Umberto Triacca; Andrei Volodin

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ON A CHARACTERIZATION OF ORTHOGONALITY WITH RESPECT TO PARTICULAR SEQUENCES OF RANDOM VARIABLES IN L^2

Umberto Triacca, L'Aquila, Andrei Volodin, Regina

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Abstract. This note deals with the orthogonality between sequences of random variables. The main idea of the note is to apply the results on equidistant systems of points in a Hilbert space to the case of the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of real square integrable random variables. The main result gives a necessary and sufficient condition for a particular sequence of random variables (elements of which are taken from sets of equidistant elements of $L^2(\Omega, \mathcal{F}, \mathbb{P})$) to be orthogonal to some other sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. The result obtained is interesting from the point of view of the time series analysis, since it can be applied to a class of sequences random variables that exhibit a monotonically increasing variance. An application to ergodic theorem is also provided.

Keywords: Hilbert space, orthogonality, ergodic theorem

MSC 2010: 60F15, 60G50

1. Introduction

In Wermuth [1] a few results on equidistant systems of vectors in a Hilbert space are given (cf. Lemma 1 below). While the arguments that lead to the proof of Wermuth's results are in some sense elementary, they are by all means extremely clever, elegant, and beautiful. In this note we apply Wermuth's results to some questions of Probability Theory, mainly to orthogonal systems of random variables. An application to ergodic theorem is given.

In time series analysis, most of the models are based on the assumption of covariance stationarity. However, in the applications, this is often not a reasonable

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assumption. In fact, many time series show a time-varying second-order structure. That is, variance and covariance are likely to change over time. Examples may be found in a growing number of fields, such as hydrological time series, biomedicine and economics.

In this note, we consider a situation in which only the variance is time-varying. In particular, we provide a necessary and sufficient condition for a sequence of random variables that exhibits a monotonically increasing variance to be orthogonal to some other sequence of random variables. In other words, we use Hilbert space methods to obtain a necessary and sufficient condition for a particular nonstationary sequence of random variables, $\{x_n,\, n \geqslant 1\}$, to be orthogonal to some other sequence. The nonstationarity is due to the fact that the variance of the sequence $\{x_n,\, n \geqslant 1\}$ increases with n to a specific constant.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. We consider the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of all real square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, that is all real random variables which satisfy $E(z^2) < \infty$, E being the mathematical expectation operator. The inner product in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is defined by $\langle z, w\rangle = E(zw)$ for any $z, w \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. The space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a normed space, the norm is given by $\|w\| = [E(w^2)]^{1/2}$ and the distance measure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is provided by the norm of the difference, that is $d(z, w) = \|z - w\|$.

A sequence $\{z_n\} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge to a limit point $z \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ if $d(z_n, z) \to 0$ as $n \to \infty$. A point $z \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a limit point of a set $M \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ if it is a limit point of a sequence from M. In particular, M is said to be closed if it contains all its limit points. If S is an arbitrary subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then the set of all $\alpha_1 z_1 + \ldots + \alpha_n z_n$ ($n \geqslant 1$, $\alpha_1, \ldots, \alpha_n$ are arbitrary real numbers; $z_1, \ldots, z_n$ are arbitrary elements of S) is called the linear manifold spanned by S and denoted by $\operatorname{sp}(S)$. If we add to $\operatorname{sp}(S)$ all its limit points, we obtain a closed set that we call the closed linear manifold or subspace spanned by S, denoted by $\overline{\operatorname{sp}}(S)$.

Two elements $z, w \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ are called orthogonal, and we write $z \perp w$, if $\langle z, w\rangle = 0$. If S is any subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then we write $x \perp S$ if $x \perp s$ for all $s \in S$. For a given $z \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and a closed subspace M of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, we define the orthogonal projection of z on M, denoted by $P(z|M)$, as the unique element of M such that $\|z - P(z|M)\| \leqslant \|z - w\|$ for any $w \in M$.

The distance $d(x, S)$ of a point $x \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ to the set $S \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is defined by

$$d(x, S) = \inf\{\|x - y\|;\ y \in S\}.$$

Let $\delta > 0$ be a real number. An array $X = \{x_{n,k},\, n \geqslant 1,\, 1 \leqslant k \leqslant n+1\}$ of random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is called *rowwise δ-equidistant, zero-sum* if

$$\|x_{n,j} - x_{n,k}\| = \delta > 0 \quad (1 \leqslant j < k \leqslant n+1)$$

and

$$x_{n,1} + \ldots + x_{n,n+1} = 0.$$

We say that $\{x_n,\, n \geqslant 1\}$ is *a sequence of random variables based on the δ-equidistant, zero-sum array X*, if for all $n \geqslant 1$ there exists k, $1 \leqslant k \leqslant n+1$ such that $x_n = x_{n,k}$. In other words, we pick one element from each row of the array.

The spread of each row of the array is determined by δ. We note that δ is the diameter of any row of the array, that is

$$\sup\{d(x_{n,i}, x_{n,j});\, x_{n,i}, x_{n,j} \in X\} = \delta \text{ for any } n \geqslant 1.$$

In this section we offer a characterization of the condition of orthogonality between the sequence of random variables $\{x_n,\, n \geqslant 1\}$ based on the δ-equidistant, zero-sum array X and any other sequence of random variables $\{y_n,\, n \geqslant 1\}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We start by reminding that two sequences $\{x_n;\, n = 1, 2, \ldots\}$ and $\{y_n;\, n = 1, 2, \ldots\}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ are said to be *orthogonal* if $x_n \perp y_m$ for all n and m, that is, $E(x_n y_m) = 0$.

The following proposition is simply a reformulation of Theorem 1 of Wermuth [1] for the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 1. *Let $\{x_n,\, n \geqslant 1\}$ be a sequence of random variables based on a rowwise δ-equidistant, zero-sum array X. Then for all $n \geqslant 1$*

$$\|x_n\| = \frac{\delta}{\sqrt{2}} \sqrt{1 - \frac{1}{n+1}}.$$

Remark 1. If $E(x_n) = 0$ for all $n \geqslant 1$, then by Proposition 1 the sequence $\{x_n,\, n \geqslant 1\}$ has a monotonically increasing variance and

$$\lim_{n\to\infty} \operatorname{Var}(x_n) = \frac{\delta^2}{2}.$$

We are now ready to formulate and prove the main result of this section.

Proposition 2. *Let $\{x_n,\, n \geqslant 1\}$ be a sequence of random variables based on a rowwise δ-equidistant, zero-sum array X and let $\{y_n,\, n \geqslant 1\}$ be a sequence in*

$L^2(\Omega, \mathcal{F}, \mathbb{P})$. *The sequences* $\{x_n,\, n \geqslant 1\}$ *and* $\{y_n,\, n \geqslant 1\}$ *are orthogonal if and only if for any* $n \geqslant 1$

$$d(x_n, H_y) = \frac{\delta}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}},$$

where $H_y = \overline{\mathrm{sp}}(y_n,\, n \geqslant 1)$.

P r o o f. *Necessity.* If $\{x_n,\, n \geqslant 1\}$ and $\{y_n,\, n \geqslant 1\}$ are orthogonal, then $x_n \perp H_y$ for all $n \geqslant 1$ and so $P(x_n|H_y) = 0$ for all $n \geqslant 1$. Since

$$\begin{aligned} d(x_n, H_y) &= \inf\{\|x_n - y\|;\, y \in H_y\} \\ &= \|x_n - P(x_n|H_y)\|, \end{aligned}$$

we have for all $n \geqslant 1$

$$d(x_n, H_y) = \|x_n\|.$$

On the other hand, by Proposition 1, we have for all $n \geqslant 1$

$$\|x_n\| = \frac{\delta}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}}.$$

Thus, we can conclude that

$$d(x_n, H_y) = \frac{\delta}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}}.$$

Sufficiency. If $\{x_n,\, n \geqslant 1\}$ and $\{y_n,\, n \geqslant 1\}$ are not orthogonal, then $x_n \not\perp H_y$ for some n. Let $\gamma \geqslant 1$ be a natural number. Without loss of generality, assume $x_\gamma \not\perp H_y$ and so there exists a $y \in H_y$ such that $\langle x_\gamma, y\rangle \neq 0$. Thus $\langle x_\gamma, y\rangle^2 > 0$. Now, we note that $\Delta = 4\langle x_\gamma, y\rangle^2$ is the discriminant of the polynomial in λ

$$P(\lambda) = \|y\|^2\lambda^2 - 2\langle x_\gamma, y\rangle\lambda.$$

Since $\|y\|^2 > 0$ and $\Delta > 0$, we have that for all $\lambda \in (\lambda_1, \lambda_2)$

$$\|y\|^2\lambda^2 - 2\langle x_\gamma, y\rangle\lambda < 0,$$

where

$$\lambda_1 = \begin{cases} 0 & \text{if } \langle x_\gamma, y\rangle > 0, \\ 2\langle x_\gamma, y\rangle/\|y\| & \text{if } \langle x_\gamma, y\rangle < 0, \end{cases}$$

and

$$\lambda_2 = \begin{cases} 2\langle x_\gamma, y\rangle/\|y\| & \text{if } \langle x_\gamma, y\rangle > 0, \\ 0 & \text{if } \langle x_\gamma, y\rangle < 0. \end{cases}$$

Now, we remind that

$$\begin{aligned}\|x_\gamma - \lambda y\|^2 &= \langle x_\gamma - \lambda y, x_\gamma - \lambda y\rangle \\ &= \langle x_\gamma, x_\gamma\rangle - 2\lambda\langle x_\gamma, y\rangle + \lambda^2\langle y, y\rangle \\ &= \|x_\gamma\|^2 - 2\lambda\langle x_\gamma, y\rangle + \lambda^2\|y\|^2\end{aligned}$$

and hence,

$$\|y\|^2\lambda^2 - 2\lambda\langle x_\gamma, y\rangle = \|x_\gamma - \lambda y\|^2 - \|x_\gamma\|^2.$$

Thus the condition

$$\|x_\gamma - \lambda y\| < \|x_\gamma\| = \frac{\delta}{\sqrt{2}}\sqrt{1 - \frac{1}{\gamma + 1}}$$

is equivalent to

$$\|y\|^2\lambda^2 - 2\lambda\langle x_\gamma, y\rangle < 0.$$

We can conclude that

$$\|x_\gamma - \lambda y\| < \frac{\delta}{\sqrt{2}}\sqrt{1 - \frac{1}{\gamma + 1}} \quad \text{for all } \lambda \in (\lambda_1, \lambda_2)$$

and so

$$d(x_\gamma, H_y) < \frac{\delta}{\sqrt{2}}\sqrt{1 - \frac{1}{\gamma + 1}}.$$

This completes the proof. □

Proposition 2 shows that if the distance of a fixed random variable from a closed linear subset equals the norm of this random variable, then it is orthogonal to the set and vice versa.

Corollary 1. *Let $\alpha > 0$ and $\beta > 0$ be two real numbers. Let $\{x_n,\, n \geqslant 1\}$ be a sequence of random variables based on a α-rowwise equidistant, zero-sum array X and let $\{y_n,\, n \geqslant 1\}$ be a sequence of random variables based on another rowwise β-equidistant, zero-sum array Y. The relation*

$$d(x_n, H_y) = \frac{\alpha}{\sqrt{2}}\sqrt{1 - \frac{1}{n + 1}} \quad \textit{for all } n \geqslant 1$$

is true, if and only if for all $n \geqslant 1$

$$d(y_n, H_x) = \frac{\beta}{\sqrt{2}}\sqrt{1 - \frac{1}{n + 1}}.$$

P r o o f. If for all $n \geqslant 1$

$$d(x_n, H_y) = \frac{\alpha}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}},$$

then, by Proposition 2, it follows that $\{x_n,\, n \geqslant 1\}$ and $\{y_n,\, n \geqslant 1\}$ are orthogonal and so $H_x \perp y_n$ for all $n \geqslant 1$. Thus, by the Pythagorean theorem, we have for all $x \in H_x$ and $n \geqslant 1$

$$\|x - y_n\|^2 = \|x\|^2 + \|y_n\|^2.$$

This implies that for all $x \in H_x$

$$\|x - y_n\| \geqslant \|y_n\|.$$

Since

$$\|y_n\| = \frac{\beta}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}},$$

we can conclude that for all $n \geqslant 1$

$$d(y_n, H_x) = \frac{\beta}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}}.$$

In similar way we can prove that

$$d(y_n, H_x) = \frac{\beta}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}}$$

implies that for all $n \geqslant 1$

$$d(x_n, H_y) = \frac{\alpha}{\sqrt{2}}\sqrt{1 - \frac{1}{n+1}}.$$

□

The following proposition is an ergodic theorem that follows from Theorem 2 of Wermuth [1] for the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Note that this proposition is an immediate reformulation of Wermuth's result, so no proof is required.

Proposition 3. *Let $\{x_n,\, n \geqslant 1\}$ be a sequence of random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$E(x_n - x_k)^2 = \delta^2 > 0 \quad (n, k \geqslant 1,\ n \neq k),$$

then there exists a random variable $y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties:

(i) *The ergodic theorem is true for the sequence $\{x_n,\, n \geqslant 1\}$, that is,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = y.$$

(ii) $E(x_n - y)^2 = \delta^2/2$ *for all $n \geqslant 1$.*

Note that we do not impose any independence assumptions on the sequence $\{x_n,\, n \geqslant 1\}$ in this proposition.

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Authors' addresses: *U. Triacca*, Department of Economics, University of L'Aquila, P. le del Santuario 19, 67040 Roio Poggio, Aquila, Italy, e-mail: `triacca@ec.univaq.it`; *A. Volodin*, Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada, e-mail: `volodin@math.uregina.ca`.