset$. This is equivalent to $\dim(\mathcal{M}_1\cap\mathcal{M}_2)=n$, and we can apply Theorem 3 to the set

$$\mathcal{M}_1\cap\mathcal{M}_2=\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{A}\boldsymbol{x}\leqslant\boldsymbol{b},\ \boldsymbol{C}\boldsymbol{x}\leqslant\boldsymbol{d}\}$$

with the additional inequality $\boldsymbol{\lambda}^T\boldsymbol{x}\leqslant\nu$. □

Notice that the description (3.4) for the set $\mathcal{U}$ cannot be used in the case when $0 < \dim(\mathcal{M}_1 \cap \mathcal{M}_2) < n$. Notice also that the inclusion $\mathcal{P}' \subseteq \mathcal{U}$ holds, the proof of which is an easy exercise.

4. Description of separating hyperplanes

Let us introduce

$$\mathcal{Q}^*(\boldsymbol{\lambda}, \nu) \equiv \left\{ (\boldsymbol{u}, u_m, \boldsymbol{v}, v_{l+1}) \in \mathbb{R}^{m+l+1} \colon \ \boldsymbol{Z}(\boldsymbol{\lambda}, \nu) \begin{pmatrix} \boldsymbol{u} \\ u_m \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} = \boldsymbol{z}, \ \begin{pmatrix} \boldsymbol{u} \\ u_m \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} \geqslant \boldsymbol{0} \right\},$$

where

$$\boldsymbol{Z}(\boldsymbol{\lambda}, \nu) \equiv \begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{\lambda} & \boldsymbol{C}^T & \boldsymbol{0} \\ \boldsymbol{b}^T & \nu & \boldsymbol{d}^T & 1 \\ \boldsymbol{1}^T & 1 & \boldsymbol{1}^T & 0 \end{pmatrix}, \quad \boldsymbol{z} \equiv \begin{pmatrix} \boldsymbol{0} \\ 0 \\ 1 \end{pmatrix}.$$

For each $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$, the set $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$ represents a convex polyhedral set that plays a crucial role in checking separability of $\mathcal{M}_1(\boldsymbol{\lambda}, \nu)$ and $\mathcal{M}_2$. In particular, it enables us to explicitly describe all hyperplanes that separate $\mathcal{M}_1(\boldsymbol{\lambda}, \nu)$ and $\mathcal{M}_2$.

The following proposition is a direct consequence of Hladík [10, Assertion 4], which has the origin in Grygarová [5], [6].

Proposition 2. *Let $(\boldsymbol{\lambda}, \nu) \in \mathcal{U} \setminus \mathcal{P}'$ and $(\boldsymbol{u}, u_m, \boldsymbol{v}, v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$. Suppose that $\boldsymbol{u}^T \boldsymbol{A} + u_m \boldsymbol{\lambda}^T \neq \boldsymbol{0}^T$, and $\eta \in [0, v_{l+1}]$ is arbitrary. Then*

$$\mathcal{R} = \{\boldsymbol{x} \in \mathbb{R}^n \colon \ (\boldsymbol{u}^T \boldsymbol{A} + u_m \boldsymbol{\lambda}^T)\boldsymbol{x} - (\boldsymbol{u}^T \boldsymbol{b} + u_m \nu) = \eta\} \tag{4.1}$$

represents a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\lambda}, \nu)$, $\mathcal{M}_2$. Conversely, any separating hyperplane $\mathcal{R}$ of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\lambda}, \nu)$ and $\mathcal{M}_2$ can be expressed in the form (4.1) for certain $(\boldsymbol{u}, u_m, \boldsymbol{v}, v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$, $\boldsymbol{u}^T \boldsymbol{A} + u_m \boldsymbol{\lambda}^T \neq \boldsymbol{0}^T$ and $\eta \in [0, v_{l+1}]$.

5. Stability sets

We define stability sets in a way similar to Hladík [10], [11]. That is, stability sets consist of parameters $\boldsymbol{\lambda}$, ν under which feasibility of all bases of $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$ are preserved. This is a natural approach, as the set $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$ closely relates to the separability of $\mathcal{M}_1(\boldsymbol{\lambda}, \nu)$ and $\mathcal{M}_2$. Moreover, such stability sets often have a geometric interpretation (cf. Hladík [10]).

First recall the definition of a basis of a convex polyhedral set, and the definition of stability sets follows.

Definition 3. A *basis* of a convex polyhedral set described by $\boldsymbol{Mx} = \boldsymbol{v}$, $\boldsymbol{x} \geqslant \boldsymbol{0}$ (with $\boldsymbol{M} \in \mathbb{R}^{m\times n}$, $\boldsymbol{v} \in \mathbb{R}^m$, $m \leqslant n$) is any vector $B \in \{1, \ldots, n\}^m$ for which $\operatorname{rank}(\boldsymbol{M}_B) = m$ (where $\boldsymbol{M}_B$ means the restriction of the matrix $\boldsymbol{M}$ to the basic columns). A basis B is *feasible* if $\boldsymbol{M}_B^{-1}\boldsymbol{v} \geqslant \boldsymbol{0}$.

Definition 4. Let $(\boldsymbol{r}, s) \in \mathcal{U} \setminus \mathcal{P}'$ and denote by $\mathcal{S}$ the system of all feasible bases of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{r}, s)$. Then the *stability set* corresponding to the system $\mathcal{S}$ is the intersection of the solution set $\mathcal{U} \setminus \mathcal{P}'$ and the closure of the set of all $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ for which all bases $B \in \mathcal{S}$ remain feasible also for the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$.

Notice that we have defined stability sets as closures to simplify their description, and one can easily extend it to non-closed sets. The additional points do not change the properties of stability sets.

Our aim is to derive a description of stability sets. Let $(\boldsymbol{r}, s) \in \mathcal{U} \setminus \mathcal{P}'$ be fixed and let B be an arbitrary basis of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{r}, s)$. If $m \notin B$, then the basis B remains feasible for all $(\boldsymbol{\lambda}, \nu) \in \mathcal{U} \setminus \mathcal{P}'$. Thus consider the case when $m \in B$, i.e. $m = B_k$ for a certain $k \in \{1, \ldots, n+2\}$. By $\boldsymbol{D}(\boldsymbol{\lambda}, \nu) \equiv \boldsymbol{Z}_B(\boldsymbol{\lambda}, \nu)$ denote the restriction of the matrix $\boldsymbol{Z}(\boldsymbol{\lambda}, \nu)$ to the basic columns. For the sake of simplicity denote also $\boldsymbol{D} \equiv \boldsymbol{D}(\boldsymbol{r}, s)$ and $\boldsymbol{q} \equiv (\boldsymbol{\lambda}^T, \nu, 0)^T$.

The following Lemma 1 characterizes (by means of inequalities) the set of parameters $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ for which the basis B remains feasible. Putting all these inequalities together for all feasible bases B, we obtain the description of the stability set, which is summarized in Theorem 5.

Lemma 1. *Let B be a basis of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{r}, s)$ such that $B_k = m$ for a certain $k \in \{1, \ldots, n+2\}$. Then B is a feasible basis of $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$ for all parameters satisfying*

$$(\boldsymbol{D}^{-1}_{\cdot,n+2}\boldsymbol{D}^{-1}_{k,\cdot})\boldsymbol{q} - \boldsymbol{D}^{-1}_{k,n+2}\boldsymbol{D}^{-1}\boldsymbol{q} \geqslant \boldsymbol{0}, \tag{5.1}$$

$$\boldsymbol{D}^{-1}_{k,\cdot}\boldsymbol{q} + \boldsymbol{D}^{-1}_{k,n+2} > 0. \tag{5.2}$$

Moreover, the kth inequality of (5.1) *is redundant and can be omitted.*

Proof. The basis B remains feasible for all $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ satisfying

$$(\boldsymbol{D}(\boldsymbol{\lambda}, \nu))^{-1}\boldsymbol{z} \geqslant \boldsymbol{0}. \tag{5.3}$$

Denote $\widetilde{\boldsymbol{q}} \equiv (\boldsymbol{\lambda}^T - \boldsymbol{r}^T, \nu - s, 0)^T$. Under the assumption that $1 + \boldsymbol{e}_k^T \boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}} \neq 0$ we have according to the well-known Sherman-Morrison formula

$$(\boldsymbol{D}(\boldsymbol{\lambda}, \nu))^{-1} = (\boldsymbol{D} + \widetilde{\boldsymbol{q}} \boldsymbol{e}_k^T)^{-1} = \boldsymbol{D}^{-1} - \frac{\boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}} \boldsymbol{e}_k^T \boldsymbol{D}^{-1}}{1 + \boldsymbol{e}_k^T \boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}}}.$$

For the choice $\boldsymbol{\lambda} = \boldsymbol{r}$, $\nu = s$, the denominator $1 + \boldsymbol{e}_k^T \boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}} = 1$ is a positive number, thus we consider in addition the condition

$$1 + \boldsymbol{e}_k^T \boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}} > 0. \tag{5.4}$$

Rearrange the expression (5.3):

$$\begin{gathered}
\boldsymbol{D}^{-1}(\boldsymbol{\lambda}, \nu) \boldsymbol{z} \geqslant \boldsymbol{0}, \\
\left(\boldsymbol{D}^{-1} - \frac{\boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}} \boldsymbol{e}_k^T \boldsymbol{D}^{-1}}{1 + \boldsymbol{e}_k^T \boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}}} \right) \boldsymbol{e}_{n+2} \geqslant \boldsymbol{0}, \\
\boldsymbol{D}^{-1}_{\cdot, n+2} - \frac{\boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}} \boldsymbol{D}^{-1}_{k, n+2}}{1 + \boldsymbol{D}^{-1}_{k, \cdot} \widetilde{\boldsymbol{q}}} \geqslant \boldsymbol{0}.
\end{gathered}$$

Under the assumption (5.4) we obtain that

$$\boldsymbol{D}^{-1}_{\cdot, n+2} + (\boldsymbol{D}^{-1}_{k, \cdot} \widetilde{\boldsymbol{q}}) \boldsymbol{D}^{-1}_{\cdot, n+2} - \boldsymbol{D}^{-1}_{k, n+2} (\boldsymbol{D}^{-1} \widetilde{\boldsymbol{q}}) \geqslant \boldsymbol{0}. \tag{5.5}$$

For the vector $\widetilde{\boldsymbol{q}}$ we have $\widetilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{D}_{\cdot, k} + \boldsymbol{e}_{n+2}$. Thus the absolute term (value at $\boldsymbol{\lambda} = \boldsymbol{0}$, $\nu = 0$) of the expression (5.5) is equal to

$$\begin{aligned}
&\boldsymbol{D}^{-1}_{\cdot, n+2} + (\boldsymbol{D}^{-1}_{k, \cdot} (\boldsymbol{e}_{n+2} - \boldsymbol{D}_{\cdot, k})) \boldsymbol{D}^{-1}_{\cdot, n+2} - \boldsymbol{D}^{-1}_{k, n+2} (\boldsymbol{D}^{-1} (\boldsymbol{e}_{n+2} - \boldsymbol{D}_{\cdot, k})) \\
&\quad = \boldsymbol{D}^{-1}_{\cdot, n+2} + \boldsymbol{D}^{-1}_{k, n+2} \boldsymbol{D}^{-1}_{\cdot, n+2} - \boldsymbol{D}^{-1}_{\cdot, n+2} - \boldsymbol{D}^{-1}_{k, n+2} \boldsymbol{D}^{-1}_{\cdot, n+2} + \boldsymbol{D}^{-1}_{k, n+2} \boldsymbol{e}_k \\
&\quad = \boldsymbol{D}^{-1}_{k, n+2} \boldsymbol{e}_k = \begin{cases} 0 & \text{for any row} \neq k, \\ \boldsymbol{D}^{-1}_{k, n+2} & \text{for the row } k. \end{cases}
\end{aligned}$$

The remaining terms of the expression (5.5) are

$$(\boldsymbol{D}^{-1}_{k, \cdot} \boldsymbol{q}) \boldsymbol{D}^{-1}_{\cdot, n+2} - \boldsymbol{D}^{-1}_{k, n+2} \boldsymbol{D}^{-1} \boldsymbol{q};$$

in particular, the kth element reduces to

$$(\boldsymbol{D}^{-1}_{k, \cdot} \boldsymbol{q}) \boldsymbol{D}^{-1}_{k, n+2} - \boldsymbol{D}^{-1}_{k, n+2} \boldsymbol{D}^{-1}_{k, \cdot} \boldsymbol{q} = 0.$$

Since the basis B is feasible for the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{r}, s)$, the inequality $\boldsymbol{D}^{-1}_{k,n+2} \geqslant 0$ holds. Therefore, the kth inequality of (5.5) is redundant and the resulting system of inequalities is (without the kth inequality)

$$(\boldsymbol{D}^{-1}_{k,\cdot}\boldsymbol{q})\boldsymbol{D}^{-1}_{\cdot,n+2} - \boldsymbol{D}^{-1}_{k,n+2}\boldsymbol{D}^{-1}\boldsymbol{q} \geqslant \boldsymbol{0}.$$

Now we investigate the expression (5.4). It is equivalent to

$$\begin{aligned} 1 + \boldsymbol{e}_k^T\boldsymbol{D}^{-1}(\boldsymbol{q} + \boldsymbol{e}_{n+2} - \boldsymbol{D}_{\cdot,k}) &> 0,\\ 1 + \boldsymbol{D}^{-1}_{k,\cdot}(\boldsymbol{q} + \boldsymbol{e}_{n+2} - \boldsymbol{D}_{\cdot,k}) &> 0,\\ \boldsymbol{D}^{-1}_{k,\cdot}\boldsymbol{q} + \boldsymbol{D}^{-1}_{k,n+2} &> 0. \end{aligned}$$

□

Theorem 5 (Description of the stability set). *Let $(\boldsymbol{r}, s) \in \mathcal{U}\setminus\mathcal{P}'$ and denote by $\mathcal{S}$ the system of all feasible bases of the set $\mathcal{Q}^*(\boldsymbol{r}, s)$. The stability set corresponding to the system $\mathcal{S}$ is the set of all $(\boldsymbol{\lambda}, \nu) \in \mathcal{U}\setminus\mathcal{P}'$ satisfying the inequality systems* (5.1) *for all feasible bases B of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{r}, s)$ with the property $m \in B$.*

P r o o f. Multiply the system (5.1) by the vector $\boldsymbol{D}_{n+2,\cdot} \geqslant \boldsymbol{0}$. We obtain

$$\boldsymbol{D}^{-1}_{k,\cdot}\boldsymbol{q} - \boldsymbol{D}^{-1}_{k,n+2}\boldsymbol{e}^T_{n+2}\boldsymbol{q} \geqslant 0$$

or

$$\boldsymbol{D}^{-1}_{k,\cdot}\boldsymbol{q} \geqslant 0.$$

Hence, the constraint (5.2) is redundant (since the stability set is closed by its definition). The rest simply follows from Definition 4 and Lemma 1. □

E x a m p l e 1. Given convex polyhedral sets (see Fig. 1)

$$\mathcal{M}_1 = \left\{\boldsymbol{x} \in \mathbb{R}^2\colon \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} \boldsymbol{x} \leqslant \begin{pmatrix} 0\\ 0 \end{pmatrix}\right\},$$

$$\mathcal{M}_2 = \left\{\boldsymbol{x} \in \mathbb{R}^2\colon \begin{pmatrix} 1 & 0\\ 0 & 1\\ -1 & -1 \end{pmatrix} \boldsymbol{x} \leqslant \begin{pmatrix} 5\\ 3\\ 1 \end{pmatrix}\right\}$$

and the family of convex polyhedral sets

$$\mathcal{M}_1(\boldsymbol{\lambda}, \nu) = \{\boldsymbol{x} \in \mathcal{M}_1\colon\ \boldsymbol{\lambda}^T\boldsymbol{x} \leqslant \nu\}, \quad (\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3,$$

we compute the solution set and all stability sets. Since $\operatorname{int}\mathcal{M}_1 \cap \operatorname{int}\mathcal{M}_2 \neq \emptyset$, we proceed as follows.

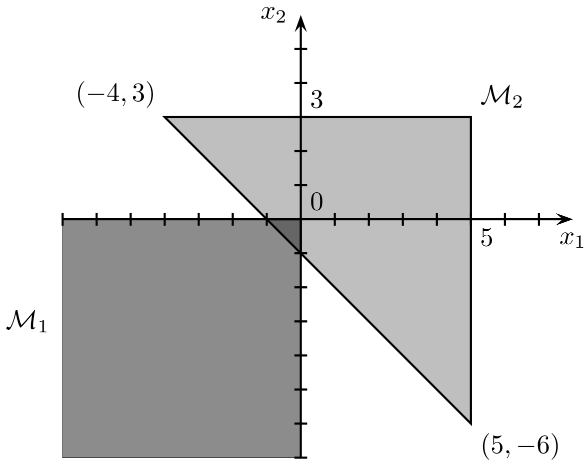

Figure 1. Illustration to Example 1.

The convex polyhedral set $\mathcal{M}_1$ contains only one vertex $(0,0)^T$, and the extremal directions of $\mathcal{M}_1$ are $(-1,0)^T$ and $(0,-1)^T$. The lineality space $\mathcal{L}$ equals $\{\mathbf{0}\}$. According to (3.2), we have

$$\mathcal{P}' = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ 0 \geqslant \nu,\ -\lambda_1 \geqslant 0,\ -\lambda_2 \geqslant 0\}.$$

The convex polyhedral set $\mathcal{M}_1 \cap \mathcal{M}_2$ contains three vertices $(0,0)^T$, $(-1,0)^T$, and $(0,-1)^T$, but no unbounded edge. The lineality space $\mathcal{L}_{12}$ equals $\{\mathbf{0}\}$. Using (3.4), we obtain that

$$\mathcal{U} = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ 0 \geqslant \nu,\ -\lambda_1 \geqslant \nu,\ -\lambda_2 \geqslant \nu\}.$$

Using Theorem 4, we get the description of the solution set

$$\begin{aligned}(5.6)\qquad \mathcal{U} \setminus \mathcal{P}' &= \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ 0 \geqslant \nu,\ -\lambda_1 \geqslant \nu,\ -\lambda_2 \geqslant \nu\} \\ &\quad \setminus \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ 0 \geqslant \nu,\ -\lambda_1 \geqslant 0,\ -\lambda_2 \geqslant 0\} \\ &= \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ 0 \geqslant \nu,\ -\lambda_1 \geqslant \nu,\ -\lambda_2 \geqslant \nu,\ \lambda_1 > 0\} \\ &\quad \cup \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ 0 \geqslant \nu,\ -\lambda_1 \geqslant \nu,\ -\lambda_2 \geqslant \nu,\ \lambda_2 > 0\} \\ &= \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ -\nu \geqslant \lambda_1 > 0,\ -\nu \geqslant \lambda_2\} \\ &\quad \cup \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \colon\ -\nu \geqslant \lambda_1,\ -\nu \geqslant \lambda_2 > 0\}.\end{aligned}$$

Now we compute the stability sets according to Theorem 5.

(1) We choose $(\lambda_1^1, \lambda_2^1, \nu^1) \in \mathcal{U} \setminus \mathcal{P}'$, for instance as $(\lambda_1^1, \lambda_2^1, \nu^1) = (1, 2, -4)$. Next we determine all feasible bases of the convex polyhedral set $\mathcal{Q}^*(\lambda_1^1, \lambda_2^1, \nu^1)$ that contain the index $m = 3$, and compute the corresponding systems of inequalities (5.1):

basis	the corresponding system
$(1,2,3,6)$	$-\lambda_1 - \nu \geqslant 0,\ \ -\lambda_2 - \nu \geqslant 0,\ \ -\nu \geqslant 0$
$(1,3,4,6)$	$-\lambda_2 - \nu \geqslant 0,\ \ \lambda_2 \geqslant 0,\ \ -5\lambda_1 + 6\lambda_2 + \nu \geqslant 0$
$(1,3,5,6)$	$-4\lambda_1 + 3\lambda_2 - \nu \geqslant 0,\ \ -\lambda_2 - \nu \geqslant 0,\ \ 3\lambda_2 - \nu \geqslant 0$
$(1,3,6,7)$	$-\lambda_1 + \lambda_2 \geqslant 0,\ \ -\lambda_2 - \nu \geqslant 0,\ \ \lambda_2 \geqslant 0$

Putting all these inequalities together with (5.6) and omitting the redundant ones, we get the final system that describes the first stability set:

$$-\lambda_2 - \nu \geqslant 0, \quad \lambda_2 > 0, \quad -5\lambda_1 + 6\lambda_2 + \nu \geqslant 0.$$

(2) Choose $(\lambda_1^2, \lambda_2^2, \nu^2) \in \mathcal{U} \setminus \mathcal{P}'$, but not from the first stability set. For instance, $(\lambda_1^2, \lambda_2^2, \nu^2) = (1, 2, -9)$. The stability set for $(\lambda_1^2, \lambda_2^2, \nu^2)$ is described by

$$-\lambda_1 + \lambda_2 \geqslant 0, \quad \lambda_2 > 0, \quad 5\lambda_1 - 6\lambda_2 - \nu \geqslant 0.$$

(3) Choose $(\lambda_1^3, \lambda_2^3, \nu^3) \in \mathcal{U} \setminus \mathcal{P}'$, but not from the first or second stability set. For instance, $(\lambda_1^3, \lambda_2^3, \nu^3) = (1, -1, -9)$. The stability set for $(\lambda_1^3, \lambda_2^3, \nu^3)$ is described by

$$\lambda_1 - \lambda_2 \geqslant 0, \quad \lambda_1 > 0, \quad -4\lambda_1 + 3\lambda_2 - \nu \geqslant 0.$$

(4) Choose $(\lambda_1^4, \lambda_2^4, \nu^4) \in \mathcal{U} \setminus \mathcal{P}'$, but not from the previous stability sets. For instance $(\lambda_1^4, \lambda_2^4, \nu^4) = (1, -1, -2)$. The stability set for $(\lambda_1^4, \lambda_2^4, \nu^4)$ is described by

$$-\lambda_1 - \nu \geqslant 0, \quad \lambda_1 > 0, \quad 4\lambda_1 - 3\lambda_2 + \nu \geqslant 0.$$

The union of the above stability sets forms the whole solution set.

Tables 1–2 contain other examples. For the pseudorandomly generated input matrices $\boldsymbol{A}$, $\boldsymbol{C}$, and vectors $\boldsymbol{b}$, $\boldsymbol{d}$ the tables involve the corresponding number of stability sets and the computing time. Our source code was written in MATLAB 6.5. The results were carried out on PC (x86), Pentium 4, 2.6 GHz, 512 MB RAM, Gentoo Linux.

Matrix $\boldsymbol{A}$	vector $\boldsymbol{b}$	Matrix $\boldsymbol{C}$	vector $\boldsymbol{d}$	number of stab sets	computing time
$\begin{pmatrix} -2 & 5 \\ -2 & 8 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ -8 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -7 & -1 \\ 4 & 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix}$	19	10 s
$\begin{pmatrix} 0 & 5 \\ 8 & -4 \\ 9 & -3 \\ 7 & -4 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \\ -3 \\ -6 \end{pmatrix}$	$\begin{pmatrix} 3 & 7 \\ -3 & -3 \\ 8 & -3 \\ -9 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 1 \\ 3 \\ 7 \end{pmatrix}$	33	37 s
$\begin{pmatrix} 6 & 1 \\ -1 & -1 \\ 9 & -7 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 7 \\ 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 8 & -9 \\ 9 & 4 \\ -7 & -6 \\ -5 & 6 \\ 7 & -1 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 4 \\ 6 \\ 6 \\ -1 \end{pmatrix}$	65	2 min 39 s
$\begin{pmatrix} 4 & -2 \\ 7 & -7 \\ 4 & -5 \\ 3 & -9 \\ 0 & 6 \\ -7 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 6 \\ 14 \\ 10 \\ 2 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ -3 & -2 \\ -4 & 0 \\ -6 & 8 \\ -1 & -2 \\ -7 & 1 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 7 \\ -1 \\ 7 \\ 2 \\ 14 \end{pmatrix}$	93	17 min 49 s

Table 1. Examples in $\mathbb{R}^2$.

Matrix $\boldsymbol{A}$	vector $\boldsymbol{b}$	Matrix $\boldsymbol{C}$	vector $\boldsymbol{d}$	number of stab sets	computing time
$\begin{pmatrix} -3 & 5 & 1 \\ 5 & 1 & 2 \\ -6 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 13 \\ 13 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -7 & 6 & -7 \\ -6 & -8 & -5 \\ 7 & -5 & 8 \end{pmatrix}$	$\begin{pmatrix} 14 \\ 12 \\ 14 \end{pmatrix}$	37	18 s
$\begin{pmatrix} 1 & -9 & -4 \\ -8 & 0 & -6 \\ 7 & 3 & -9 \\ -6 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 10 \\ -1 \\ 4 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 2 & 7 & -6 \\ 0 & 4 & -4 \\ 2 & 4 & 8 \\ -5 & -8 & 7 \end{pmatrix}$	$\begin{pmatrix} 12 \\ 13 \\ 5 \\ 3 \end{pmatrix}$	263	3 min 28 s
$\begin{pmatrix} 4 & 6 & -3 \\ -4 & 4 & -4 \\ 9 & -3 & -5 \\ -7 & -9 & -7 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 0 \\ 3 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 7 & -3 & -6 \\ 6 & -7 & 1 \\ -5 & 2 & -1 \\ 1 & 5 & 3 \\ -1 & -5 & -9 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 11 \\ 0 \\ -3 \\ 5 \end{pmatrix}$	569	22 min 25 s
$\begin{pmatrix} -4 & 5 & -1 & 0 \\ 7 & 7 & -7 & -1 \\ -3 & -8 & 3 & -3 \\ 2 & 4 & -7 & 6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 4 \\ 4 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 7 & -2 & 7 & -6 \\ 3 & -1 & -5 & 4 \\ -7 & 6 & 6 & -1 \\ 0 & -8 & -5 & 9 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 9 \\ 10 \\ 4 \end{pmatrix}$	882	28 min 13 s

Table 2. Examples in $\mathbb{R}^3$ and $\mathbb{R}^4$.

6. Separating supporting hyperplanes

Supporting separation is a special case of separation of two sets, where we require that there is a hyperplane that separates the sets and simultaneously has a non-empty intersection with both of them. Such an approach has applications, for instance, in optimization (see e.g. Grygarová [7], [8]).

In this section we derive the description of the set of all parameters $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which there exists a separating hyperplane of convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ and $\mathcal{M}_2$ supporting their faces determined by given sub-bases. The notion of a sub-basis generalizes the traditional notion of a basis and the geometric interpretation is that a sub-basis determines a face of a convex polyhedral set. Informally, a sub-basis is a list of row indices of some inequality system and the corresponding rows are linearly independent. The formal definition follows.

Definition 5. A *sub-basis* of the convex polyhedral set described by $\boldsymbol{M}\boldsymbol{x} \leqslant \boldsymbol{v}$ ($\boldsymbol{M} \in \mathbb{R}^{m\times n}$, $\boldsymbol{v} \in \mathbb{R}^m$) is any vector $S \in \{1,\dots,m\}^k$, $1 \leqslant k \leqslant m$, for which $\operatorname{rank}(\boldsymbol{M}_S) = k$ holds (where $\boldsymbol{M}_S$ stands for the restriction of the matrix $\boldsymbol{M}$ to the sub-basic rows). A sub-basis S is called *feasible* if $\{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{M}_S\boldsymbol{x} = \boldsymbol{v}_S, \boldsymbol{M}_N\boldsymbol{x} \leqslant \boldsymbol{v}_N\} \neq \emptyset$ with $N \equiv \{1,\dots,m\} \setminus S$.

Denote by $\mathcal{S}_{B^1B^2}$ the set of parameters $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which there exists a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ and $\mathcal{M}_2$ which supports the faces of $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ and $\mathcal{M}_2$ determined by sub-bases B^1 and B^2, respectively.

In order to derive a characterization of $\mathcal{S}_{B^1B^2}$ we have to state some preliminary results first. We introduce $\mathcal{S}_{B^1}$ as the set of $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which the sub-basis B^1 of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ is feasible. The description of the set $\mathcal{S}_{B^1}$ follows from Lemmas 2 and 3.

Lemma 2. *Let B^1 be a sub-basis of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ and suppose $m \notin B^1$. Denote $N^1 \equiv \{1,\dots,m-1\} \setminus B^1$. Let $\boldsymbol{g}_k$, $k \in L$, be a basis of the lineality space $\mathcal{L} \equiv \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$. Denote by $\boldsymbol{x}_i$, $i \in V$, all vertices and by $\boldsymbol{h}_j$, $j \in H$, all extremal directions of the convex polyhedral set*

$$\{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1},\ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}\} \cap \mathcal{L}^{\perp}.$$

Then

$$\mathcal{S}_{B^1} = \mathbb{R}^{n+1} \setminus \mathcal{P}_{B^1},$$

where

$$\mathcal{P}_{B^1} = \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1} \colon \boldsymbol{x}_i^T\boldsymbol{\lambda} > \mu\ \forall\, i \in V,\ \boldsymbol{h}_j^T\boldsymbol{\lambda} \geqslant 0\ \forall\, j \in H,\ \boldsymbol{g}_k^T\boldsymbol{\lambda} = 0\ \forall\, k \in L\}.$$

P r o o f. The sub-basis B^1 of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ is feasible for all $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ such that the set $\{\boldsymbol{x} \in \mathbb{R}^n \colon \ \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1}, \ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}, \ \boldsymbol{\lambda}^T\boldsymbol{x} \leqslant \mu\}$ is non-empty. Now we just apply the statement of Theorem 2. □

Lemma 3. *Let B^1 be a sub-basis of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ and suppose $m \notin B^1$. Denote $N^1 \equiv \{1, \ldots, m-1\} \setminus B^1$. Let $\boldsymbol{g}_k$, $k \in L$, be a basis of the lineality space $\mathcal{L} \equiv \{\boldsymbol{x} \in \mathbb{R}^n \colon \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$. Denote by $\boldsymbol{x}_i$, $i \in V$, all vertices and by $\boldsymbol{h}_j$, $j \in H$, all extremal directions of the convex polyhedral set*

$$\{\boldsymbol{x} \in \mathbb{R}^n \colon \ \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1}, \ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}\} \cap \mathcal{L}^{\perp}.$$

Then

$$\mathcal{S}_{B^1 \cup \{m\}} = \mathbb{R}^{n+1} \setminus (\mathcal{P}_{B^1} \cup -\mathcal{P}_{B^1}),$$

where $\mathcal{P}_{B^1}$ is the convex cone from Lemma 2.

P r o o f. The sub-basis $B^1 \cup \{m\}$ of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\lambda}, \mu)$ is feasible for all $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which the set $\{\boldsymbol{x} \in \mathbb{R}^n \colon \ \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1}, \ \boldsymbol{\lambda}^T\boldsymbol{x} = \mu, \ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}\}$ is not empty. In other words, both the sets $\{\boldsymbol{x} \in \mathbb{R}^n \colon \ \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1}, \ \boldsymbol{\lambda}^T\boldsymbol{x} \leqslant \mu, \ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}\}$ and $\{\boldsymbol{x} \in \mathbb{R}^n \colon \ \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1}, \ -\boldsymbol{\lambda}^T\boldsymbol{x} \leqslant -\mu, \ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}\}$ must be simultaneously non-empty. From Lemma 2 we obtain the description of the set $\mathcal{S}_{B^1 \cup \{m\}}$ as $(\mathbb{R}^{n+1} \setminus \mathcal{P}_{B^1}) \cap (\mathbb{R}^{n+1} \setminus -\mathcal{P}_{B^1}) = \mathbb{R}^{n+1} \setminus (\mathcal{P}_{B^1} \cup -\mathcal{P}_{B^1})$. □

Consider the family of convex polyhedral sets

$$\mathcal{M}(\boldsymbol{\xi}) \equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \colon \ \boldsymbol{M}\boldsymbol{x} + \boldsymbol{\xi}x_n = \boldsymbol{v}, \ \boldsymbol{x} \geqslant \boldsymbol{0}, \ x_n \geqslant 0\},$$

where $\boldsymbol{M} \in \mathbb{R}^{m \times (n-1)}$, $\boldsymbol{v} \in \mathbb{R}^m$ are fixed and $\boldsymbol{\xi}$ is an m-dimensional vector of parameters. The second preliminary result is stated in Proposition 3 and describes the set $\mathcal{S}^{\mathcal{M}}$ of all parameters $\boldsymbol{\xi} \in \mathbb{R}^m$ for which the set $\mathcal{M}(\boldsymbol{\xi})$ is non-empty.

Proposition 3. *Denote by $\boldsymbol{h}_k$, $k \in L$, any basis of the lineality space $\mathcal{L} \equiv \{\boldsymbol{y} \in \mathbb{R}^m \colon \ \boldsymbol{M}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{v}^T\boldsymbol{y} = \boldsymbol{0}\}$. For the convex polyhedral cone*

$$\{\boldsymbol{y} \in \mathbb{R}^m \colon \ \boldsymbol{M}^T\boldsymbol{y} \leqslant \boldsymbol{0}, \ \boldsymbol{v}^T\boldsymbol{y} \geqslant \boldsymbol{0}\} \cap \mathcal{L}^{\perp}$$

denote by $\boldsymbol{g}_i$, $i \in I_1$, its extremal directions with the property $\boldsymbol{g}_i^T\boldsymbol{v} > 0$ and by $\boldsymbol{f}_j$, $j \in I_2$, its extremal directions with the property $\boldsymbol{f}_j^T\boldsymbol{v} = 0$. If $I_1 = \emptyset$, then $\mathcal{S}^{\mathcal{M}} = \mathbb{R}^m$. Otherwise

$$\mathcal{S}^{\mathcal{M}} = \{\boldsymbol{\xi} \in \mathbb{R}^m \colon \ \boldsymbol{g}_i^T\boldsymbol{\xi} > 0 \ \forall\, i \in I_1, \ \boldsymbol{f}_j^T\boldsymbol{\xi} \geqslant 0 \ \forall\, j \in I_2, \ \boldsymbol{h}_k^T\boldsymbol{\xi} = 0 \ \forall\, k \in L\}. \tag{6.1}$$

P r o o f. $\mathcal{S}^{\mathcal{M}}$ is the set of all $\boldsymbol{\xi} \in \mathbb{R}^m$ for which $\mathcal{M}(\boldsymbol{\xi}) \neq \emptyset$, i.e ., the problem

$$\min\{\mathbf{0}^T\boldsymbol{x} + 0x_n\colon\ \boldsymbol{M}\boldsymbol{x} + \boldsymbol{\xi}x_n = \boldsymbol{v},\ \boldsymbol{x} \geqslant \mathbf{0},\ x_n \geqslant 0\}$$

has an optimal solution. Using the duality theorems in linear programming this is true if and only if the problem

$$\max\{\boldsymbol{v}^T\boldsymbol{y}\colon\ \boldsymbol{M}^T\boldsymbol{y} \leqslant \mathbf{0},\ \boldsymbol{\xi}^T\boldsymbol{y} \leqslant 0\} \tag{6.2}$$

has an optimal solution. Since the set of feasible solutions to the problem (6.2) represents a convex polyhedral cone (with one vertex at the origin), we can formulate this situation as

$$\{\boldsymbol{y} \in \mathbb{R}^m\colon\ \boldsymbol{M}^T\boldsymbol{y} \leqslant \mathbf{0},\ \boldsymbol{\xi}^T\boldsymbol{y} \leqslant 0,\ \boldsymbol{v}^T\boldsymbol{y} > 0\} = \emptyset. \tag{6.3}$$

If $I_1 = \emptyset$, then clearly $\mathcal{S}^{\mathcal{M}} = \mathbb{R}^m$. Suppose that $I_1 \neq \emptyset$.

Let $\boldsymbol{\xi}_0 \in \mathbb{R}^m$ be such that $\boldsymbol{g}_i^T\boldsymbol{\xi}_0 > 0$ for all $i \in I_1$, $\boldsymbol{f}_j^T\boldsymbol{\xi}_0 \geqslant 0$ for all $j \in I_2$ and $\boldsymbol{h}_k^T\boldsymbol{\xi}_0 = 0$ for all $k \in L$. Each point $\boldsymbol{y} \in \{\boldsymbol{y} \in \mathbb{R}^m\colon\ \boldsymbol{M}^T\boldsymbol{y} \leqslant \mathbf{0},\ \boldsymbol{v}^T\boldsymbol{y} > 0\}$ can be written as a linear combination

$$\boldsymbol{y} = \sum_{i\in I_1} \alpha_i\boldsymbol{g}_i + \sum_{j\in I_2} \beta_j\boldsymbol{f}_j + \sum_{k\in L} \gamma_k\boldsymbol{h}_k$$

for certain $\alpha_i, \beta_j \geqslant 0$, $\sum\limits_{i\in I_1} \alpha_i > 0$, and $\gamma_k \in \mathbb{R}$. Then

$$\boldsymbol{y}^T\boldsymbol{\xi}_0 = \sum_{i\in I_1} \alpha_i\boldsymbol{g}_i^T\boldsymbol{\xi}_0 + \sum_{j\in I_2} \beta_j\boldsymbol{f}_j^T\boldsymbol{\xi}_0 + \sum_{k\in L} \gamma_k\boldsymbol{h}_k^T\boldsymbol{\xi}_0 > 0.$$

Therefore, the condition (6.3) holds.

Conversely, let $\boldsymbol{\xi}_0 \in \mathbb{R}^m$ be such that (6.3) holds. Then for all $\boldsymbol{y} \in \{\boldsymbol{y} \in \mathbb{R}^m\colon$ $\boldsymbol{M}^T\boldsymbol{y} \leqslant \mathbf{0},\ \boldsymbol{v}^T\boldsymbol{y} > 0\}$ we have $\boldsymbol{y}^T\boldsymbol{\xi}_0 > 0$. In particular, $\boldsymbol{g}_i^T\boldsymbol{\xi}_0 > 0$ for all $i \in I_1$. For any sufficiently small $\varepsilon > 0$ we also have that the convex combination $(1-\varepsilon) \times$ $\boldsymbol{f}_j^T\boldsymbol{\xi}_0 + (\varepsilon/|I_1|) \sum\limits_{i\in I_1} \boldsymbol{g}_i^T\boldsymbol{\xi}_0 > 0$ for all $j \in I_2$. Hence, $(1-\varepsilon)\boldsymbol{f}_j^T\boldsymbol{\xi}_0 \geqslant 0$ and therefore $\boldsymbol{f}_j^T\boldsymbol{\xi}_0 \geqslant 0$ for all $j \in I_2$. Analogously we can prove $\boldsymbol{h}_k^T\boldsymbol{\xi}_0 = 0$ for all $k \in L$. Hence $\boldsymbol{\xi}_0$ belongs to the set (6.1). □

We derive the description of the set $\mathcal{S}_{B^1B^2}$. Consider the system

$$\boldsymbol{Z}(\boldsymbol{\lambda}, \mu)_B\boldsymbol{w} = \boldsymbol{z}, \quad \boldsymbol{w} \geqslant \mathbf{0}, \tag{6.4}$$

where B is a sub-basis of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\lambda}, \mu)$. Define $\mathcal{S}_{B^1B^2}^{\mathcal{Q}}$ as the set of $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which the system (6.4) with $B \equiv B^1 \cup \big(B^2 + m\big)$ is solvable.

The description of the set $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ follows from Proposition 3 if for $B_r \equiv B^1_r \cup (B^2+m)$ we assign

$$\boldsymbol{M} \equiv \boldsymbol{Z}(\boldsymbol{\lambda},\mu)_{B_r}, \quad \boldsymbol{v} \equiv \boldsymbol{z}, \quad \boldsymbol{\xi} \equiv (\boldsymbol{\lambda}^T,\mu,1)^T.$$

The sense of the set $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ is to ensure the existence of a separating supporting hyperplane of $\mathcal{M}_1(\boldsymbol{\lambda},\mu)$ and $\mathcal{M}_2$ (supporting the given faces), while the sense of the set $\mathcal{S}_{B^1}$ is to preserve feasibility of the sub-basis B^1 of the set $\mathcal{M}_1(\boldsymbol{\lambda},\mu)$.

Now we are ready to characterize the set $\mathcal{S}_{B^1B^2}$; the description is a direct consequence of our definitions and lemmas.

Theorem 6 (Description of the set $\mathcal{S}_{B^1B^2}$)**.**

(1) *Let $m \notin B^1$. If the system* (6.4) *with $B \equiv B^1 \cup (B^2+m)$ has no solution, then $\mathcal{S}_{B^1B^2} = \emptyset$. Otherwise,*

$$\mathcal{S}_{B^1B^2} = \mathcal{S}_{B^1},$$

where $\mathcal{S}_{B^1}$ is described according to Lemma 2.

(2) *Let $m \in B^1$, i.e. $B^1 = B^1_r \cup \{m\}$ for a certain sub-basis B^1_r. Then*

$$\mathcal{S}_{B^1B^2} = \mathcal{S}_{B^1} \cap \mathcal{S}^{\mathcal{Q}}_{B^1B^2},$$

where $\mathcal{S}_{B^1} = \mathcal{S}_{B^1_r\cup\{m\}}$ is described according to Lemma 3.

Example 2. Given convex polyhedral sets (see Fig. 2)

$$\mathcal{M}_1 = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix} \boldsymbol{x} \leqslant \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \right\},$$

$$\mathcal{M}_2 = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \colon \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \boldsymbol{x} \leqslant \begin{pmatrix} 2 \\ -8 \\ -8 \end{pmatrix} \right\},$$

a family of convex polyhedral sets

$$\mathcal{M}_1(\boldsymbol{\lambda},\mu) = \{\boldsymbol{x} \in \mathcal{M}_1 \colon \boldsymbol{\lambda}^T\boldsymbol{x} \leqslant \mu\}, \quad (\boldsymbol{\lambda},\mu) \in \mathbb{R}^3,$$

and bases $B^1 = (2,4)$ and $B^2 = (1,2)$ of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\lambda},\mu)$ and $\mathcal{M}_2$, respectively. We aim at calculating the description of the set $\mathcal{S}_{B^1B^2}$. Since $4 \in B^1$, we proceed according to the second paragraph of Theorem 6.

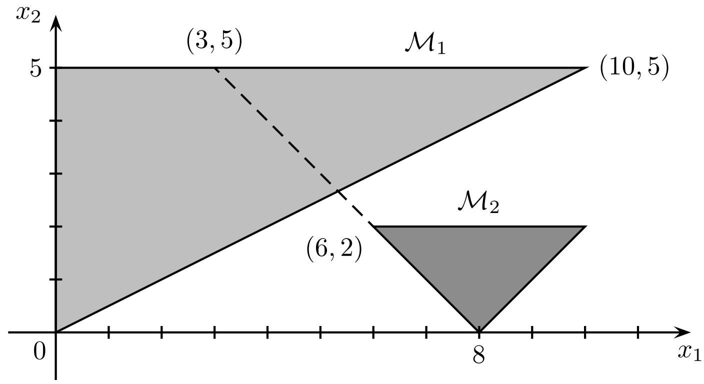

Figure 2. Illustration to Example 2.

First, we deal with the set $\mathcal{S}_{B^1}$. From Lemma 3 we have $\mathcal{S}_{B^1} = \mathcal{S}_{(2,4)} = \mathbb{R}^3 \setminus (\mathcal{P}_{(2)} \cup -\mathcal{P}_{(2)})$. The convex polyhedral set

$$\{\boldsymbol{x} \in \mathbb{R}^2\colon\ \boldsymbol{A}_{B^1}\boldsymbol{x} = \boldsymbol{b}_{B^1},\ \boldsymbol{A}_{N^1}\boldsymbol{x} \leqslant \boldsymbol{b}_{N^1}\} = \{\boldsymbol{x} \in \mathbb{R}^2\colon\ x_2 = 5,\ -x_1 \leqslant 0,\ x_1 - 2x_2 \leqslant 0\}$$

has two vertices $\boldsymbol{x}_1 = (0,5)^T$ and $\boldsymbol{x}_2 = (10,5)^T$. Hence, we obtain

$$\begin{aligned}\mathcal{P}_{(2)} &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3\colon\ 5\lambda_2 > \mu,\ 10\lambda_1 + 5\lambda_2 > \mu\},\\ \mathcal{S}_{(2,4)} &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3\colon\ 5\lambda_2 \geqslant \mu,\ 10\lambda_1 + 5\lambda_2 \leqslant \mu\}\\ &\qquad \cup \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3\colon\ 5\lambda_2 \leqslant \mu,\ 10\lambda_1 + 5\lambda_2 \geqslant \mu\}.\end{aligned}$$

The description of the set $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ follows from Proposition 3. The convex polyhedral cone described as

$$\begin{pmatrix} 0 & 1 & 5 & 1\\ 0 & 1 & 2 & 1\\ -1 & -1 & -8 & 1\end{pmatrix}\boldsymbol{y} \leqslant \boldsymbol{0}, \quad (0\ \ 0\ \ 0\ \ 1)\,\boldsymbol{y} \geqslant 0$$

has edges in the directions of vectors $\boldsymbol{g}_1 = (2,-1,0,1)^T$, $\boldsymbol{f}_1 = (6,2,-1,0)^T$, $\boldsymbol{f}_2 = (-3,-5,1,0)^T$, and $\boldsymbol{f}_3 = (1,0,0,0)^T$. Hence,

$$\begin{aligned}\mathcal{S}^{\mathcal{Q}}_{B^1B^2} &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3\colon\ 2\lambda_1 - \lambda_2 + 1 > 0,\ 6\lambda_1 + 2\lambda_2 - \mu \geqslant 0,\ -3\lambda_1 - 5\lambda_2 + \mu \geqslant 0,\\ &\qquad \lambda_1 \geqslant 0\}\\ &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3\colon\ 6\lambda_1 + 2\lambda_2 - \mu \geqslant 0,\ -3\lambda_1 - 5\lambda_2 + \mu \geqslant 0,\ \lambda_1 \geqslant 0\}.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathcal{S}_{B^1B^2} &= \mathcal{S}_{B^1} \cap \mathcal{S}^{\mathcal{Q}}_{B^1B^2}\\ &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3\colon\ 10\lambda_1 + 5\lambda_2 - \mu \geqslant 0,\ 6\lambda_1 + 2\lambda_2 - \mu \geqslant 0,\\ &\qquad - 3\lambda_1 - 5\lambda_2 + \mu \geqslant 0\}.\end{aligned}$$

7. A permanent separating hyperplane

Consider $\mathcal{M}_1(\boldsymbol{\lambda},\mu)$ from (2.1) with the property $(\boldsymbol{\lambda},\mu)\in\mathcal{Z}$, where $\mathcal{Z}\subset\mathbb{R}^{n+1}$ is a convex polytope. Assume that $\mathcal{Z}\cap\mathcal{P}'=\emptyset$. The question that we ask in this section is whether there exists a separating hyperplane $\mathcal{R}=\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{r}^T\boldsymbol{x}=s\}$ such that

$$\mathcal{M}_1(\boldsymbol{\lambda},\mu)\subseteq\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{r}^T\boldsymbol{x}\leqslant s\}\ \forall\,(\boldsymbol{\lambda},\mu)\in\mathcal{Z},\quad \mathcal{M}_2\subseteq\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{r}^T\boldsymbol{x}\geqslant s\}.$$

Such a separating hyperplane $\mathcal{R}$ is called *permanent*. We check the existence of a permanent separating hyperplane in the following manner: We compute the convex hull $\operatorname{conv}\Big(\bigcup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}}\mathcal{M}_1(\boldsymbol{\lambda},\mu)\Big)$ and check separability of this convex hull and the set $\mathcal{M}_2$. Proposition 4 says how to compute $\bigcup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}}\mathcal{M}_1(\boldsymbol{\lambda},\mu)$. This set represents a polyhedral set, but not convex in general; see Example 3.

Proposition 4. *Let $(\boldsymbol{r}_i,s_i)$, $i\in V$, be all vertices of the convex polytope $\mathcal{Z}$. Then*

$$\bigcup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}}\mathcal{M}_1(\boldsymbol{\lambda},\mu)=\bigcup_{i\in V}\mathcal{M}_1(\boldsymbol{r}_i,s_i).$$

P r o o f. Let $(\boldsymbol{\lambda}_1,\mu_1),(\boldsymbol{\lambda}_2,\mu_2)\in\mathcal{Z}$. It is sufficient to prove that for a convex combination $(\boldsymbol{\lambda}_c,\mu_c)\equiv q(\boldsymbol{\lambda}_1,\mu_1)+(1-q)(\boldsymbol{\lambda}_2,\mu_2)$, $q\in(0,1)$, the inclusion

$$\mathcal{M}_1(\boldsymbol{\lambda}_c,\mu_c)\subseteq(\mathcal{M}_1(\boldsymbol{\lambda}_1,\mu_1)\cup\mathcal{M}_1(\boldsymbol{\lambda}_2,\mu_2))$$

holds. To prove this inclusion it is sufficient to prove the relation

$$\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{\lambda}_c^T\boldsymbol{x}\leqslant\mu_c\}\subseteq(\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{\lambda}_1^T\boldsymbol{x}\leqslant\mu_1\}\cup\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \boldsymbol{\lambda}_2^T\boldsymbol{x}\leqslant\mu_2\}).$$

We prove this relation by contradiction. Suppose that for a certain point $\boldsymbol{x}_0\in\mathbb{R}^n$

$$\boldsymbol{\lambda}_1^T\boldsymbol{x}_0>\mu_1,\quad \boldsymbol{\lambda}_2^T\boldsymbol{x}_0>\mu_2,\quad\text{and}\quad \boldsymbol{\lambda}_c^T\boldsymbol{x}_0\leqslant\mu_c$$

hold. Multiplying the first inequality by a number $q>0$ and the second inequality by a number $1-q>0$, we obtain

$$q\boldsymbol{\lambda}_1^T\boldsymbol{x}_0+(1-q)\boldsymbol{\lambda}_2^T\boldsymbol{x}_0>q\mu_1+(1-q)\mu,$$

i.e. $\boldsymbol{\lambda}_c^T\boldsymbol{x}_0>\mu_c$, which contradicts our assumption. □

Example 3. Given

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ -5 & 2 \\ -1 & 6 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 28 \end{pmatrix},$$
$$\mathcal{Z} = \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^3 \colon \ \lambda_1 = 1, \ |\lambda_2| \leqslant 1, \ \mu = 2\lambda_2 + 4\},$$

we compute the set $\operatorname{conv}\Bigl(\bigcup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} \mathcal{M}_1(\boldsymbol{\lambda},\mu)\Bigr)$ by the proposed method; see Fig. 3. The convex polytope $\mathcal{Z}$ contains two vertices $(\boldsymbol{r}_1, s_1) = (1, 1, 6)$ and $(\boldsymbol{r}_2, s_2) = (1, -1, 2)$.

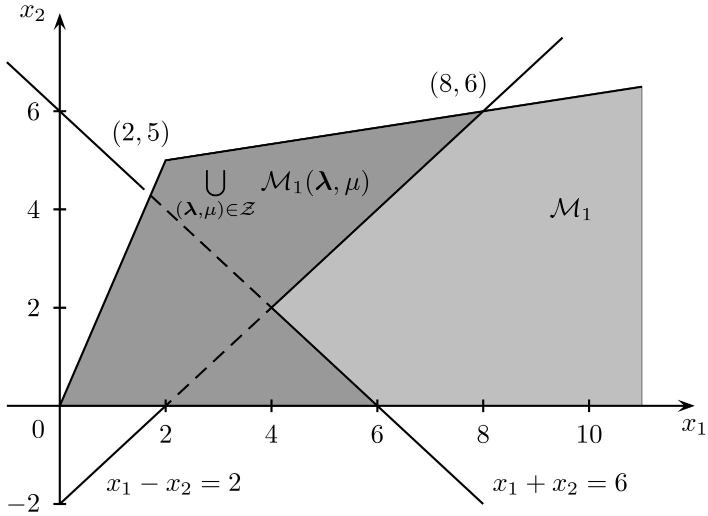

Figure 3. Illustration to Example 3.

Hence, the convex hull $\operatorname{conv}\Bigl(\bigcup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} \mathcal{M}_1(\boldsymbol{\lambda},\mu)\Bigr)$ is equal to $\operatorname{conv}(\mathcal{M}_1(\boldsymbol{r}_1, s_1) \cup \mathcal{M}_1(\boldsymbol{r}_2, s_2))$ (for an explicit description of the convex hull see Grygarová [6]), which represents the convex polytope with vertices $(0,0)$, $(6,0)$, $(8,6)$, $(2,5)$ and is described by the system of inequalities

$$\begin{pmatrix} 0 & -1 \\ -5 & 2 \\ -1 & 6 \\ 3 & -1 \end{pmatrix} \boldsymbol{x} \leqslant \begin{pmatrix} 0 \\ 0 \\ 28 \\ 18 \end{pmatrix}.$$

8. Application in multiobjective linear programming

Now, we show how the proposed theory can be applied in multiobjective linear programming the problems of which involve parameters. Consider a multiobjective linear program

$$\max\{\boldsymbol{C}\boldsymbol{x}\colon \boldsymbol{x} \in \mathcal{M}\},$$

where $\mathcal{M} \equiv \{\boldsymbol{x} \in \mathbb{R}^n\colon \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\}$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{C} \in \mathbb{R}^{l\times n}$, and $\boldsymbol{b} \in \mathbb{R}^m$. Let $\boldsymbol{x}_0 \in \mathcal{M}$ be a weakly efficient solution, i.e., there is no $\boldsymbol{x} \in \mathcal{M}$ with $\boldsymbol{C}\boldsymbol{x} > \boldsymbol{C}\boldsymbol{x}_0$. Alternatively, weak efficiency of $\boldsymbol{x}_0$ can be characterized as separability by a hyperplane of two convex polyhedral sets,

$$\mathcal{M} \quad \text{and} \quad \{\boldsymbol{x} \in \mathbb{R}^n\colon \boldsymbol{C}\boldsymbol{x} \geqslant \boldsymbol{C}\boldsymbol{x}_0\}, \tag{8.1}$$

or, after translation,

$$\{\boldsymbol{x} \in \mathbb{R}^n\colon \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0\} \quad \text{and} \quad \{\boldsymbol{x} \in \mathbb{R}^n\colon \boldsymbol{C}\boldsymbol{x} \geqslant \boldsymbol{0}\}. \tag{8.2}$$

Provided there are certain uncertainties in one row of the cost matrix coefficients, they can be modelled by row parameters and the theory derived in the previous sections is applicable to the pair of convex polyhedral sets (8.2). The pair of sets (8.1) can be used in the case that parameters appear in one row of the constraint matrix $\boldsymbol{A}$.

In this example, the solution set contains only such values of parameters for which $\boldsymbol{x}_0$ remains a weakly efficient solution. Due to the condition of full dimensionality in the definition of separation (Definition 1) we do not cover all weakly efficient points, however, the remaining ones have usually zero measure. Stability sets represent well defined convex subsets of the solution set.

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